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FRACTIONAL DIFFERENCE INEQUALITIES

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ABSTRACT. The basic inequalities for fractional difference equations have been obtained using a modified definition of the fractional difference operator [1].

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1. INTRODUCTION

Though the importance of fractional derivative to model a variety of real world problems became obvious few decades ago, the study of the theory of fractional differential equations was initiated and some basic results have been obtained recently [8]. In fact, the study of the theory of fractional dynamic systems is more global than the theory of classical ordinary differential equations. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent.

In [2] Diaz and Osler defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the n^{th} difference, to be any real or complex number.

In this paper, the definition of fractional difference of a function u_n given by [1], is slightly modified using which the fractional difference of function u_n is expressed in terms of the function at the previous arguments. Using the modified definition some important difference inequalities are obtained.

2. PRELIMINARIES

Definition 2.1. The backward difference operator Δ_{-n} is defined as $\Delta_{-n} = \varepsilon^{-1}(1 - B)$ where $Bf_n = f_{n-1}$ is standard backward shift operator and ε is interval length.

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In [3] Gray and Zhang gave a definition of the fractional difference as follows:

Definition 2.2. For any complex number α and f defined over the integer set $\{a-p, a-p+1, \ldots, n\}$, the α^{th} order difference of $f(n)$ over $\{a, a+1, \ldots, n\}$ is defined by

$$\nabla^{\alpha} f(n) = \frac{\nabla^{p}}{\Gamma(p-\alpha)} \sum_{k=0}^{n-a} \frac{\Gamma(k+p-\alpha)}{\Gamma(k+1)} f(n-k). \tag{2.1}$$

Later, Hirota [4] took the first n terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-B)^{\alpha}$ and gave the following definition.

Definition 2.3. Let $\alpha \in \mathbb{R}$. Then difference operator of order α is defined by

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j}, & \alpha \neq 1, 2, \ldots \\ \varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j}, & \alpha = m \in \mathbb{Z}_{>0}. \end{cases} \tag{2.2}$$

Here $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0. \end{cases} \tag{2.3}$$

In 2002, Nagai [1] introduced another definition of fractional difference which is a slight modification of Hirota's fractional difference operator.

Definition 2.4. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator $\Delta_{*,-n}$ of order α is defined as

$$\Delta_{*,-n}^{\alpha} u_n = \Delta_{-n}^{\alpha-m} \Delta_{-n}^{m} u_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \Delta_{-(n-j)}^{m} u_{n-j}. \tag{2.4}$$

Definition 2.5. Let $f(n,r)$ be any function defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and consider the initial value problem

$$\nabla^{\alpha} u_{n+1} = f(n, u_n), \qquad u(0) = u_0. \tag{2.5}$$

A function v_n defined on $\mathbb{N}_0^+$ is said to be an under function with respect to the initial value problem (2.5) if $\nabla^{\alpha} v_{n+1} \leq f(n, v_n)$. Similarly any function w_n defined on $\mathbb{N}_0^+$ is said to be a over function with respect to the initial value problem (2.5) if $\nabla^{\alpha} w_{n+1} \geq f(n, w_n)$.

3. MAIN RESULTS

In this paper, we consider a particular case of (2.4). By taking the interval length $\varepsilon = 1$ and $m = 1$, (2.4) becomes

$$\Delta^{\alpha}_{*,-n} u_n = \sum_{j=0}^{n-1} \binom{\alpha-1}{j} (-1)^j \Delta_{-(n-j)} u_{n-j}.$$

Since for $0 < \alpha \leq 1$, $\binom{\alpha-1}{j} = (-1)^j \binom{j-\alpha}{j}$,

$$\Delta^{\alpha}_{*,-n} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \Delta_{-(n-j)}\ u_{n-j}.$$

For convenience, we denote the backward difference operator $\Delta_{*,-n}$ by ∇. Then the fractional difference operator of order α $(0 < \alpha \leq 1)$ is given by

$$\nabla^{\alpha} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j}. \tag{3.1}$$

Throughout this paper we use (3.1) as the fractional difference operator of order α $(0 < \alpha \leq 1)$.

