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ON EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF LINEAR DELAY DIFFERENCE EQUATIONS

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ABSTRACT. A necessary and sufficient condition is established for the existence of periodic solutions for a class of linear delay difference equations.

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1. INTRODUCTION

It is well known (see eg., [1, page 306]) that the nonhomogeneous linear equation $x'(t) = A(t)x(t) + f(t)$ has periodic solutions if and only if

$$\int_0^\omega y^T(t)f(t)\mathrm{d}t = 0 \tag{1.1}$$

for all periodic solutions $y(t)$ of period ω of the adjoint equation $y'(t) = -A^T(t)y(t)$ where $A \in C(\mathbb{R}, \mathbb{R}^{m\times m})$ and $f \in C(\mathbb{R}, \mathbb{R}^m)$ are periodic of period ω. 'T' denotes the transposition and $\mathbb{R}^m$ denotes the $m-$dimensional Euclidean space. This result was extended in [2, page 423] to delay differential equations of the form

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), \tag{1.2}$$

where A, B and f satisfy the same conditions and $\tau > 0$ is a fixed real number. Indeed, Halanay proved that (1.2) has periodic solutions if and only if (1.1) holds for all periodic solutions $y(t)$ of period ω of the adjoint equation

$$y'(t) = -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), \tag{1.3}$$

which was constructed in terms of

$$\langle y(t), x(t)\rangle = y^T(t)x(t) + \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)\mathrm{d}s. \tag{1.4}$$

Recently, the above result has been carried out for linear impulsive delay differential equations [3] and for linear impulsive differential equations with distributed delay [4]. In this paper, however, we shall consider a discrete time analogue of the above result and establish

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a necessary and sufficient condition for the existence of periodic solutions for a class of linear delay difference equations.

For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points. However, it is impossible to satisfy this requirement unless we can explicitly solve both equations. Often, it is desirable that a difference equation when derived from a differential equation preserves the dynamical features of the corresponding continuous time model. If such discrete models can be derived from continuous time delay models, then the discrete time models can be used without any loss of functional similarities of continuous models. There are several methods for deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the methods is based on appropriate modifications of the models. For this technique, differential equations with piecewise constant arguments prove helpful, see [5] for more information.

Assume that the average growth rate in (1.2) changes at regular intervals of time, then we can incorporate this aspect in (1.2) and obtain the following modified equation with piecewise constant arguments

$$x'(t) = A([t])x([t]) + B([t])x([t-\tau]), \quad t \geq 0, \tag{1.5}$$

where $[t]$ denotes the integer part of t for $t \geq 0$. This equation occupies a position midway between differential and difference equations. For more details on the theory of this type of equations, see the recent interesting papers [6, 7, 8]. Integrating (1.5) on any interval of the form $[n, n+1)$, $n = 0, 1, 2, \ldots$, we obtain

$$x(t) - x(n) = A(n)x(n) + B(n)x(n-\tau)\big(t-n\big).$$

Letting $t \to n+1$, we have

$$\Delta x(n) = A(n)x(n) + B(n)x(n-\tau), \tag{1.6}$$

where $\Delta x(n) := x(n+1) - x(n)$. Equation (1.6) is considered to be a discrete analogue of equation (1.2). For the sake of convenience, however, we shall consider equation of the form

$$\Delta x(n) = A(n)x(n) + B(n+1)x(n-j+1), \quad n \geq 0. \tag{1.7}$$

In the previous two decades, the study of qualitative properties of delay difference equations has attracted significant interest by many researchers. This is due, in a large part, to the rapidly increasing number of applications of the theory of these equations to various fields of applied sciences and technology [9, 10, 11, 12, 13, 14, 15]. In particular, existence of periodic solutions for delay difference equations has been extensively developed, see for instance [14, 16, 17, 18, 19]. This paper contributes to the theory of delay difference equations by proving a well known result using easily formulated algebraic analysis.

