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NON-EXISTENCE OF GLOBAL SOLUTIONS TO SYSTEMS OF NON-AUTONOMOUS NONLINEAR PARABOLIC EQUATIONS

SEBTI KERBAL

Department of Mathematics and Statistics, Sultan Qaboos University
Muscat,P.O. Box 36, PC 126, Al-Khod, Oman
E-mail: skerbal@squ.edu.om

ABSTRACT. We consider the non-autonomous system of nonlinear parabolic equations

$$\begin{cases} u_t + t^r\,\Delta_\alpha u = |v|^q \\ v_t + t^s\,\Delta_\beta v = |u|^p \end{cases}$$

posed in $Q := (0,\infty)\times\mathbb{R}^N$, subject to the initial data $(u(0,x) = u_0(x), v(0,x) = v_0(x))$, where $p > 1$ and $q > 1$ are positive real numbers, $\alpha,\beta \in]0,2]$ and $\Delta_\gamma := (-\Delta)^{\gamma/2}$ is the $(-\Delta)^{\gamma/2}$ fractional power of $-\Delta$ in the x variable defined via the Fourier transform $\mathfrak{F}$ and its inverse $\mathfrak{F}^{-1}$ by $(-\Delta)^{\gamma/2}w(x,t) = \mathfrak{F}^{-1}\left(|\xi|^\gamma\mathfrak{F}(w)(\xi)\right)(x,t)$, where $r > -1$ and $s > -1$.

The Fujita critical exponent which separates the case of blowing-up solutions from the case of globally in time existing solutions is determined.

Keywords. Non-autonomous reaction-diffusion systems, critical exponent

AMS (MOS) Subject Classification. 35B33, 35K57, 35K65.

1. INTRODUCTION

The equation

$$u_t - \Delta u = |u|^p$$

has been considered by Fujita in his pioneering article [7]. He determined a critical exponent $p_c = 1 + \frac{2}{N}$ (called since then the Fujita critical exponent) such that:

- For $p < p_c$, any positive solution blows-up at a finite time.
- However for $p > p_c$, there are solutions that blow-up, and under a certain restriction solutions exist globally in time.

The critical case $p = p_c$ has been decided by Hayakawa [10] when $N = 2$, and by Kobayashi, Sirao and Tanaka [14] for any $N \geq 1$. Of course, many generalizations followed these important articles, see [4] for a long list of references. Nagasawa and Sirao [18] used a probabilistic treatment of blowing-up solutions to equations with fractional powers of the Laplacian of the form

$$u_t - \Delta_\alpha u = c(x)\,|u|^p$$

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for a certain positive function $c(x)$. Sugitani [20] treated the same equation with $c(x) = 1$, while Guedda and Kirane [9] considered

$$u_t - \Delta_\alpha u = d(t)\,|u|^p$$

for a certain positive function $d(t)$. The very recent article of Kirane, Laskri and Tatar [12] treated the more general equation

$$D^\delta_{0|t} u - \Delta_\alpha u = h(t,x)\,|u|^p$$

where $h(t,x) = O(t^\sigma\,|x|^\rho)$ for large $|x|$ and $D^\delta_{0|t} u$ is the time fractional derivative of u in the sense of Caputo [12].

The case of the system (RDS) has been studied when $r = s = 0$ by Escobedo and Herrero [5]; they showed that when $pq > 1$ and $(\mu+1)/(pq-1) \geq N/2$ with $\mu = \max\{p,q\}$, any nontrivial positive solution to system (RDS) blows-up in a finite time. Certain generalizations have been considered in [6, 9, 13].

In this paper, we consider a more general case and present the Fujita exponent for the system (RDS) using the test function method due to Mitidieri and Pohozaev [16, 17, 19].

The study of fractional diffusion equation is motivated by the use of fractional models in different fields of science such as transport theory, plasma physics, porous media and so on (see [2, 15] and the reference therein)

This article deals with the non-autonomous reaction-diffusion system

$$(RDS)\begin{cases} u_t + t^r\,\Delta_\alpha u = |v|^q \\ v_t + t^s\,\Delta_\beta v = |u|^p \end{cases}$$

posed in $Q := (0,\infty)\times\mathbb{R}^N$, subject to the initial data $(u(0,x) = u_0(x), v(0,x) = v_0(x))$, where $p > 1$ and $q > 1$ are real numbers, $\alpha, \beta \in\,]0,2]$, $r > -1$, $s > -1$.

