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FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING CAUSAL OPERATORS

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ABSTRACT. In this paper we develop some existence results for fractional equations involving causal operators.

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1. INTRODUCTION

In this paper, we introduce fractional differential equations with causal operators. The importance of the theory of differential equations involving causal operators resides in its powerful quality of unifying various dynamic systems, such as ordinary differential equations, differential equations with delay and integro-differential equations, to name only a few. Fractional differential equations have the inherent advantage of providing a better framework for modeling many physical phenomena in biological and social sciences. In this paper, we combine these two fields by considering fractional differential equations in the set up of Caputo fractional derivative and proving some existence results.

2. PRELIMINARIES

We begin with some definitions. Let $t_0 \geq 0$ and $T > t_0$ be arbitrary and let $E = C[[t_0, T), \mathbb{R}^n]$ be a function space. The map $Q : E \to E$ is said to be a causal or

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a nonanticipative map if $x, y \in E$ have the property that if $x(s) = y(s)$, $t_0 \le s \le t$, then $(Qx)(s) = (Qy)(s)$, $t_0 \le s \le t$, $t < T$ [1].

Next, we give the definition of and relation between the Riemann-Liouville and Caputo fractional differential equations. The Riemann-Liouville fractional differential equation is given by

$$D^q x = (Qx)(t),\ x(t_0) = x^0 = x(t)(t - t_0)^{1-q}|_{t=t_0}, \quad t_0 \le t < T, \tag{2.1}$$

where $0 < q < 1$ and $\Gamma(q)$ is the standard gamma function. The corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (Qx)(s)\ ds,$$

where $x^0(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)}$ [8].

The Caputo derivative has the main advantage that the initial condition of the corresponding initial-value problem has the same form as that of ordinary differential equations, and also the derivative of a constant is zero. Hence, it is convenient to use the Caputo fractional derivative.

The fractional differential equation of Caputo type is given by

$$\left.\begin{array}{rcl} {}^cD^q x & = & (Qx)(s) \\ x(t_0) & = & x_0 \end{array}\right\} \tag{2.2}$$

where $0 < q < 1$. If $x \in C^q([t_0, t_0 + a], \mathbb{R}^n)$ satisfies (2.2), it also satisfies the Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (Qx)(s)\ ds, \tag{2.3}$$

and vice versa.

The relation between the two types of fractional derivatives is given by

$${}^cD^q x(t) = D^q (x(t) - x(t_0)).$$

Next, we state some results from [3, 5, 6] that are needed to prove our main theorems. These results are stated for fractional differential equations of Riemann-Liouville type, but they can be readily extended to those of Caputo type. Let $p = 1 - q$ and $C_p([t_0, T], \mathbb{R}) = \{u : u \in C((t_0, T], \mathbb{R}) \text{ and } (t - t_0)^p u(t) \in C([t_0, T], \mathbb{R})\}$.

Consider the initial-value problem (IVP)

$$D^q x = f(t, x), \quad x(t_0) = x^0 = x(t)\ (t - t_0)^{1-q}|_{t-t_0} \tag{2.4}$$

where $f \in C(R_0, \mathbb{R}^n)$, $R_0 = \{(t, x) : t_0 \le t \le t_0 + a \ \text{ and } |x - x^0(t)| \le b\}$, and $x^0(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)}$.

Lemma 2.1. *Let $m \in C_p([t_0, T], \mathbb{R})$ be locally Hölder continuous with exponent $\lambda > q$, and for any $t_1 \in (t_0, T]$,*

$$m(t_1) = 0 \text{ and } m(t) \le 0 \quad \text{for } t_0 \le t \le t_1.$$

Then,

$$D^q m(t_1) \ge 0. \tag{2.5}$$

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for $\epsilon > 0$, such that*

$$D^q x_\epsilon(t) = f(t, x_\epsilon(t))$$
$$x_\epsilon^0 = x_\epsilon(t)(t - t_0)^{1-q}|_{t=t_0},$$

and $|f(t, x_\epsilon(t))| \le M$ for $t_0 \le t \le T$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.