2. ADJOINT EQUATION AND SOLUTIONS REPRESENTATIONS

Let $\mathbb{N}$ be, as usual, the set of natural numbers. In equation (1.7), it is assumed that $2 \leq j$ is a fixed positive integer number and $A, B : \mathbb{N} \to \mathbb{R}^{m\times m}$. For any $a, b \in \mathbb{N}$, define $\mathbb{N}(a) = \{a, a+1, \ldots\}$ and $\mathbb{N}(a, b) = \{a, a+1, \ldots, b\}$ where $a \leq b$. By a solution of (1.7), we mean a sequence $x(n)$ of elements in $\mathbb{R}^m$ which is defined for all $n \in \mathbb{N}(n_0 - j)$ and satisfies (1.7) for $n \in \mathbb{N}(n_0)$ for some $n_0 \in \mathbb{N}$. It is easy to see that for any given $n_0 \in \mathbb{N}$ and initial conditions of the form

$$x(n) = \phi(n), \quad n \in \mathbb{N}(n_0 - j, n_0), \tag{2.1}$$

(1.7) has a unique solution $x(n)$ which is defined for $n \in \mathbb{N}(n_0 - j)$ and satisfies the initial conditions (2.1).

We shall construct the adjoint equation of (1.7) with respect to a function resembles (1.4). It turns out that the discrete analogue of (1.4) has the form

$$\langle y(n), x(n)\rangle = y^T(n)x(n) + \sum_{k=n+1}^{n+j-1} y^T(k)B(k)x(k-j). \tag{2.2}$$

We should remark that no periodicity condition is used throughout the results of this section.

Lemma 2.1. *Let $x(n)$ be any solution of* (1.7) *and $y(n)$ be any solution of*

$$\Delta y(n) = -A^T(n)y(n+1) - B^T(n+j)y(n+j), \tag{2.3}$$

then

$$\langle y(n), x(n)\rangle = \text{constant}. \tag{2.4}$$

Proof. Clearly, it suffices to show that $\Delta\langle y(n), x(n)\rangle = 0$. It follows that

$$\begin{aligned}\Delta\langle y(n), x(n)\rangle = {} & y^T(n+1)\Delta x(n) + \Delta y^T(n)x(n) + y^T(n+d)B(n+j)x(n)\\ & - y^T(n+1)B(n+1)x(n-j+1).\end{aligned}$$

In view of equations (1.7) and (2.3), we have

$$\begin{aligned}\Delta\langle y(n), x(n)\rangle = {} & y^T(n+1)[A(n)x(n) + B(n+1)x(n-j+1)]\\ & - [y^T(n+1)A(n) + y^T(n+j)B(n+j)]x(n)\\ & + y^T(n+j)B(n+j)x(n) - y^T(n+1)B(n+1)x(n-j+1) = 0.\end{aligned}$$

The proof is finished.

In virtue of Lemma 2.1, we may say that equation (2.3) is an adjoint of (1.7). It is easy to verify also that the adjoint of (2.3) is (1.7), i.e., they are mutually adjoint of each other.

Definition 2.2. A matrix solution $X(n, \alpha)$ of (1.7) satisfying $X(\alpha, \alpha) = I$, (I is an identity matrix), and $X(n, \alpha) = 0$ for $n < \alpha$ is called a fundamental matrix of (1.7).

Definition 2.3. A matrix solution $Y(n,\alpha)$ of (2.3) satisfying $Y(\alpha,\alpha)=I$ and $Y(n,\alpha)=0$ for $n>\alpha$ is called a fundamental matrix of (2.3).

Consider the nonhomogeneous equation

$$\Delta x(n)=A(n)x(n)+B(n+1)x(n-j+1)+f(n),\quad n\geq 0, \tag{2.5}$$

where $f:\mathbb{N}\to\mathbb{R}^m$. It is to be noted that the construction of function (2.2) is of special interest in itself. Besides, it is used to derive the adjoint equation in Lemma 2.1, solutions representations of equations (1.7), (2.3) and (2.5) can also be obtained using this function. In view of (2.4), we may write

$$\langle y(n),x(n)\rangle=\langle y(n_0),x(n_0)\rangle. \tag{2.6}$$

Replacing $y(\alpha)$ by $Y(\alpha,n)$ and using the properties of the fundamental matrix, we have the following result.

