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A CLASS OF NONSTANDARD PARTIALLY OBSERVED STOCHASTIC SYSTEMS ON A HILBERT SPACE AND THEIR OPTIMAL STRUCTURAL FEEDBACK CONTROL

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ABSTRACT. In this paper we consider the question of weak compactness of the set of attainable measures on a Hilbert space induced by a class of partially observed nonstandard stochastic systems. The system is perturbed not only by Brownian motion but also by an arbitrary second order random process taking values from a Hilbert space. Structural controls used are measures with values from the space of bounded linear operators, $\mathcal{L}(Y, X)$ where X, Y are the state and output spaces, respectively. We consider several control problems and prove existence of optimal policies.

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1. INTRODUCTION

In this paper we are interested in partially observed stochastic control problem with structural controls which are operator valued measures. This is described by a stochastic differential equation on Hilbert space coupled with an algebraic equation representing noisy measurement process as follows:

$$dx = Ax\,dt + B(dt)y(t-) + \sigma(t)dW(t), \quad t \in I \equiv [0,T], x(0) = x_0, \tag{1.1}$$

$$y(t) = C(t)x(t) + \xi(t), \quad t \in I. \tag{1.2}$$

The process x is the state, y is the observation, and W and ξ are random processes to be described shortly. The system is called "partially observed" since the state x is not accessible; only the noisy measured output y is available for control. The operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X and B is an operator valued measure defined on the sigma algebra Σ of subsets of the set I with values in the space of bounded linear operators from Y to X. Any change of this operator valued measure means a structural change of the system or perturbation of the operator A. This operator valued measure is considered here as the decision or control variable which is called the structural control. Our concern here is to study the properties of the set of attainable measures induced by the system on the state space X and prove existence of optimal controls for several interesting control problems.

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This is a nonstandard stochastic system in several ways. First, it includes direct feedback control and its optimal choice. Second, the controls are operator valued measures much more general than mere operator valued functions. Third, the system is corrupted not only by Brownian motion; the measurement process is corrupted by a general Hilbert space valued second order random process, not the standard martingales. All these features allow a wider class of systems covering the classical ones. These class of systems are not covered in the literature including the monograms due to Da Prato and Zabczyk [1], [2].

There are both physical and theoretical motivations for study of this class of problems. For details see [4]. We mention here some physical motivations leading to such models. It is well known in aerospace engineering, that a change of physical configuration or structure of an aircraft can significantly alter the flight dynamics. This is done by appropriate maneuver of ailerons, rudders, elevators, wing flaps etc. These are structural controls. In material sciences, structural changes of molecules, for example polymerization, can produce new materials with desired properties. Many more examples are given in [4].

In addition to the above physical motivations, there is considerable theoretical interest in modeling hybrid systems and their control. See, for example, the special issue of the journal of hybrid Systems [3, p. 490–509], [8, p. 544–567] and [4-7, 9] and the extensive references there in. Except the recent paper [4], it seems not much work has been done on stochastic hybrid systems driven by structural controls based on partial information. Here, we consider this problem under relaxed assumptions compared to those used in [4] and prove existence of optimal structural feedback controls based on an entirely different approach involving functional analysis on the space of measures on Hilbert space and weak compactness property of the set of attainable measures.

The rest of the paper is organized as follows. In section 2 some basic notations and definitions are presented. Precise formulation of the control problem is given in section 3. In section 4, we study the properties of the set of attainable measures on the state space. Results on the existence of optimal structural feedback controls are presented in section 5 for several different objective functionals. The paper is concluded with some open problems.

2. NOTATIONS

Let $\mathcal{L}(Y,X)$ denote the space of bounded linear operators from a Hilbert space Y to a Hilbert space X. Furnished with the standard operator norm topology (uniform operator topology), it is a Banach space. Let $I \equiv [0,T]$ be a finite interval and Σ the sigma algebra of subsets of the set I. Let $M_{ba}(\Sigma, \mathcal{L}(Y,X))$ denote the space of all finitely additive bounded operator valued measures furnished with the total variation norm. For the uniform operator topology, the variation of $B \in M_{ba}(\Sigma, \mathcal{L}(Y,X))$ on any set $J \in \Sigma$ is given by