2. PRELIMINARIES AND NOTATIONS

Let $S_\alpha(t,x)$ be the semi-group associated with the heat equation

$$u_t + \Delta_\alpha u = 0, \quad 0 < \alpha \leq 2, \quad t > 0, \quad x \in \mathbb{R}^N.$$

It is known that $S_\alpha(t,x)$ is defined by

$$S_\alpha(t,x) =: S_\alpha(t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{ix\xi - t|\xi|^\alpha} d\xi$$

satisfying the following properties

- $S_\alpha(t) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$
- $S_\alpha(t) \geq 0$ and $\int_{\mathbb{R}^N} S_\alpha(t)dx = 1$, $x \in \mathbb{R}^N$, $t > 0$.

and the following estimates:

- $\|S_\alpha(t) * u_0\|_p \leq \|u_0\|_p$, $u_0 \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, $t > 0$.

- $\|S_\alpha(t) * u_0\|_q \leq ct^{-\frac{N}{\alpha}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_p$.
- $\|\nabla S_\alpha(t)\|_q \leq ct^{-\frac{N}{\alpha}(1-\frac{1}{q})-\frac{1}{\alpha}}$ for any $u_0 \in L^p(\mathbb{R}^N)$, $1 \leq p < q \leq \infty$, $t > 0$.

In the sequel,the L^∞ and L^r norms will be denoted by $\|\cdot\|$ and $\|\cdot\|_r$ respectively. H^s will denote the usual Sobolev space and $\int$ will denote the integral over $\mathbb{R}^N$.

3. EXISTENCE OF LOCAL SOLUTIONS

The local existence is given by the following Theorem

Theorem 3.1. *Let $u_0 \in L^p(\mathbb{R}^N) \cap H^\alpha(\mathbb{R}^N)$ for $1 \leq p < \infty$, $0 < \alpha \leq 2$ and $v_0 \in L^q(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)$ for $1 \leq q < \infty$, $0 < \beta \leq 2$. Then, there is a $T_{\max} > 0$ such that the system (RDS) has a unique mild solution $(u, v) \in C([0, T_{\max}); L^p(\mathbb{R}^N) \cap H^\alpha(\mathbb{R}^N)) \times C\left([0, T_{\max}); L^q(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)\right)$. Moreover, if $u_0 \geq 0$, $v_0 \geq 0$, then $u > 0$ and $v > 0$.*

Proof. It is natural to associate to the system (RDS) the corresponding pair of integral equations

$$u(t,x) = \mathcal{U}(t,0)u_0(x) + \int_0^t \mathcal{U}(t,s)|v|^q(s,x)\,ds, \quad t > 0, \quad x \in \mathbb{R}^N, \tag{3.1}$$

and

$$v(t,x) = \mathcal{V}(t,0)v_0(x) + \int_0^t \mathcal{V}(t,s)|u|^p(s,x)\,ds, \quad t > 0, \quad x \in \mathbb{R}^N, \tag{3.2}$$

where $\{\mathcal{U}(t,s)_{t>s\geq 0}\}$ and $\{\mathcal{V}(t,s)_{t>s\geq 0}\}$ are the evolution families on $C_B(\mathbb{R}^N)$ that describe the solutions to the homogeneous Cauchy problem for the families of generators $\{t^r\Delta_\alpha\}_{t\geq 0}$ and $\{t^s\Delta_\beta\}_{t\geq 0}$, respectively. We know from [8, 1] that

$$\mathcal{U}(t,\sigma) = \mathcal{S}\left(\frac{t^{r+1} - \sigma^{r+1}}{r+1}\right), \quad t \geq \sigma \geq 0$$

and

$$\mathcal{V}(t,\sigma) = \mathcal{T}\left(\frac{t^{s+1} - \sigma^{s+1}}{s+1}\right), \quad t \geq \sigma \geq 0,$$

where $\{\mathcal{S}(t)\}_{t\geq 0}$ and $\{\mathcal{T}(t)\}_{t\geq 0}$ are the semigroups with infinitesimal generators Δ_α and Δ_β, respectively.