Lemma 2.4. *Let $X(n,\alpha)$ be a fundamental matrix of* (1.7) *and $n_0\in\mathbb{N}$. If $x(n)$ is a solution of* (1.7)*, then*

$$x(n)=X(n,n_0)x(n_0)+\sum_{k=n_0+1}^{n_0+j-1}X(n,k)B(k)x(k-j).$$

One can also obtain the solutions representation of equation (2.5). Indeed,

Lemma 2.5. *Let $X(n,\alpha)$ be a fundamental matrix of* (1.7) *and $n_0\in\mathbb{N}$. If $x(n)$ is a solution of* (2.5)*, then*

$$x(n)=X(n,n_0)x(n_0)+\sum_{k=n_0+1}^{n_0+j-1}X(n,k)B(k)x(k-j)+\sum_{k=n_0}^{n-1}X(n,k+1)f(k).$$

Upon replacing $x(\alpha)$ by $X(\alpha,n)$ in relation (2.6), one can similarly derive the solutions representation of the adjoint equation (2.3). Namely,

Lemma 2.6. *Let $Y(n,\alpha)$ is a fundamental matrix of* (2.3) *and $n_0\in\mathbb{N}$. If $y(n)$ is a solution of* (2.3)*, then*

$$y(n)=Y(n,n_0)y(n_0)+\sum_{k=n_0+1}^{n_0+j-1}Y(n,k-j)B^T(k)y(k).$$

Furthermore, relation (2.6) tells us that $X(n,n_0)=Y^T(n_0,n)$ which can be seen by replacing $x(n)$ by $X(n,n_0)$ and $y(n)$ by $Y(n,n_0)$.

3. PRELIMINARY ASSERTIONS

With regard to equation (2.5), the following conditions are assumed to be valid throughout the rest of the paper.

(i) $A, B : \mathbb{N} \to \mathbb{R}^{m\times m}$ are p periodic sequences, $p > j$;

(ii) $f : \mathbb{N} \to \mathbb{R}^m$ is p periodic sequence, $p > j$.

Let $x(n) = x(n;\varphi)$ be the solution of equation (2.5) defined for $n \geq 0$ such that $x(n)$ coincides with φ on $[-j, 0]$. The periodicity of the equation implies that $x(n+p;\varphi)$ is likewise a solution of the equation defined for $n+p \geq j$. If this solution coincides with φ in $[-j, 0]$, then on the basis of the uniqueness theorem it follows that $x(n+p;\varphi) = x(n;\varphi)$ for all $n \geq -j$ and the solution is periodic. Thus the periodicity condition of the solution is written as $x(n+p;\varphi) = \varphi(n)$ for $n \in [-j, 0]$. If W is defined by $W\varphi = x(n+p;\varphi)$, $n \in [-j, 0]$, then it follows that $x(n)$ is periodic if and only if $W\varphi = \varphi$, i.e., φ is a fixed point of W.

Let $z(n) = z(n;\varphi)$ be the solution of (1.7) defined for $n \geq 0$ such that $z(n) = \varphi(n)$ on $[-j, 0]$. Then by Lemma 2.5,