$$|B|_u(J) \equiv \sup_{\pi} \sum_{\sigma \in \pi} \|B(\sigma)\|_{\mathcal{L}(Y,X)} \tag{2.1}$$

where π denotes any finite, disjoint, Σ measurable partition of the set J. The supremum is taken over all such partitions. It is well known [10, 11] that, furnished with the total variation norm,

$$|B|_v \equiv \sup\{|B|_u(\sigma), \sigma \in \Sigma\}, \tag{2.2}$$

$M_{ba}(\Sigma, \mathcal{L}(Y,X))$ is a Banach space. Let $M_{ca}(\Sigma, \mathcal{L}(Y,X))$ denote the class of bounded countably additive members of this space. Furnished with the total variation norm, as defined above, this is a closed subspace of $M_{ba}(\Sigma, \mathcal{L}(Y,X))$ and hence also a Banach space. We denote this space by $M_{cabv}(\Sigma, \mathcal{L}(Y,X))$ indicating that its members are countably additive having bounded variation. The set $M_{rcabv}(\Sigma, \mathcal{L}(Y,X))$ will denote the class of regular measures contained in $M_{cabv}(\Sigma, \mathcal{L}(Y,X))$.

Throughout the rest of the paper $\{X, Y\}$ will denote a pair of separable Hilbert spaces. This is what we need for study of stochastic systems. For a fixed but arbitrary measure $\varrho \in M^+_{cabv}(\Sigma)$, we introduce the following class of operator valued measures

$$M_\varrho \equiv \left\{ B \in M_{cabv}(\Sigma, \mathcal{L}(Y,X)) : \sup_\pi \{ \sum_{\sigma \in \pi} \big(\|B(\sigma)\|_{\mathcal{L}(Y,X)} / \varrho(\sigma) \big) \varrho(\sigma) \} \equiv |B|_\varrho < \infty \right\},$$

where π denotes any finite and disjoint Σ-measurable partition of the interval I. Here we use the convention $0/0 = 1$. Since X, Y are Hilbert spaces, for every $B \in M_\varrho$, there exists an $\mathcal{L}(Y,X)$ valued measurable function F_B defined on I such that

$$B(\sigma)\zeta = \int_\sigma F_B(t)\zeta \; \varrho(dt), \quad \zeta \in Y, \quad \sigma \in \Sigma,$$

with $|B|_\varrho = \int_I \|F_B(t)\|_{\mathcal{L}(Y,X)} \varrho(dt)$. It is easy to see that M_ϱ is a Banach space with respect to this norm topology. A set $\Gamma \subset M_\varrho$ is bounded if there exists a finite positive number C_Γ such that

$$\sup\{|B|_\varrho, B \in \Gamma\} \leq C_\Gamma.$$

For admissible structural controls, we choose a weakly compact subset Γ of M_ϱ.

Definition 2.1 A bounded set $\Gamma \subset M_\varrho$ is said to be conditionally weakly compact if for every generalized sequence $\{B_n\} \in \Gamma$, there exists a generalized subsequence, relabeled as the original sequence, and an element $B_o \in M_\varrho$ such that

$$\int_I \langle g(t), B_n(dt) f(t) \rangle_X \longrightarrow \int_I \langle g(t), B_o(dt) f(t) \rangle_X \tag{2.3}$$

for every $f \in B_\infty(I, Y)$ and $g \in B_\infty(I, X)$. And it is said to be weakly compact if the limit $B_o \in \Gamma$.

For details on vector measures, the reader is refereed to the well known books of Diestel and Uhl [10] and Dunford and Schwartz [11].

For any Banach space Z, the space of nuclear operators (also called trace class operators) is a subset of the space of bounded linear operators $\mathcal{L}(Z)$ and it is denoted by $\mathcal{L}_1(Z)$. The collection of positive members of $\mathcal{L}_1(Z)$ is denoted by $\mathcal{L}_1^+(Z)$. We use $B_\infty(I, Z)$ to denote the vector space of bounded Σ-measurable functions with values in Z. Furnished with the supnorm topology it is a Banach space.

Additional notations will be introduced as and when required.