Let us first show the positivity of the solutions in case they exist locally in time. Assume that $u_0 \geq 0$ and $v_0 \geq 0$, then from the representation of the mild solution

$$u(t) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)|v|^q(s)\,ds$$

and

$$v(t) = \mathcal{V}(t,0)v_0 + \int_0^t \mathcal{V}(t,s)|u|^p(s)\,ds,$$

one immediately has

$$u(t) \geq \mathcal{U}(t,0)u_0$$

and

$$v(t) \geq \mathcal{V}(t,0)v_0.$$

But as $\mathcal{U}$ and $\mathcal{V}$ are positively preserving since $\mathcal{S}$ and $\mathcal{T}$ have the same property, then

$$u(t) > 0 \text{ whenever } u_0 > 0$$

and

$$v(t) > 0 \text{ whenever } v_0 > 0.$$

So in case a solution (u,v) exists, it is positive and even more bounded from below by $(\mathcal{U}(t,0)u_0, \mathcal{V}(t,0)v_0)$ componentwise.

The propagators $\mathcal{U}(t,0)$ and $\mathcal{V}(t,0)$ are Markovian from $L^\infty(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$, that is: $\|\mathcal{U}(t,0)\| \leq 1$ and $\|\mathcal{V}(t,0)\| \leq 1$. Now, we take $T > 0$ fixed, but otherwise arbitrary, and consider the set

$$\mathcal{X}_T = \left\{(u,v) : [0,T] \to L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N) \quad \text{such that } |||(u,v)||| < +\infty\right\}$$

where $|||(u,v)||| = \sup_{0\leq t\leq T} (\|u(t)\| + \|v(t)\|)$.

Clearly, $\mathcal{X}_T$ is a Banach space and

$$\mathcal{P}_T = \{(u,v) \in \mathcal{X}_T : u \geq 0, v \geq 0\}$$

is a closed subset of $\mathcal{X}_T$. Let

$$\mathcal{B}_T = \{(u,v) \in \mathcal{X}_T : |||(u,v)||| < R\}.$$

If we set

$$\Phi(v) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)v^q \, ds,$$

$$\Psi(u) = \mathcal{V}(t,0)v_0 + \int_0^t \mathcal{V}(t,s)u^p \, ds$$

and

$$F(u,v) = (\Phi(v), \Psi(u))$$

then, if $R > 0$ is large enough and $T > 0$ is sufficiently small, $F(u,v)$ is a strict contraction of $\mathcal{B}_T \cap \mathcal{X}_T$ into itself. Indeed

$$\begin{aligned} F(u,v) \quad - \quad & F(\overline{u},\overline{v}) = \\ & \left(\int_0^t \mathcal{U}(t,s)(v^q(s) - \overline{v}^q(s))\, ds \, , \int_0^t \mathcal{V}(t,s)(u^p(s) - \overline{u}^p(s))\, ds\right). \end{aligned}$$

Using the mean value theorem, it follows that

$$F(u,v) - F(\overline{u},\overline{v}) = (F_1(v,\overline{v}), F_2(\overline{u},\overline{u}))$$

where

$$F_1(v,\overline{v}) = q\int_0^t \mathcal{U}(t,s)(\theta v(s) + (1-\theta)\overline{v}(s))^{q-1}(v - \overline{v})\, ds$$

and

$$F_2(u,\overline{u}) = p\int_0^t \mathcal{V}(t,s)(\lambda u(s) + (1-\lambda)\overline{u}(s))^{p-1}(u(s)-\overline{u}(s))\,ds$$

for some $\theta = \theta(s) \in (0,1)$ and $\lambda = \lambda(s) \in (0,1)$. Where upon