$$x(n;\varphi) = z(n;\varphi) + \sum_{k=0}^{n-1} X(n, k+1)f(k).$$

Define U by $U\varphi = z(n+p;\varphi),\ n \in [-j, 0]$. Then, since

$$W\varphi = U\varphi + \sum_{k=0}^{n+p-1} X(n+p, k+1)f(k),$$

the periodicity condition reads as

$$\varphi = U\varphi + \sum_{k=0}^{n+p-1} X(n+p, k+1)f(k). \tag{3.1}$$

Let $y(n) = y(n;\psi)$ be the solution of (2.3) defined for $n \leq p+j$ such that $y(n) = \psi(n)$ on $[p, p+j]$. Similarly, we conclude that if $y(n-p;\psi)$ coincides with ψ in $[p, p+j]$ then $y(n-p;\psi) = y(n;\psi)$ and hence the solution is periodic. From Lemma 2.6, we get

$$\psi(n) = X^T(p, n-p)\psi(p) + \sum_{k=p+1}^{p+j-1} X^T(k-j, n-p)B^T(k)\psi(k),$$

for $n \in [p, p+j]$. Let $\tilde{\varphi}(s) = \psi(s+p+j)$ for $s \in [-j, 0]$. Setting $\eta = k-p-j$, we find out

$$\tilde{\varphi}(s) = X^T(p, s+j)\tilde{\varphi}(-j) + \sum_{\eta=-j+1}^{-1} X^T(\eta+p, s+j)B^T(\eta+j)\tilde{\varphi}(\eta).$$

For convenience, we also use the notation

$$\langle \Psi(s), \Phi(s)\rangle = \Psi^T(-j)\Phi(0) + \sum_{k=-j+1}^{-1} \Psi^T(k)B(k+j)\Phi(k), \tag{3.2}$$

for matrix sequences Ψ and Φ defined on $[-j,0]$ as long as multiplication is possible. Note that $\langle\Psi(s),\Phi(s)\rangle$ is either a number or a vector or a matrix, depending on the sizes of Ψ and Φ.

The following lemma, which is a discrete analogue of Lemma 4 in [3], plays a key role in our later analysis. Its proof is straightforward and can be achieved directly by changing the order of summations.

Lemma 3.1. *For any matrix sequences $N, M, L \in \mathbb{R}^{m\times m}$, we have*

$$\langle\langle L(\sigma), M(\alpha,\sigma)\rangle^T, N(\alpha)\rangle = \langle L(\sigma), \langle M^T(\alpha,\sigma), N(\alpha)\rangle\rangle.$$

By using this notation, the operator U can be written as

$$U\varphi = \langle X^T(p+s,\eta+j),\varphi(\eta)\rangle.$$

If we define $\tilde{U}\tilde{\varphi} = \langle\tilde{\varphi}(\eta), X(p+\eta,s+j)\rangle^T$, then in view of Lemma 3.1 we obtain

$$\langle\tilde{U}\tilde{\varphi},\varphi\rangle = \langle\tilde{\varphi}(\eta),\langle X^T(p+\eta,s+j),\varphi(s)\rangle\rangle = \langle\tilde{\varphi},U\varphi\rangle.$$

Let $\tilde{V}\psi = y(n_0-p;\psi)$ for $n_0\in[p,p+j]$. That is,

$$\tilde{V}\psi = X^T(p,n_0-p)\psi(p) + \sum_{k=p+1}^{p+j-1} X^T(k-\tau,n_0-p)B^T(k)\psi(k),$$

for $n_0\in[p,p+j]$. If ρ is an eigenvalue of $\tilde{V}$, then there exists a nonzero solution of

$$\rho\tilde{\varphi}(s) = X^T(p,s+j)\tilde{\varphi}(-j) + \sum_{k=-j+1}^{-1} X^T(\eta+p,s+j)B^T(\eta+p)\tilde{\varphi}(\eta),$$

where $\tilde{\varphi}(s) = \psi(s+p+j)$, $s\in[-j,0]$. The right side of the above equation is nothing but $\tilde{U}\tilde{\varphi}$. Thus the eigenvalues of the operators $\tilde{U}$ and $\tilde{V}$ coincide and in addition, if ψ is an eigenfunction for $\tilde{V}$, then $\tilde{\varphi} = \psi(s+p+j)$ is an eigenfunction for $\tilde{U}$.