3. ATTAINABLE SET OF MEASURES

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathcal{P})$ be a complete filtered probability space. The filtration $\{\mathcal{F}_t, t \geq 0\}$ is a family of increasing right continuous complete subsigma algebras of the sigma-algebra $\mathcal{F}$ with $\mathcal{P}$ being the probability measure defined on $\mathcal{F}$. According to standard notation, the expected value of a random variable h is denoted by $E(h) \equiv \int_\Omega h(\omega)\mathcal{P}(dw) \equiv \bar{h}$. Let $\{X, Y, H\}$ denote three separable real Hilbert spaces, X denoting the state space, Y the output space and H the space where the Brownian motion takes values from. Consider the partially observed stochastic system on the Hilbert spaces $\{X, Y, H\}$ given by

$$dx = Ax\,dt + B(dt)y(t-) + \sigma(t)dW(t), \quad t \in I \equiv [0, T], \quad x(0) = x_0, \tag{3.1}$$

$$y(t) = C(t)x(t) + \xi(t), \quad t \in I, \tag{3.2}$$

where x is the state and y is the observation. The system is called "partially observed" since the state x is not accessible; only the noisy measured output y is available for control. Our general assumptions are as follows. The operator A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, on X and $B \in M_{cabv}(\Sigma, \mathcal{L}(Y, X))$, $\sigma \in B_\infty(I, \mathcal{L}(H, X))$, W is an H valued Brownian motion with covariance $Q \in \mathcal{L}_s^+(H)$, (the class of positive self adjoint linear operators in X), $C \in B_\infty(I, \mathcal{L}(X, Y)) \cap C(I, \mathcal{L}(X, Y))$ and ξ is a measurable random process taking values from Y. Without further notice, it is always assumed that all the random processes are adapted to the filtration $\mathcal{F}_t, t \geq 0$.

Let $\Gamma \subset M_\varrho \subset M_{cabv}(\Sigma, \mathcal{L}(Y, X))$ denote the class of admissible controls. The operator valued measure $B \in \Gamma$ represents structural control which is activated by observation y controlling the process x.

We need the following result.

Lemma 3.1 Suppose A generates a C_0-semigroup $S(t), t \geq 0$, on X and $C \in B_\infty(I, \mathcal{L}(X, Y)) \cap C(I, \mathcal{L}(X, Y))$. Then for every $B \in M_{cabv}(\Sigma, \mathcal{L}(Y, X))$ there exists a unique strongly measurable bounded evolution operator $U_B(t, s)$, $0 \leq s < t \leq T$, such that the evolution equation

$$dz = Azdt + B(dt)C(t)z(t-), \quad z(s) = \zeta \in X, \quad t \in [s, T], \tag{3.3}$$

has a unique mild solution $z \in B_\infty([s, T], X)$ given by $z(t) = U_B(t, s)\zeta$, $t \geq s$. Further it is easy to verify that

$$\sup\{\|U_B(t, s)\|_{\mathcal{L}(X)}, B \in \Gamma\} \leq b \equiv M \exp\{MCC_\Gamma\},$$

where $M \equiv \sup\{\|S(t)\|, t \in I\}$, $C \equiv \sup\{\|C(t)\|_{\mathcal{L}(X,Y)}, t \in I\}$, and $C_\Gamma \equiv \sup\{|B|_\varrho, B \in \Gamma\}$.

Proof. The proof is based on Banach fixed point theorem similar to that given in [4]. For a-priori bounds, it uses a generalized Gronwall type inequality proved in [9, Lemma 5]. □

Substituting the observation process y given by equation (3.2) into equation (3.1) we obtain the feedback system

$$dx = Ax\,dt + B(dt)C(t)x(t-) + B(dt)\xi(t) + \sigma(t)dW(t), x(0) = x_0, \quad t \in I. \tag{3.4}$$

It follows from Lemma 3.1 and Dhumels formula that for every $B \in M_{cabv}(\Sigma, \mathcal{L}(Y, X))$ the mild solution of equation (3.4) is given by

$$\begin{aligned} x(t) = x^B(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)B(ds)\xi(s) \\ + \int_0^t U_B(t,s)\sigma(s)dW(s), \quad t \in I. \end{aligned} \tag{3.5}$$

Let $\mathcal{B}(X)$ denote the sigma algebra of Borel subsets of the (separable) Hilbert space X and $\mathcal{M}_1(X)$ the space of probability measures on $\mathcal{B}(X)$. For each $B \in \Gamma$ and $t \in I$, define the measure on $\mathcal{B}(X)$ by

$$\mu_t^B(\sigma) \equiv Prob.\{x^B(t) \in \sigma\}, \quad \sigma \in \mathcal{B}(X).$$

We are interested in the attainable set of measures

$$\mathcal{A}(t) \equiv \{\mu_t^B, B \in \Gamma\}, \quad t \in I \tag{3.6}$$

and their application to control problems.