$$\begin{aligned}|||F(u,v) \quad - \quad & F(\overline{u},\overline{v})||| = q\|v-\overline{v}\| \int_0^t \|(\theta\, v(s) + (1-\theta)\overline{v}(s))^{q-1}\|\,ds + \\ & p\|u-\overline{u}\| \int_0^t \|(\lambda u(s) + (1-\lambda)\overline{u}(s))^{p-1}\|\,ds \\ & \leq q\|v-\overline{v}\|(\theta R + (1-\theta)R^{q-1})T + p\|u-\overline{u}\|(\lambda R + (1-\lambda)R^{p-1})T \\ & \leq q\|v-\overline{v}\|R^{q-1}T + p\|u-\overline{u}\|R^{p-1}T \\ & \leq |||(u,v)-(\overline{u},\overline{v})|||(pR^{p-1}+qR^{q-1})T.\end{aligned}$$

If we choose $R > 0$ large enough and $T > 0$ small enough, we ensure that the mapping $F(u,v)$ is a contraction from $\mathcal{B}_T \cap \mathcal{X}_T \cap \mathcal{P}_T$ into itself. Hence a local solution exists in $\mathcal{X}_T$. The regularity of the solution is improved as usual leading to $u \in C([0,T_{\max});\, L^p(\mathbb{R}^N) \cap H^\alpha(\mathbb{R}^N))$ and $C\left([0,T_{\max}); L^q(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)\right)$. $\square$

4. NON-EXISTENCE OF GLOBAL SOLUTIONS

For the sake of the reader, we recall the following proposition from ([11, Proposition 3.3]) which will be used in the proof of our main result.

Proposition 4.1 ([11])**.** *Suppose that $\delta \in [0,2]$, $\beta + 1 \geq 0$, and $\theta \in C_0^\infty(\mathbb{R}^N)$. Then, the following point-wise inequality holds:*

$$|\theta(x)|^\beta\, \theta(x)(-\Delta)^{\delta/2}\theta(x) \geq \frac{1}{\beta+2}\,(-\Delta)^{\delta/2}|\theta(x)|^{\beta+2}. \tag{4.1}$$

Proof. The proof given in N. Ju ([11, Proposition 3.3]) for $N = 2$, making use of the Riesz potential representation of the operator $(-\Delta)^{\delta/2}$ is motivated by the proof of the Proposition 3.2 of A. Cordoba and D. Cordoba [3].

For the sake of the reader, we will reproduce Ju's Proof in dimension N. When $\delta = 0$ or $\delta = 2$, the result is obvious. Now, we consider the case $\delta \in (0,2)$. Then by proposition 3.3 [11],

$$(-\Delta)^{\delta/2}\theta(x) = C_\delta P.V. \int \frac{\theta(x)-\theta(y)}{|x-y|^{N+\delta}}dy.$$

Therefore,

$$|\theta(x)|^\beta\theta(x)(-\Delta)^{\delta/2}\theta(x) = C_\delta P.V. \int \frac{|\theta(x)|^{\beta+2} - |\theta(x)|^\beta\theta(x)\theta(y)}{|x-y|^{N+\delta}}dy.$$

By Young's inequality, if $\beta + 1 > 0$, then

$$|\theta(x)|^\beta\theta(x)\theta(y) \leq |\theta(x)|^{\beta+1}|\theta(y)| \leq \frac{\beta+1}{\beta+2}|\,\theta(x)|^{\beta+2} + \frac{1}{\beta+2}|\theta(y)|^{\beta+2}.$$

Thus,

$$|\theta(x)|^{\beta}\theta(x)(-\Delta)^{\delta/2}\theta(x) \geq C_{\delta}\frac{1}{\beta+2}P.V.\int \frac{|\theta(x)|^{\beta+2}-|\theta(y)|^{\beta+2}}{|x-y|^{N+\delta}}dy$$
$$= \frac{1}{\beta+2}(-\Delta)^{\delta/2}|\theta(x)|^{\beta+2}.$$

The case $\beta=-1$ is still valid from the above proof, without using Young's inequality. $\square$