Lemma 3.2. *Equations* (1.7) *and* (2.3) *have the same number of linearly independent periodic solutions of period $p>j$.*

Proof. Consider the equation

$$\rho\varphi(s) - U\varphi(s) = F(s). \tag{3.3}$$

It is obvious that the fundamental matrix X can be written as a linear combination of linearly independent vectors. That is,

$$X(p+s,\eta+j) = \sum_{k=1}^{m} a_k(s)b_k(\eta) + K_1(s,\eta), \quad \text{for } s,\eta\in[-j,0]\times[-j,0],$$

where $a_k(s)$ are column and $b_k(\eta)$ are row linearly independent vectors, K_1 is a matrix such that $|K_1|$ is chosen small. Clearly, we have

$$X^T(p+s,\eta+j) = \sum_{k=1}^{m} b_k^T(\eta)a_k^T(s) + K_1^T(s,\eta).$$

Then, by using the fact that $\langle b_k^T(\eta)a_k^T(s), \varphi(s)\rangle = a_k(s)\langle b_k^T(\eta), \varphi(s)\rangle$, (3.3) becomes

$$\rho\varphi(s) - \sum_{k=1}^{m} a_k(s)\langle b_k^T(\eta), \varphi(\eta)\rangle - \langle K_1^T(s,\eta), \varphi(\eta)\rangle = F(s).$$

Setting

$$\nu(s) = \frac{1}{\rho}\sum_{k=1}^{m} a_k(s)\langle b_k^T(\eta), \varphi(\eta)\rangle + \frac{1}{\rho}F(s), \tag{3.4}$$

we obtain

$$\nu(s) = \varphi(s) - \frac{1}{\rho}\langle K_1^T(s,\eta), \varphi(\eta)\rangle. \tag{3.5}$$

Now consider equation of the form

$$\nu(s) = \varphi(s) - \lambda\langle K_1^T(s,\eta), \varphi(\eta)\rangle. \tag{3.6}$$

We seek a solution of the form $\varphi(s) = \sum_{i=0}^{\infty}\lambda^i\varphi_i(s)$. Substituting this into (3.6) and identifying the coefficients of the powers of λ, we obtain

$$\varphi_0(s) = \nu(s) \quad \text{and} \quad \varphi_i(s) = \langle K_1^T(s,\alpha), \varphi_{i-1}(\alpha)\rangle, \quad i = 1, 2, \ldots.$$

It follows that $|\varphi_i(s)| \leq M^i \sup\limits_s |\nu(s)|$, where $M = \sup|K_1^T|$ and $i = 1, 2, \ldots$. Therefore, the series converges if $|\lambda|M < 1$. We have

$$\varphi_1(s) = \langle K_1^T(s,\alpha), \nu(\alpha)\rangle.$$

By the induction principle, we obtain

$$\varphi_l(s) = \langle K_l^T(s,\alpha), \nu(\alpha)\rangle,$$

where $K_l(s,\eta) = \langle K_1^T(s,\alpha), K_{l-1}(\alpha,\eta)\rangle$. Indeed, we have

$$\varphi_{l+1}(s) = \langle K_1^T(s,\alpha), \varphi_l(\alpha)\rangle = \langle K_1^T(s,\alpha), \langle K_l^T(\alpha,\eta), \nu(\eta)\rangle\rangle.$$

Using Lemma 3.1, we get

$$\varphi_{l+1}(s) = \langle\langle K_1^T(s,\alpha), K_l(\alpha,\eta)\rangle^T, \nu(\eta)\rangle = \langle K_{l+1}^T(s,\eta), \nu(\eta)\rangle.$$

It follows that, if $|\lambda| < \frac{1}{M}$ then the solution of equation (3.6) can be written as

$$\varphi(s) = \nu(s) + \sum_{l=1}^{\infty}\lambda^l\varphi_l(s) = \nu(s) + \sum_{l=1}^{\infty}\lambda^l\langle K_l^T(s,\alpha), \nu(\alpha)\rangle.$$