Theorem 2.3. *Assume that $m \in C_p([t_0, T],\ \mathbb{R}_+)$ is locally Hölder continuous, $g \in C([t_0, T] \times \mathbb{R}_+, \mathbb{R})$ and*

$$D^q m(t) \le g(t, m(t)), \quad t_0 \le t \le T.$$

Let $r(t)$ be the maximal solution of the IVP

$$D^q u(t) = g(t, u(t)), \quad u(t)(t - t_0)^{1-q}|_{t=t_0} = u^0 \ge 0, \tag{2.6}$$

existing on $[t_0, T]$, such that $m^0 \le u^0$, where $m^0 = m(t)(t - t_0)^{1-q}|_{t=t_0}$. Then, we have

$$m(t) \le r(t), \quad t_0 \le t \le T.$$

Lemma 2.4. *Assume that $f \in C[\Omega, \mathbb{R}]$, where Ω is an open set in $\mathbb{R}^2$, $(t_0, x^0) \in \Omega$, with $x^0 = x(t)(t-t_0)^{1-q}|_{t=t_0}$. Suppose that $[t_0, t_0+a)$ is the largest interval of existence of the maximal solution $r(t)$ of the fractional differential equation (2.4). Assume that $[t_0, t_1]$ is a compact interval of $[t_0, t_0 + a)$. Then, there is an $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, \epsilon)$ of*

$$D^q x = f(t, x) + \epsilon \text{ with initial value } x^0 + \epsilon, \tag{2.7}$$

where $x^0 = x(t)(t - t_0)^{1-q}|_{t=t_0}$, exists on $[t_0, t_1]$, and $\lim_{\epsilon \to 0} r(t, \epsilon) = r(t)$, uniformly on $[t_0, t_1]$.

3. EXISTENCE RESULTS

We begin with the theory of fractional differential inequalities.

Theorem 3.1. *Let $\alpha, \beta \in C^q(J, \mathbb{R}]$ be Hölder continuous with exponent $\lambda > q$, such that*

$$
\begin{aligned}
{}^cD^q\alpha(t) &\leq (Q\alpha)(t), \quad (3.1)\\
{}^cD^q\beta(t) &\geq (Q\beta)(t), \quad (3.2)
\end{aligned}
$$

with one of the inequalities (3.1) *or* (3.2) *being strict and $\alpha(t_0) < \beta(t_0)$. Then $\alpha(t) < \beta(t)$, $t \in J$.*

Proof. Suppose the conclusion does not hold. Then there exists a $t_1 > t_0$ such that $\alpha(t_1) = \beta(t_1)$ and $\alpha(t) < \beta(t)$, $t_0 \leq t < t_1$. Now set $m(t) = \alpha(t) - \beta(t)$. Then $m(t_1) = 0$ and $m(t) < 0$, $t_0 \leq t < t_1$.

Now, observe that ${}^cD^q m(t) = D^q[m(t) - m(t_0)]$, where $D^q m(t)$ is the Riemann-Liouville fractional derivative and also that $m(t_0) < 0$ implies $-D^q m(t_0) > 0$. Thus, by Lemma 2.1, we have ${}^cD^q m(t_1) \geq D^q m(t_1) \geq 0$. This yields

$$(Q\alpha)(t_1) \geq {}^cD^q\alpha(t_1) \geq {}^cD^q\beta(t_1) > (Q\beta)(t_1),$$

a contradiction. Here, we have used (3.2) with a strict inequality. The contradiction validates the conclusion and the proof is complete. □

Having proved the basic result for strict differential inequalities, we are now in a position to prove it for nonstrict inequalities.

Theorem 3.2. *Assume that the hypothesis of Theorem* 3.1 *holds with nonstrict inequalities. Further, assume that*

$$(Qx)(t) - (Qy)(t) \leq L \max_{t_0 \leq s \leq t} |x(s) - y(s)| \quad \text{for } x \geq y.$$

Then, $\alpha(t) \leq \beta(t)$ on J, provided $\alpha(t_0) \leq \beta(t_0)$.

Proof. Set $\beta_\epsilon(t) = \beta(t) + \epsilon E_q(2L(t - t_0)^q)$. Then, $\beta_\epsilon(t_0) = \beta(t_0) + \epsilon > \alpha(t_0)$. Further,

$$
\begin{aligned}
{}^cD^q\beta_\epsilon(t) &= {}^cD^q\beta(t) + 2L\epsilon E_q(2L(t - t_0)^q)\\
&\geq (Q\beta)(t) + 2L\epsilon E_q(2L(t - t_0)^q)\\
&\geq (Q\beta_\epsilon)(t) + L\epsilon E_q(2L(t - t_0)^q),
\end{aligned}
$$

which gives

$$
{}^cD^q\beta_\epsilon(t) > (Q\beta_\epsilon)(t). \quad (3.3)
$$

Now, applying Theorem 3.1 to (3.1) and (3.3), we obtain that $\alpha(t) < \beta_\epsilon(t)$. Taking the limit as $\epsilon \to 0$, we arrive at $\alpha(t) \leq \beta(t)$, and the conclusion holds. □

Next, we shall prove a general uniqueness theorem using successive approximations.