The main result on nonexistence of global solution is given by

Theorem 4.2. *Assume that* $\int u_0(x) > 0$, $\int v_0(x) > 0$, *and let* $N \geq 1$ *and* $p > 1$, $q > 1$, $\alpha = 2(r+1)$, $\beta = 2(s+1)$. *If*

$$(pq-1)\,N \leq \max\left\{\frac{\alpha(q+1)}{r+1}, \frac{\beta(p+1)}{s+1}\right\} \tag{4.2}$$

then problem (RDS) *admits no global weak nonnegative solutions.*

Proof. The proof is by contradiction. Indeed we assume that the solution is global. Multiplying the first equation scalarly in L^2 and the second equation of the system (RDS) by φ^{θ}, where φ is a test function satisfying $\varphi(x,T)=0$ and integrating by parts, we obtain

$$\int_Q |v|^q\varphi^{\theta} + \int u_0(x)\varphi_0^{\theta}(x) = -\theta\int_Q u\varphi^{\theta-1}\varphi_t + \int_Q t^r u\Delta_{\alpha}\varphi^{\theta} \tag{4.3}$$

and

$$\int_Q |u|^q\varphi^{\theta} + \int v_0(x)\varphi_0^{\theta}(x) = -\theta\int_Q v\varphi^{\theta-1}\varphi_t + \int_Q t^r v\Delta_{\alpha}\varphi^{\theta}. \tag{4.4}$$

According to Ju's inequality, we have $\Delta_{\alpha}\varphi^{\theta} \leq \theta\varphi^{\theta-1}\Delta_{\alpha}\varphi$, and since u and v are positive, we may write

$$\int_Q v^q\varphi^{\theta} + \int_{supp\{\varphi\}} u_0(x)\varphi_0^{\theta}(x) \leq -\theta\int_{supp\{\varphi_t\}} u\varphi^{\theta-1}\varphi_t + \int_{supp\{\Delta_{\alpha}\varphi\}} t^r u\Delta_{\alpha}\varphi^{\theta} \tag{4.5}$$

and

$$\int_Q u^q\varphi^{\theta} + \int_{supp\{\varphi\}} v_0(x)\varphi_0^{\theta}(x) \leq -\theta\int_{supp\{\varphi_t\}} v\varphi^{\theta-1}\varphi_t + \int_{supp\{\Delta_{\beta}\varphi\}} t^r v\Delta_{\alpha}\varphi^{\theta} \tag{4.6}$$

where $supp$ stands for support. We are going to estimate terms in the right hand side of (4.5) and (4.6) using Hölder's inequality. For, we first estimate

$$-\int_Q u\varphi^{\theta-1}\varphi_t \leq \left(\int_Q |u|^p\varphi^{\theta}\right)^{1/p}\mathcal{I}_1 \tag{4.7}$$

and

$$-\int_Q t^r u\varphi^{\theta-1}\Delta_{\alpha}\varphi \leq \left(\int_Q |u|^p\varphi^{\theta}\right)^{1/p}\mathcal{I}_2 \tag{4.8}$$

where

$$\mathcal{I}_1 = \left(\int_{supp\{\varphi\}} \varphi^{(\theta-1-\frac{\theta}{p})p}|\varphi_t|^{p'}\right)^{1/p'}$$

and

$$\mathcal{I}_2 = \left(\int_{supp\{\Delta_\alpha \varphi\}} t^{p'r} \varphi^{(\theta-1-\frac{\theta}{p})p} |\Delta_\alpha \varphi|^{p'} \right)^{1/p'}$$

and $p + p' = pp'$.

Similarly, one can verify that

$$-\int_Q v\varphi^{\theta-1}\varphi_t \le \left(\int_Q |v|^q \varphi^\theta \right)^{1/q} \mathcal{J}_1 \tag{4.9}$$

and

$$-\int_Q t^s v\varphi^{\theta-1}\Delta_\beta \varphi \le \left(\int_Q |v|^q \varphi^\theta \right)^{1/q} \mathcal{J}_2 \tag{4.10}$$

where

$$\mathcal{J}_1 = \left(\int_{supp\{\varphi\}} \varphi^{(\theta-1-\frac{\theta}{q})q'} |\varphi_t|^{q'} \right)^{1/q'}$$

and

$$\mathcal{J}_2 = \left(\int_{supp\{\Delta_\beta \varphi\}} t^{q's} \varphi^{(\theta-1-\frac{\theta}{q})q'} |\Delta_\beta \varphi|^{q'} \right)^{1/q'}$$

and $q + q' = qq'$.