Thus, $\varphi(s) = \nu(s) + \langle\Gamma^T(s,\alpha), \nu(\alpha)\rangle$ where $\Gamma^T(s,\alpha) = \sum_{l=1}^{\infty}\lambda^l K_l^T(s,\alpha)$. Therefore, if $\frac{1}{|\rho|} < \frac{1}{M}$ and $\sup|K_1^T| < |\rho|$, we deduce that

$$\varphi(s) = \nu(s) + \langle\Gamma^T(s,\alpha), \nu(\alpha)\rangle \tag{3.7}$$

is a solution of (3.5).

On the other hand, consider the equation

$$\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0,$$

which can be written as

$$\rho\tilde{\varphi}(s) = \sum_{k=1}^{m} b_k^T(s)\langle\tilde{\varphi}(\alpha), a_k(\alpha)\rangle^T + \langle\tilde{\varphi}(\alpha), K_1(\alpha, s)\rangle^T.$$

Setting

$$\tilde{\nu}(s) = \frac{1}{\rho}\sum_{k=1}^{m} b_k^T(s)\langle\tilde{\varphi}(\alpha), a_k(\alpha)\rangle^T, \tag{3.8}$$

we obtain

$$\tilde{\nu}(s) = \tilde{\varphi}(s) - \frac{1}{\rho}\langle\tilde{\varphi}(\alpha), K_1(\alpha, s)\rangle^T. \tag{3.9}$$

Following similar analysis, we obtain that the solution of (3.9) is in the form

$$\tilde{\varphi}(s) = \tilde{\nu}(s) + \langle\tilde{\nu}(\alpha), \tilde{\Gamma}(\alpha, s)\rangle^T, \tag{3.10}$$

where $\tilde{\Gamma}(\alpha, s) = \sum_{l=1}^{\infty} \lambda^l \tilde{K}_l(\alpha, s)$ and $\tilde{K}_l(\eta, s) = \langle K_{l-1}^T(\eta, \alpha), K_1(\alpha, s)\rangle$. However, using the induction principle and Lemma 3.1, it is easy to verify that $\tilde{K}_l(\eta, s) = K_l(\eta, s)$ by which one can see that

$$\tilde{\Gamma}(\eta, s) = \Gamma(\eta, s). \tag{3.11}$$

In view of equation (3.4), we have

$$\rho\nu(s) = \sum_{k=1}^{m} a_k(s)\langle b_k^T(\eta), \varphi(\eta)\rangle + F(s). \tag{3.12}$$

Since $\varphi(s) = \nu(s) + \langle\Gamma^T(s, \alpha), \nu(\alpha)\rangle$, we have

$$\rho\nu(s) = \sum_{k=1}^{m} a_k(s)\langle b_k^T(\eta), \nu(\eta) + \langle\Gamma^T(\eta, \alpha), \nu(\alpha)\rangle\rangle + F(s),$$

which can be written as

$$\rho\nu(s) = \sum_{k=1}^{m} a_k(s)\Big(\langle b_k^T(\eta), \nu(\eta)\rangle + \langle b_k^T(\eta), \langle\Gamma^T(\eta, \alpha), \nu(\alpha)\rangle\rangle\Big) + F(s).$$

Using Lemma 3.1, we get

$$\rho\nu(s) = \sum_{k=1}^{m} a_k(s)\langle b_k^T(\alpha) + \langle b_k^T(\eta), \Gamma(\eta, \alpha)\rangle^T, \nu(\alpha)\rangle + F(s).$$