Theorem 3.3. *Assume that*

1. *$Q \in C[B, E]$ is a causal map where $B = B(x_0, b) = \{x \in E : \max_J |x(t) - x_0| \leq b\}$, $J = [t_0, T]$ and $|(Qx)| \leq M_0$ on B;*
2. *$g \in C(J \times [0, 2b], \mathbb{R}_+)$, $g(t, u) \leq M_1$ on $J \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, u)$ is nondecreasing in u for each $t \in J$, and $u(t) \equiv 0$ is the only solution of*

$$^cD^q u \; = \; g(t, u), \quad u(t_0) = 0 \text{ on } J; \tag{3.4}$$

and

3. *$|(Qx)(t) - (Qy)(t)| \leq g(t, |x - y|_0(t))$ on B, where $|x - y|_0(t) = \max_{t_0 \leq s \leq t} |x(t) - y(t)|$.*

Then, the successive approximations defined by

$$x_{n+1}(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (Qx_n)(s) ds, \quad n = 0, 1, 2, \ldots \tag{3.5}$$

exist and are continuous on $I_0 = [t_0, t_0 + \alpha]$, with $\alpha = \min(T - t_0, (\frac{b\Gamma(1+q)}{M})^{\frac{1}{q}})$ and $M = \max\{M_0, M_1\}$, and converge uniformly to the unique solution $x(t)$ of (2.2).

Proof. By our choice of α, we have, for $t \in I_0$,

$$|x_1(t) - x_0| \leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} |f(s, x_0)| ds.$$

$$\leq \frac{M(t - t_0)^q}{\Gamma(q + 1)} \leq \frac{M\alpha^q}{\Gamma(1 + q)} \leq b.$$

Hence, using induction, one can show that the successive approximations are continuous and satisfy

$$|x_n(t) - x_0| \leq b, \quad n = 0, 1, 2, \ldots \;. \tag{3.6}$$

Next, we shall define the successive approximations for the IVP (3.4) as follows:

$$u_0(t) = \frac{M(t - t_0)^q}{\Gamma(1 + q)},$$

$$u_{n+1}(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} g(s, u_n(s)) ds \quad \text{ on } I_0. \tag{3.7}$$

Since $g(t, u)$ is assumed to be nondecreasing in u for each t, using induction we can show that the successive approximations (3.7) are well defined and satisfy

$$0 \leq u_{n+1}(t) \leq u_n(t) \text{ on } I_0.$$

Moreover, $|D^q u_{n+1}(t)| = g(t, u_{n-1}(t)) \leq M$, and equicontinuity follows from Lemma 2.2. Thus, using Ascoli-Arzela theorem and the monotonicity of the sequence $\{u_n(t)\}$, we obtain $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on I_0. Clearly, $u(t)$ satisfies (3.4). Hence, by assumption (b), $u(t) \equiv 0$ on $[t_0, t_0 + \alpha] = I_0$.

We first note that $|x_1(t) - x_0| \leq M\frac{(t-t_0)^q}{\Gamma(1+q)} \equiv u_0(t)$, which gives $|x_1 - x_0)|_0(t) \leq u_0(t)$. Then, assuming $|x_k - x_{k-1}|_0(t) \leq u_{k-1}(t)$ for some k, we have

$$|x_{k+1}(t) - x_k(t)| \leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |(Qx_k)(s) - (Qx_{k-1}(s)| ds.$$

Using condition (c) and the monotone character of $g(t,u)$ in u, we get

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, |x_k - x_{k-1}|_0) ds \\ &\equiv u_k(t). \end{aligned}$$

Hence, $|x_{k+1} - x_k|_0(t) \leq u_k(t)$. Thus, by induction, the inequality $|x_{n+1} - x_n|_0(t) \leq u_n(t)$ on I_0 holds for all n. Also,

$$\begin{aligned} |{}^cD^q x_{n+1}(t) - {}^cD^q x_n(t)| &= |(Qx_n)(t) - (Qx_{n-1})(t)| \\ &\leq g(t, |x_n - x_{n-1}|_0(t)) \leq g(t, u_n(t)). \end{aligned}$$

Let $n \leq m$. Then,

$$\begin{aligned} {}^cD^{+q}|x_n(t) - x_m(t)| &\leq |\, {}^cD^q x_n(t) - {}^cD^q x_m(t)| \\ &\leq g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) + g(t, |x_n - x_m|_0(t)). \end{aligned}$$