Define $\mathcal{I} = \left(\int_Q |u|^p \varphi^\theta \right)^{1/p}$ and $\mathcal{J} = \left(\int_Q |v|^q \varphi^\theta \right)^{1/q}$. Using the above estimates, we obtain the following inequalities

$$\mathcal{J}^q + \int u_0(x)\varphi_0^\theta(x) \le \mathcal{I}\left(\mathcal{I}_1 + \mathcal{I}_2\right) \tag{4.11}$$

and

$$\mathcal{I}^p + \int v_0\varphi_0^0(x) \le \mathcal{J}\left(\mathcal{J}_1 + \mathcal{J}_2\right). \tag{4.12}$$

At this stage, we choose

$$\varphi(t,x) = \chi\left(\frac{t}{R^2} + \frac{|x|^2}{R^2} \right) \tag{4.13}$$

where $\chi \in C^2(\mathbb{R})$ is defined by

$$\chi(\xi) = \begin{cases} 1 & \text{if } 0 \le \xi \le 1 \\ \searrow & \text{if } 1 \le \xi \le 2 \\ 0 & \text{if } \xi \ge 2. \end{cases}$$

In order to estimate the integrals $\mathcal{I}_1, \mathcal{I}_2$, $\mathcal{J}_1$, and $\mathcal{J}_2$, we use the change of variables

$$t = R^2\tau \quad \text{and} \quad x = Ry. \tag{4.14}$$

Thus

$$\mathcal{I}_1 = \theta\left(\int_Q \varphi^{(\theta-1-\frac{\theta}{p})p'} R^{-2p'} \varphi_\tau R^{2+N} dyd\tau \right)^{1/p'} \le C_1 R^{-2+\frac{2+N}{p'}} \tag{4.15}$$

and

$$\mathcal{I}_2 = \left(\int_Q \tau^{rp'} R^{r2p'} \varphi^{(\theta-1-\frac{\theta}{p})p'} R^{-\alpha p'} R^{2+N} \Delta_\alpha^y \varphi \, dy \, d\tau \right)^{1/p'} \le C_2 R^{-\alpha+\frac{2+N}{p'}+r2} \tag{4.16}$$

while

$$\mathcal{J}_1 = \theta \left(\int_Q \varphi^{(\theta-1-\frac{\theta}{q})q'} R^{-2q'} R^{2+N} \varphi_\tau \, dy \, d\tau \right)^{1/q'} \leq D_1 R^{-2+\frac{2+N}{q'}} \tag{4.17}$$

and

$$\mathcal{J}_2 = \left(\int_Q \tau^{sq'} R^{s2q'} \varphi^{(\theta-1-\frac{\theta}{q})q'} R^{-\beta q'} R^{2+N} \Delta_\beta^y \varphi \, dy \, d\tau \right)^{1/q'} \leq D_2 R^{-\beta+\frac{2+N}{q'}+s2}. \tag{4.18}$$

Observe that $\mathcal{I}_1$ and $\mathcal{I}_2$ in one hand, and $\mathcal{J}_1$ and $\mathcal{J}_2$ on the other hand are of the same order in R as $-2 + \frac{N+2}{p'} = 2r - \alpha + \frac{N+2}{p'}$ and $-2 + \frac{2+N}{q'} = 2s - \beta + \frac{2+N}{q'}$; i.e, $2 = \frac{\alpha}{r+1} = \frac{\beta}{s+1}$ as required where C_2, C_2, D_1, D_2 are positive constants. Now, since

$$\int u_0 > 0 \quad \text{and} \int v_0 > 0$$

then

$$\int u_0 \phi_0 \geq 0 \quad \text{and} \quad \int v_0 \phi_0 > 0 \quad \text{for } R \text{ large}.$$