Hence

$$\rho\nu(s) = \sum_{k=1}^{m} a_k(s)\langle\bar{b}_k^T(\alpha), \nu(\alpha)\rangle + F(s), \tag{3.13}$$

where $\bar{b}_k^T(\alpha) = b_k^T(\alpha) + \langle b_k^T(\eta), \Gamma(\eta, \alpha)\rangle^T$. Setting $\lambda_k = \langle\bar{b}_k^T(\alpha), \nu(\alpha)\rangle$, it follows from (3.13) that

$$\rho\nu(s) - F(s) = \sum_{k=1}^{m} \lambda_k a_k(s) \tag{3.14}$$

which is of the form of the solution of (3.13). Analogously, the solution of

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{m} b_k^T(s)\langle\tilde{\nu}(\eta), \bar{a}_k(\eta)\rangle^T, \tag{3.15}$$

has the form

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{m} \mu_k b_k^T(s), \tag{3.16}$$

where $\mu_k = \langle \tilde{\nu}(\eta), \bar{a}_k(\eta)\rangle^T$ and $\bar{a}_k(\eta) = a_k(\eta) + \langle \tilde{\Gamma}^T(\eta,\alpha), a_k(\alpha)\rangle$. In view of (3.13), (3.14) becomes

$$\sum_{k=1}^{m} \lambda_k a_k(s) = \sum_{k=1}^{m} a_k(s)\langle \bar{b}_k^T(\alpha), \frac{1}{\rho}F(\alpha) + \frac{1}{\rho}\sum_{j=1}^{m} \lambda_j a_j(\alpha)\rangle. \tag{3.17}$$

Similarly, equation (3.15) implies that (3.16) can be written as

$$\sum_{k=1}^{m} \mu_k b_k^T(s) = \sum_{k=1}^{m} b_k^T(s)\langle \frac{1}{\rho}\sum_{j=1}^{m} \mu_j b_j^T(\eta), \bar{a}_k(\eta)\rangle^T. \tag{3.18}$$

Taking into account that the vectors $\{a_k\}$ are linearly independent, we obtain from (3.17) the algebraic equation

$$\rho\lambda_k = \sum_{j=1}^{m} \gamma_{kj}\lambda_j + f_k, \tag{3.19}$$

where $\gamma_{kj} = \langle \bar{b}_k^T(\alpha), a_j(\alpha)\rangle$ and $f_k = \langle \bar{b}_k^T(\alpha), F(\alpha)\rangle$. Similarly, we get from (3.18) the algebraic equation

$$\rho\mu_k = \sum_{j=1}^{m} \tilde{\gamma}_{jk}^T \mu_j, \tag{3.20}$$

where $\tilde{\gamma}_{jk}^T = \langle b_j^T(\eta), \bar{a}_k(\eta)\rangle$. We know that equation (3.19) for λ_k has a solution if and only if

$$\sum_{k=1}^{m} \mu_k f_k = 0 \tag{3.21}$$

for all the solutions μ_k of the equation

$$\rho\mu_k = \sum_{j=1}^{m} \gamma_{jk}\mu_j. \tag{3.22}$$

By employing Lemma 3.1 and relation (3.11), however, we can obtain that $\tilde{\gamma}_{jk}^T = \gamma_{jk}$. Thus, equations (3.20) and (3.22) coincide.

Therefore, we conclude that the equations

$$\rho\lambda_k = \sum_{j=1}^{m} \gamma_{kj}\lambda_j \tag{3.23}$$

and

$$\rho\mu_k = \sum_{j=1}^{m} \gamma_{jk}\mu_j \tag{3.24}$$

have the same number of linearly independent solutions. To a solution of (3.23) corresponds $\nu(s) = \frac{1}{\rho}\sum_{k=1}^{m} \lambda_k a_k(s)$ and to this corresponds the solution $\varphi(s) = \nu(s) +$

$\langle \Gamma^T(s,\alpha), \nu(\alpha)\rangle$ for the equation $\rho\varphi(s) - U\varphi(s) = 0$, linearly independent solutions corresponding to the linearly independent solutions of equation (3.23). Likewisely, a solution of the equation $\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$ will correspond to a solution of equation (3.20) which coincides with (3.24), linearly independent solutions corresponding to linearly independent solutions. It follows from here that the equations $\rho\varphi(s) - U\varphi(s) = 0$ and $\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$ have the same number of independent solutions, which implies in particular the fact that U and $\tilde{U}$ have the same eigenvalues, hence if ρ is a multiplier of the equation, $\frac{1}{\rho}$ is a multiplier of the adjoint equation. The proof of Lemma 3.2 is completed.