In particular, we consider the following control problems for the feedback system (3.4).

P1: Let $D \subset X$ be a given closed target set. Our problem is to find a structural control $B^o \in \Gamma$ for the feedback system (3.4) such that the probability of hitting this target at time T is maximum. Mathematically this can be formulated as follows. Define $J_1(B) \equiv \mu_T^B(D)$ and find a $B^o \in \Gamma$ such that

$$J_1(B^o) \geq J_1(B)\ \forall\ B \in \Gamma. \tag{3.7}$$

P2: Another problem of similar nature is the evasion problem. Here an open set $\mathcal{O} \subset X$ is given which is hazardous (forbidden) and must be avoided as far as possible. The problem is to find a structural control $B^o \in \Gamma$ for the feedback system (3.4) that minimizes the encounter probability with the set $\mathcal{O}$ given by $J_2(B) \equiv \mu_T^B(\mathcal{O})$. That is, find $B^o \in \Gamma$ such that

$$J_2(B^o) \leq J_2(B))\ \forall\ B \in \Gamma. \tag{3.8}$$

P3: Let $\psi : I \times X \longrightarrow R$ be a real valued function measurable in the first argument and continuous in the second, and ν is a countably additive bounded positive measure. The problem is to find a structural control for the system (3.4) that minimizes the functional J_3 given by,

$$J_3(B) \equiv \int_I \int_X \psi(t,x)\mu_t^B(dx)\nu(dt). \tag{3.9}$$

P4: Let $\varphi_i \in BC(X), t_i$(distinct) $\in I, i = 1, 2, \ldots m$ and $F : R^m \longrightarrow R$ a lower semi-continuous function. The problem is to find a structural control for the system (3.4) that minimizes the functional $J_4(B)$ given by

$$J_4(B) \equiv F\big(\mu_{t_1}^B(\varphi_1), \mu_{t_2}^B(\varphi_2), \ldots, \mu_{t_m}^B(\varphi_m)\big), \tag{3.10}$$

where $\mu_t(\varphi) \equiv \int_X \varphi(x)\mu_t(dx)$.

We are interested in the question of existence of optimal controls for the above problems subject to the dynamic constraints (3.1)-(3.2) or equivalently equation (3.4).

4. COMPACTNESS OF ATTAINABLE SETS

In this section we prove that the attainable set of measures, as defined by the expression (3.6), is conditionally weakly compact. For this we need the following preliminary results.

Without loss of generality, we assume that the mean of the random process ξ is zero. Thus, the evolution equation satisfied by the mean of the process x is given by

$$d\bar{x} = A\bar{x}dt + B(dt)C(t)\bar{x}(t-), \bar{x}(0) = \bar{x}_0.$$

Clearly the solution of this equation is given by

$$\bar{x}(t) \equiv \bar{x}^B(t) = U_B(t,0)\bar{x}_0,$$

where U_B is the evolution operator corresponding to $B \in \Gamma$ as given by Lemma 3.1. Consequently, the error process $\{e \equiv x - \bar{x}\}$ satisfies the evolution equation

$$de = Aedt + B(dt)C(t)e(t-) + B(dt)\xi(t) + \sigma(t)dW(t), e(0) = e_0, \quad t \in I. \tag{4.1}$$

Let P_0 denote the covariance operator corresponding to x_0 given by $(P_0\zeta, \zeta) \equiv E\{(e_0, \zeta)_X^2\}$, $\zeta \in X$. Let $Q \in \mathcal{L}_s^+(H)$(space of positive self adjoint operators in H) denote the incremental covariance of the Wiener process W taking values from H and $\hat{Q}(t) \equiv \sigma(t)Q\sigma^*(t)$, $t \geq 0$, taking values from $\mathcal{L}_s^+(X)$. We assume that the random process $\{\xi(t), t \geq 0\}$ satisfies the following property:

$\mathcal{A}(\Xi)$: The process $\{\xi(t), t \geq 0\}$ is an $\mathcal{F}_t$-adapted Y valued second order centered random process and there exists a $\beta \in R$ such that for the given finite interval $I \equiv [0, T]$,

$$\sup\{E|\xi(t)|_Y^2, t \in I\} \leq \beta^2 < \infty.$$

Lemma 4.1 Consider the system (4.1) and suppose the assumptions of Lemma 3.1 hold, and the random elements $\{x_0, W, \xi\}$ are stochastically independent. Then the covariance of the process $\{x^B \equiv x(t), t \in I\}$ determined by equation (3.5), corresponding to any choice of the control measure $B \in M_{cabv}(\Sigma, \mathcal{L}(Y, X))$, is given by the following expression

$$\begin{aligned} P^B(t) = U_B(t,0)P_0U_B^*(t,0) + \int_0^t U_B(t,r)\hat{Q}(r)U_B^*(t,r)dr \\ + \int_0^t \int_0^t U_B(t,s)B(ds)\tilde{K}(s,\tau)B^*(d\tau)U_B^*(t,\tau), \\ \equiv P_\alpha^B(t) + P_\beta^B(t) + P_\gamma^B(t), \quad t \in I, \end{aligned} \tag{4.2}$$

where $\tilde{K}$ is the correlation kernel of the process $\{\xi\}$ given by

$$(\tilde{K}(s,\tau)y, z)_Y \equiv E\{(\xi(s), y)(\xi(\tau), z)\}, \quad y, z \in Y, \tag{4.3}$$

for $(s, \tau) \in I$.

Proof. Using the evolution operator U_B given by Lemma 3.1, the (mild) solution of the evolution equation (4.1) for the error process $\{e\}$ is given by

$$e(t) = U_B(t,0)e_0 + \int_0^t U_B(t,s)B(ds)\xi(s) + \int_0^t U_B(t,s)\sigma(s)dW(s), \quad t \in I. \tag{4.4}$$

Now using the standard definition of the covariance-operator valued function P^B, given by

$$(P^B(t)z, z) \equiv E\{(e(t), z)^2\}, \quad z \in X, \quad t \in I,$$

it follows from straightforward computation using (4.4) and the independence assumption for $\{x_0, W, \xi\}$ that P^B is given by the expression (4.2). □

Lemma 4.2 Consider the system (4.1) and suppose the assumptions of Lemma 3.1 hold, $P_0 \in \mathcal{L}_1^+(X)$, $\hat{Q} \in L_1(I, \mathcal{L}_1^+(X))$, and the random process ξ satisfies the assumption $\mathcal{A}(\Xi)$. The admissible set of structural controls $\Gamma \subset M_\varrho$ is bounded by C_Γ. Then for each $B \in \Gamma$, the covariance $P^B \in B_\infty(I, \mathcal{L}_1^+(X))$, and there exists a finite positive number $\hat{\pi}$ such that

$$\sup\{Tr(P^B(t)), t \in I, B \in \Gamma\} \leq \hat{\pi}.$$

Proof. By virtue of Lemma 3.1, we have

$$\sup\{\|U_B(t,s)\|, 0 \leq s \leq t \leq T, B \in \Gamma\} \leq b < \infty. \tag{4.5}$$

We show that for each $t \in I$, $P^B(t)$ is nuclear, positive, and that $P^B \in B_\infty(I, \mathcal{L}_1^+(X))$. Starting with P_α^B, it is easy to see that

$$Tr(P_\alpha^B(t)) \leq b^2 Tr(P_0), \quad t \in I, \quad B \in \Gamma, \tag{4.6}$$

and hence, by nuclearity and positivity of P_0, and finiteness of the interval I, we have $P_\alpha^B \in B_\infty(I, \mathcal{L}_1^+(X))$. Similarly, for P_β^B we have,

$$Tr(P_\beta^B(t)) \leq b^2 \int_I Tr(\hat{Q}(s))ds, \quad t \in I, \quad B \in \Gamma, \tag{4.7}$$