Remark 3.1. For any α $(0 < \alpha \leq 1)$,

$$\nabla^{-\alpha} u_n = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j}.$$

Remark 3.2. If f is defined over $\{0, 1, \ldots, n\}$, then using (2.1), the α^{th} order difference of $f(n)$ over $\{1, 2, \ldots, n\}$ can be written as

$$\begin{aligned}
\nabla^{\alpha} f(n) &= \frac{\nabla}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)} f(n-k) \\
&= \sum_{j=0}^{n-1} \frac{\Gamma(j+1-\alpha)}{\Gamma(1-\alpha)\Gamma(j+1)} \nabla f_{n-j} \\
&= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla f_{n-j}
\end{aligned}$$

which is same as (3.1). Hence (3.1) satisfies all the properties satisfied by (2.1) [3], which are given below:

i. For any real numbers α and β, $\nabla^{\alpha}\nabla^{\beta} u_n = \nabla^{\alpha+\beta} u_n$.
ii. For any constant 'c', $\nabla^{\alpha}[c u_n + v_n] = c\nabla^{\alpha} u_n + \nabla^{\alpha} v_n$.
iii. For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}(u_n v_n) = \sum_{m=0}^{n-1} \binom{\alpha}{m} [\nabla^{\alpha-m} u_{n-m}][\nabla^{\alpha} v_n]$.

Lemma 3.3. *Let $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and u_n be a function defined on $\mathbb{N}_0^+$. Then for $0 < \alpha \leq 1$*

$$\nabla^\alpha u_n = u_n - \binom{n-1-\alpha}{n-1} u_0 - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u_{n-j}. \tag{3.2}$$

Proof. We know from (3.1) that

$$\begin{aligned}
\nabla^\alpha u_n &= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j} \\
&= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} (u_{n-j} - u_{n-j-1}) \\
&= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} u_{n-j} - \sum_{j=0}^{n-1} \binom{j-\alpha}{j} u_{n-j-1} \\
&= \Bigg[u_n + \binom{1-\alpha}{1} u_{n-1} + \binom{2-\alpha}{2} u_{n-2} + \cdots \\
&\quad + \binom{n-2-\alpha}{n-2} u_2 + \binom{n-1-\alpha}{n-1} u_1 \Bigg] \\
&\quad - \Bigg[u_{n-1} + \binom{1-\alpha}{1} u_{n-2} + \binom{2-\alpha}{2} u_{n-3} + \cdots \\
&\quad + \binom{n-2-\alpha}{n-2} u_1 + \binom{n-1-\alpha}{n-1} u_0 \Bigg] \\
&= u_n - \binom{n-1-\alpha}{n-1} u_0 + \left[\binom{1-\alpha}{1} - 1 \right] u_{n-1} + \cdots \\
&\quad + \left[\binom{n-1-\alpha}{n-1} - \binom{n-2-\alpha}{n-2} \right] u_1 \\
&= u_n - \binom{n-1-\alpha}{n-1} u_0 + \sum_{j=1}^{n-1} \left[\binom{j-\alpha}{j} - \binom{j-1-\alpha}{j-1} \right] u_{n-j} \\
&= u_n - \binom{n-1-\alpha}{n-1} u_0 - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u_{n-j}.
\end{aligned}$$

Hence the proof. □

Example 3.4. For any b with $|b| > 1$, using (3.1) we get

i. $\nabla^\alpha b^n = \dfrac{b^n(b-1)}{b} \displaystyle\sum_{j=0}^{n-1} \binom{j-\alpha}{j} b^{-j}.$

ii. $\nabla^{\frac{1}{2}} b^2 = \dfrac{b^2(b-1)}{b} \displaystyle\sum_{j=0}^{1} \binom{j-\frac{1}{2}}{j} b^{-j} = b(b-1)\left[1 + \dfrac{1}{2b}\right] = b^2 - \dfrac{1}{2}b - \dfrac{1}{2}.$

iii. $\nabla^{\frac{1}{2}} b^1 = b - 1.$

Remark 3.5. We note that $\nabla^\alpha u_0 = 0$ and $\nabla^\alpha u_1 = u_1 - u_0 = \nabla u_1$.

Remark 3.6. For $\alpha \in \mathbb{R}$,

$$\nabla^\alpha \nabla^{-\alpha} u_n = \nabla^{-\alpha} \nabla^\alpha u_n = u_n - u_0.$$

Also

$$\nabla^\alpha \nabla^{-\alpha} (u_n - u_0) = \nabla^{-\alpha} \nabla^\alpha (u_n - u_0) = u_n - u_0.$$