4. THE MAIN THEOREM

We are now in a position to state and prove the main result of this paper.

Theorem 4.1. *A necessary and sufficient condition for the existence of periodic solutions of period* p *of equation* (2.5) *is that*

$$\sum_{k=0}^{p-1} y^T(k+1)f(k) = 0, \tag{4.1}$$

for all periodic solutions $y(n)$ *of period* p *of the adjoint equation* (2.3).

NECESSITY. Let $x(n)$ be p periodic solution of (2.5) and $y(n)$ p periodic solution of (2.3). It follows that $\langle y(n), x(n)\rangle$ is p periodic. In view of (2.3) and (2.5), one can conclude that

$$\Delta\langle y(n), x(n)\rangle = y^T(n+1)f(n), \quad 0 \le n \le p. \tag{4.2}$$

Summing (4.2) over the interval $[0, p-1]$ results in

$$\sum_{k=0}^{p-1} y^T(k+1)f(k) = 0,$$

which is the same as (4.1).

SUFFICIENCY. Suppose that (4.1) is satisfied for all periodic solutions $y(n)$ of period p of (2.3). In virtue of relation (3.21), Lemma 3.2 tells us that

$$\rho\varphi(s) - U\varphi(s) = F(s)$$

has solutions if and only if

$$\langle \tilde{\varphi}(\alpha), F(\alpha)\rangle = 0 \tag{4.3}$$

for all $\tilde{\varphi}$ satisfying

$$\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0.$$

Therefore, it suffices to show that (4.3) holds under condition (4.1). To see this we first observe from (3.1) that

$$F(s) = \varphi(s) - U\varphi(s) = \sum_{k=0}^{s+p-1} X(s+p, k+1)f(k).$$

It follows that

$$\langle \tilde{\varphi}(\alpha), F(\alpha) \rangle = \tilde{\varphi}^T(-j)F(0) + \sum_{k=-j+1}^{-1} \tilde{\varphi}^T(k)B(k+j)F(k). \tag{4.4}$$

Substituting F into (4.4) leads to

$$\langle \tilde{\varphi}(\alpha), F(\alpha) \rangle = \tilde{\varphi}^T(-j) \sum_{k=0}^{p-1} X(p, k+1)f(k) + \sum_{k=-j+1}^{-1} \tilde{\varphi}^T(k)B(k+j)\Big[\sum_{r=0}^{k+p-1} X(p+k, r+1)f(r)\Big].$$

Setting $\tilde{\varphi}(s) = \psi(s+p+j)$ and interchanging the order of summations, we obtain

$$\langle \tilde{\varphi}(\alpha), F(\alpha) \rangle = \psi^T(p) \sum_{k=0}^{p-1} X(p, k+1)f(k) + \sum_{r=0}^{p-1} \sum_{q=-j+1}^{-1} \psi^T(q+p+j)B(q+j)X(q+p, r+1)f(r),$$

where that $X(p+\eta, \alpha) = 0$ for $\alpha > p+\eta$ is used. Reordering the terms, we finally get

$$\langle \tilde{\varphi}(\alpha), F(\alpha) \rangle = \sum_{k=0}^{p-1} \Big[\psi^T(p)X(p, k+1) + \sum_{q=-j+1}^{-1} \psi^T(q+p+j)B(q+j)X(q+p, k+1)\Big] f(k).$$

In view of Lemma 2.6 we see that the right hand side of the above equation is nothing but

$$\sum_{k=0}^{p-1} y^T(k+1)f(k)$$

which is clearly zero by our assumption (4.1). The proof is finished.

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