Now, it follows from inequalities (4.11) and (4.12) that

$$\mathcal{J}^q \leq \mathcal{I} \left(\mathcal{I}_1 + \mathcal{I}_2 \right) \tag{4.19}$$

and

$$\mathcal{I}^p \leq \mathcal{J} \left(\mathcal{J}_1 + \mathcal{J}_2 \right) \tag{4.20}$$

respectively. Thus

$$\mathcal{J}^{q-\frac{1}{p}} \leq \left(\mathcal{I}_1 + \mathcal{I}_2 \right) \left(\mathcal{J}_1 + \mathcal{J}_2 \right)^{\frac{1}{p}} \tag{4.21}$$

and

$$\mathcal{I}^{p-\frac{1}{q}} \leq \left(\mathcal{I}_1 + \mathcal{I}_2 \right)^{\frac{1}{q}} \left(\mathcal{J}_1 + \mathcal{J}_2 \right). \tag{4.22}$$

Using the estimates of $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_1$, and $\mathcal{J}_2$, we obtain via (4.21)

$$\mathcal{J}^{q-\frac{1}{p}} \leq C \; R^{-2+\frac{N+2}{p'}} \; R^{-\frac{2}{p}+\frac{N+2}{pq'}}. \tag{4.23}$$

where C is a constant. We require

$$-2 + \frac{N+2}{p'} - \frac{2}{p} + \frac{N+2}{pq'} \leq 0 \tag{4.24}$$

giving $N \left(\frac{pq-1}{q+1} \right) \leq \frac{\alpha}{r+1}$. Similarly, we obtain via (4.22) the condition $N \left(\frac{pq-1}{p+1} \right) \leq \frac{\beta}{s+1}$. Finally, we have

$$N \left(pq - 1 \right) \leq \max \left\{ \frac{\alpha(q+1)}{r+1}, \frac{\beta(p+1)}{s+1} \right\}. \tag{4.25}$$

We consider two cases:

- The sub-critical case:

  If

$$N\,(pq-1) < \max\left\{\frac{\alpha(q+1)}{r+1}, \frac{\beta(p+1)}{s+1}\right\} \tag{4.26}$$

  then the right hand side of inequality (4.21) will go to zero as $R \to \infty$ and hence $\lim_{R\to\infty} \mathcal{J}^{\frac{pq-1}{p}} = 0$, which means that $v \equiv 0$ and hence $u \equiv 0$. This is a contradiction with our hypothesis.
- The critical case:

  If

$$N\,(pq-1) = \max\left\{\frac{\alpha(q+1)}{r+1}, \frac{\beta(p+1)}{s+1}\right\} \tag{4.27}$$

  then from estimation (4.21) we obtain

$$\mathcal{J}^{\frac{pq-1}{p}} \leq C_1 < \infty.$$

  In this case

$$\lim_{R\to\infty} \int_{\{R^2 \leq t+|x|^2 \leq 2R^2\}} \int |v|^q \phi = 0. \tag{4.28}$$

  From the estimate

$$\mathcal{I}^p \leq \mathcal{J}\,(\mathcal{C}+\mathcal{D}) \tag{4.29}$$

  one can see that the integrals are computed only on the domain

$$\Omega \equiv \left\{(x,t) : R^2 \leq t + |x|^2 \leq 2R^2\right\}.$$

  Letting $R \to \infty$ in expression (4.22), we obtain thanks to Lebesgue's dominate convergence theorem $\lim_{R\to\infty} \int\int |u|^q \phi = \int\int |u|^q \lim_{R\to\infty} \phi = 0$, giving $u \equiv 0$. Contradiction. This completes the proof.

□

Remark 4.3. One can observe that if $p = q$, $u_0 = v_0$, $r = s = 0$, and $\alpha = \beta = 2$, we obtain $u \equiv v$, and inequality (4.25) will read

$$\left(p^2-1\right) N \leq 2p+2 \Leftrightarrow p \leq \frac{2}{N} + 1 \tag{4.30}$$

which is the Fujita's exponent for the parabolic equation

$$u_t = \Delta u + |u|^p.$$

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