Theorem 3.7. *Let $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and $f(n,r)$ be a non decreasing function in r for any fixed n. Let v_n and w_n be two functions defined on $\mathbb{N}_0^+$. Suppose that for $n \geq 0$ and $0 < \alpha \leq 1$ the inequalities*

$$\nabla^\alpha v_{n+1} \leq f(n, v_n), \tag{3.3}$$

$$\nabla^\alpha w_{n+1} \geq f(n, w_n). \tag{3.4}$$

hold. Then $v_0 \leq w_0$ implies

$$v_n \leq w_n, \quad \text{for all } n \geq 0.$$

Proof. For $\alpha = 1$, fractional differences coincides with ordinary differences and hence the result is true [7]. Now consider for $0 < \alpha < 1$. Suppose that (3.5) is not true. Then because of $v_0 \leq w_0$ there exists a $k \in \mathbb{N}_0^+$ such that $v_k \leq w_k$ and $v_{k+1} > w_{k+1}$. It follows, using (3.2), (3.3), (3.4) and the monotone property of f, that

$$\begin{aligned} f(k, w_k) &\leq \nabla^\alpha w_{k+1} \\ &= w_{k+1} - \binom{k-\alpha}{k} w_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} \\ &< v_{k+1} - \binom{k-\alpha}{k} v_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} v_{k+1-j} \\ &= \nabla^\alpha v_{k+1} \\ &\leq f(k, v_k), \end{aligned}$$

which is a contradiction in view of the above assumptions and the monotonicity of $f(n,r)$ in r. Hence the proof. ☐

Remark 3.8. If we assume that $v_0 < w_0$ in Theorem 3.7 the equality in the conclusion (3.5) must be dropped.

Theorem 3.9. *Let $m_1(n,r)$ and $m_2(n,r)$ be two non negative functions defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and non decreasing with respect to r for any fixed $n \in \mathbb{N}_0^+$. Let y_n be a function defined for $n \in \mathbb{N}_0^+$ and that*

$$m_1(n, y_n) \leq \nabla^\alpha y_{n+1} \leq m_2(n, y_n) \tag{3.5}$$

for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. Let v_n and w_n be the solutions of the difference equations

$$\nabla^\alpha v_{n+1} = m_1(n, v_n), \quad v(0) = v_0, \tag{3.6}$$

$$\nabla^\alpha w_{n+1} = m_2(n, w_n), \quad w(0) = w_0. \tag{3.7}$$

and suppose that $v_0 \leq y_0 \leq w_0$. Then

$$v_n \leq y_n \leq w_n, \quad n \in \mathbb{N}_0^+. \tag{3.8}$$

Proof. Consider the second part of (3.6) and (3.8) i.e.

$$\nabla^\alpha y_{n+1} \leq m_2(n, y_n),$$
$$\nabla^\alpha w_{n+1} = m_2(n, w_n).$$

Applying Theorem (3.7), since $y_0 \leq w_0$ we obtain the right half of the inequality in (3.9) i.e. $y_n \leq w_n$. A similar argument yields the left half of the inequality (3.9). □

Theorem 3.10. *Let the functions $m_1(n, r)$ and $m_2(n, r)$ be as in Theorem 3.9 and x_n and y_n be the solutions of the difference equations*

$$\nabla^\alpha x_{n+1} = f(n, x_n), \qquad x(0) = x_0, \tag{3.9}$$

$$\nabla^\alpha y_{n+1} = g(n, y_n), \qquad y(0) = y_0. \tag{3.10}$$

where x_n and y_n are defined for $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$ and $f(n, r)$ and $g(n, r)$ are defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and satisfy the condition

$$m_1(n, |x_n - y_n|) \leq |f(n, x_n) - g(n, y_n)| \leq m_2(n, |x_n - y_n|) \tag{3.11}$$

for all $n \in \mathbb{N}_0^+$. Let v_n and w_n be the solutions of (3.7) and (3.8) and for $n \in \mathbb{N}_0^+$. Assume that $v_0 \leq |x_0 - y_0| \leq w_0$. Then

$$v_n \leq |x_n - y_n| \leq w_n, \quad \textit{for all } n \in \mathbb{N}_0^+. \tag{3.12}$$

Proof. Let $u_n = |x_n - y_n|$. Then $u_0 = |x_0 - y_0| \leq w_0$. On account of the monotonicity of $m_2(n, r)$, we obtain, using Remark 3.5,

$$\begin{aligned} u_1 &= |x_1 - y_1| \\ &= |x_0 + f(0, x_0) - y_0 - g(0, y_0)| \\ &\leq |x_0 - y_0| + |f(0, x_0) - g(0, y_0)| \\ &\leq u_0 + m_2(0, |x_0 - y_0|) \\ &\leq w_0 + m_2(0, w_0) \\ &= w_0 + \nabla^\alpha w_1 \\ &= w_1. \end{aligned}$$