Since $u_{n+1}(t) \leq u_n(t)$ for all n, it follows that

$${}^cD^{+q}|x_n(t) - x_m(t)| \leq g(t, |x_n - x_m|_0(t)) + 2(g(t, u_{n-1}(t))),$$

where ${}^cD^{+q}$ is the Caputo Dini derivative corresponding to D^+. An application of Theorem 2.3 (adjusted to the case of Caputo derivative) gives $|x_n - x_m|_0(t) \leq \gamma_n(t)$, on I_0, where $\gamma_n(t)$ is the maximal solution of the IVP

$${}^cD^q v = g(t,v) + 2g(t, u_{n-1}(t)), \quad v(t_0) = 0 \text{ for each } n.$$

Since, as $n \to \infty$, $g(t, u_{n-1}(t)) \to 0$ uniformly on I_0, using Lemma 2.4, we can conclude that $\gamma_n(t) \to 0$, uniformly on I_0. This implies that $\{x_n(t)\}$ converges uniformly to $x(t)$. Now, using the Volterra fractional integral equation (2.3), we can conclude that $x(t)$ is a solution of the IVP (2.2).

To show that the solution $x(t)$ is unique, suppose $y(t)$ is another solution of the IVP (2.2) on I_0. Define $m(t) = |x(t) - y(t)|$. Then, $m(t_0) = 0$ and, by condition (c),

$${}^cD^{+q}m(t) \leq |{}^cD^q x(t) - {}^cD^q m(t)| \leq |(Qx)(t) - (Qy)(t)| \leq g(t, |m|_0(t)).$$

Again, by Theorem 2.3, $m(t) \leq r(t, t_0, 0)$ on I_0, where $r(t)$ is the maximal solution of (3.4). But by assumption (c), $r(t) \equiv 0$. Hence, uniqueness follows and the proof is done. □

Next, assuming local existence, we prove a global existence result.

Theorem 3.4. *Let $Q \in C[E, E]$ be a causal map such that*

$$|(Qx)(t)| \leq g(t, |x|_0(t)), \tag{3.8}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t, u)$ is nondecreasing in u for each $t \in \mathbb{R}_+$, and the maximal solution $r(t) = r(t, t_0, u_0)$ of the IVP

$$^cD^q u = g(t, u), \quad u(t_0) = u_0 \geq 0 \tag{3.9}$$

exists on $[t_0, \infty)$. Suppose Q is such that the local existence of solutions of (3.4) is guaranteed for any $(t_0, x_0) \in \mathbb{R}_+ \times B$. Then, the largest interval of existence of any solution $x(t, t_0, x_0)$ of (2.2) is $[t_0, \infty)$, whenever $|x_0| \leq u_0$.

Proof. Suppose that $x(t) = x(t, t_0, x_0)$ is any solution of (2.2) existing on $[t_0, \beta)$, $t_0 < \beta < \infty$, with $|x_0| \leq u_0$, and that the value of β cannot be increased. Define $m(t) = |x(t)|$. Then, it follows that

$$^cD^{+q} m(t) \leq |^cD^q x(t)| = |(Qx)(t)| \leq g(t, |x|_0(t)) = g(t, |m|_0(t)),$$

and, using Theorem 2.3,we can conclude that $m(t) \leq r(t)$, $t_0 \leq t \leq \beta$.

Also we have

$$\begin{aligned} ^cD^g x(t) &= |(Qx)(t)| \\ &\leq g(t, |x|_0(t)) \\ &\leq g(t, |m|_0(t) \\ &\leq g(t, r(t)) \\ &\leq M, \quad t_0 \leq t \leq \beta, \end{aligned}$$

since $g(t, u) \geq 0$ and $r(t, t_0, u_0)$ is non decreasing. Now, for any t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$, we have

$$\begin{aligned} |x(t_1) - x(t_2)| &= |\frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} (Qx)(s) ds - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - s)^{q-1} (Qx)(s) ds| \\ &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| |(Qx)(s)| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} |(Qx)(s)| ds \\ &\leq \frac{M}{\Gamma(q)} [\int_{t_0}^{t_1} (t_1 - s)^{q-1} ds - \int_{t_0}^{t_1} (t_2 - s)^{q-1} ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds] \\ &= \frac{2M(t_2 - t_1)^q}{\Gamma(1 + q)}. \end{aligned}$$

Letting $t_1, t_2 \to \beta^-$ and using Cauchy criterion, we have that $\lim_{t\to\beta^-} x(t, t_0, x_0)$ exists. Set

$$x(\beta, t_0, x_0) = \lim_{t\to\beta^-} x(t, t_0, x_0),$$

and consider the IVP

$$^cD^q x = (Qx)(t), \quad x(\beta) = x(\beta, t_0, x_0).$$

The solution $x(t, t_0, x_0)$ can be continued beyond β because of our assumption of local existence. Hence, the claim is true and the proof is complete. □

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