If the inequality $u_n \leq w_n$ is fulfilled for $n = 1, 2, \dots, k$, it follows by the monotonicity of $m_2(n,r)$ that

$$\begin{aligned}
u_{k+1} &= |x_{k+1} - y_{k+1}| \\
&= \Bigg| \binom{k-\alpha}{k} x_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} x_{k+1-j} + f(k, x_k) \\
&\quad - \binom{k-\alpha}{k} y_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} y_{k+1-j} - g(k, y_k) \Bigg| \\
&= \Bigg| \binom{k-\alpha}{k} (x_0 - y_0) + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} (x_{k+1-j} - y_{k+1-j}) \\
&\quad + f(k, x_k) - g(k, y_k) \Bigg| \\
&\leq \Bigg| \binom{k-\alpha}{k} (x_0 - y_0) \Bigg| + \Bigg| \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} (x_{k+1-j} - y_{k+1-j}) \Bigg| \\
&\quad + |f(k, x_k) - g(k, y_k)| \\
&\leq \binom{k-\alpha}{k} |(x_0 - y_0)| + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} |(x_{k+1-j} - y_{k+1-j})| \\
&\quad + |f(k, x_k) - g(k, y_k)| \\
&\leq \binom{k-\alpha}{k} w_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} + m_2(k, |x_k - y_k|) \\
&= \binom{k-\alpha}{k} w_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} + m_2(k, w_k) \\
&= w_{k+1}.
\end{aligned}$$

Hence by mathematical induction we obtain $|x_n - y_n| \leq w_n$ for all $n \in \mathbb{N}_0^+$. The proof of the left half of the inequality in (3.13) is similar. $\square$

Theorem 3.11. *Let $f(n,r,s)$ be a function defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$, $0 \leq s < \infty$ is non negative and nondecreasing with respect to r and s for any fixed $n \in \mathbb{N}_0^+$. Let u_n be solution of the difference equation*

$$\nabla^\alpha u_{n+1} = f(n, u_n, u_n), \qquad u(0) = u_0 \tag{3.13}$$

for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. Suppose that the inequality

$$\nabla^\alpha x_{n+1} \leq f(n, x_n, y_n). \tag{3.14}$$

is satisfied for all $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$, *where the functions* x_n *and* y_n *are defined for* $n \in \mathbb{N}_0^+$ *such that* $x_0 \leq u_0$. *Then*

$$x_n \leq u_n \tag{3.15}$$

for all $n \in \mathbb{N}_0^+$ *provided*

$$y_n \leq u_n \tag{3.16}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Consider (3.15) and (3.14) i.e.

$$\begin{aligned} \nabla^\alpha x_{n+1} &\leq f(n, x_n, y_n), \\ \nabla^\alpha u_{n+1} &= f(n, u_n, u_n). \end{aligned}$$

Since $y_n \leq u_n$, applying Theorem 3.9, $x_0 \leq u_0$ implies $x_n \leq u_n$. □

Remark 3.12. Let u_n be any function defined on $\mathbb{N}_0^+$ and $f(n, r)$ be a function defined on $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$. Then for $n \geq 0$ and $0 < \alpha \leq 1$,

$$\nabla^\alpha u_{n+1} = f(n, u_n)$$

or

$$\nabla^{-\alpha} \nabla^\alpha u_{n+1} = \nabla^{-\alpha} f(n, u_n).$$

By using Remarks 3.1 and 3.6 we get

$$u_{n+1} - u_0 = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla f(n-j, u_{n-j})$$

or

$$u_{n+1} = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla f(n-j, u_{n-j}) + u_0.$$

Theorem 3.13. *Let* u_n, a_n *and* b_n *be nonnegative functions defined for* $n \in \mathbb{N}_0^+$. *Let* $f(n, r)$ *be a nonnegative function defined for* $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ *and non decreasing in* r *for any fixed* $n \in \mathbb{N}_0^+$. *If*

$$u_n \leq a_n + b_n \sum_{j=0}^{n-2} \binom{j+\alpha}{j} \nabla f(n-1-j, u_{n-1-j}) \tag{3.17}$$

for $n \in \mathbb{N}_0^+$, *then*

$$u_n \leq a_n + b_n r_n \tag{3.18}$$

for $n \in \mathbb{N}_0^+$, *where* r_n *is the solution of the difference equation*

$$\nabla^\alpha r_{n+1} = f(n, a_n + b_n r_n), \;\; r(0) = 0 \tag{3.19}$$

for $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$.

Proof. Define function z_n by

$$z_n = \sum_{j=0}^{n-2} \binom{j+\alpha}{j} \nabla f(n-1-j, u_{n-1-j}).$$

Then $z_0 = 0$, $u_n \leq a_n + b_n z_n$ and using Remark 3.12

$$\nabla^\alpha z_{n+1} = f(n, u_n) \leq f(n, a_n + b_n z_n).$$

By using Theorem 3.9 we have $z_n \leq r_n$. Then $u_n \leq a_n + b_n z_n \leq a_n + b_n r_n$. Hence the proof. □

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