and since $\hat{Q} \in L_1(I, \mathcal{L}_1^+(X))$ we have $P_\beta^B \in B_\infty(I, \mathcal{L}_1^+(X))$. Now we consider the last term $P_\gamma^B(t)$. Note that, for each $\tau, s \in I$, it follows from the definition of the kernel $\tilde{K}$ given by (4.3) and the assumption $\mathcal{A}(\Xi)$ that $\tilde{K}(\tau,s) \in \mathcal{L}(Y)$ and $\tilde{K}^*(\tau,s) = \tilde{K}(s,\tau)$ for all $\tau, s \in I$, and $(\tilde{K}(\tau,\tau)y,y) \geq 0$ for all $y \in Y$. Further, for any complete orthonormal basis $\{y_i\}$ of the Hilbert space Y, it follows from the second order property of ξ as mentioned above and Fubini's theorem that

$$\int_I \int_I \sum_{i=1}^{\infty} |(\tilde{K}(\tau,s)y_i, y_i)| d\tau ds \leq \ell(I) \Big(\int_I E|\xi(t)|_Y^2 dt \Big) < \infty, \tag{4.8}$$

with $\ell(I)$ denoting the Lebesgue measure of the set I. This means that $\tilde{K} \in L_1(I \times I, \mathcal{L}_1(Y))$. In other words, the kernel $\tilde{K}(\tau,s)$ defined on $I \times I$ takes values from the space of nuclear operators $\mathcal{L}_1(Y)$ and that the integral of its nuclear norm is bounded as shown in (4.8). Also, by the same assumption $(\mathcal{A}(\Xi))$, it is easy to verify that

$$\sup\{\|\tilde{K}(\tau,s)\|_{\mathcal{L}(Y)}, (\tau,s) \in I \times I\} \leq \beta^2,$$

and hence we have $\tilde{K} \in B_\infty(I \times I, \mathcal{L}(Y))$ also. Further, since I is a finite interval, this implies that $\tilde{K} \in L_2(I \times I, \mathcal{L}(Y))$. Using this kernel, we introduce the integral operator K as follows

$$(K\varphi)(t) \equiv \int_I \tilde{K}(t,s)\varphi(s) ds, \quad t \in I. \tag{4.9}$$

It is easy to verify that

$$\|K\varphi\|_{L_2(I,Y)} \leq \beta \ell(I) \|\varphi\|_{L_2(I,Y)}.$$

Thus, K is a bounded linear operator on the Hilbert space $L_2(I,Y)$. The reader can easily verify that it is also positive and selfadjoint. Further, it follows from the inequality (4.8) that it is also nuclear and so compact. Thus, it follows from the well known spectral theory for compact operators that K has discrete spectrum having the representation

$$K = \sum_i k_i \psi_i \otimes \psi_i,$$

where $\{\psi_i\}$ can be chosen as the eigenfunctions of the operator K with the corresponding eigenvalues $\{k_i\}$. The eigenvalues are all real and positive (nonnegative) with finite multiplicity. The eigenfunctions are orthogonal and we may assume that they are normalized. Clearly, $Tr(K) = \sum_{j=1}^{\infty} k_j < \infty$. Considering P_γ^B, let us define the measures

$$\mu_B^j(\sigma) \equiv \int_\sigma B(ds)\psi_j(s), \quad j \in N, \quad \sigma \in \Sigma. \tag{4.10}$$

By virtue of assumption $\mathcal{A}(\Xi)$, we have seen that $\tilde{K} \in B_\infty(I \times I, \mathcal{L}(Y))$ and the associated integral operator K is bounded and hence its spectrum is contained in a bounded subset of $[0,\infty)$. Therefore, without any loss of generality, we may assume that the set $\{\psi_i\}$, which is orthonormal in $L_2(I,Y)$, is contained in a bounded subset of $B_\infty(I,Y)$. Hence, there exists a finite positive number η so that $\sup\{\|\psi_j\|_{B_\infty(I,Y)}, j \in N\} \leq \eta$. Using this bound, it follows from (4.10) that

$$|\mu_B^j|_v \leq \|\psi_j\|_{B_\infty(I,Y)} |B|_\varrho \leq \eta |B|_\varrho. \tag{4.11}$$

Thus, $\mu_B^j \in M_{cabv}(\Sigma, X)$ for all $j \in N$ and

$$\sup\{|\mu_B^j|_v, j \in N, B \in M_\varrho\} \leq \eta C_\Gamma.$$

Now, taking the trace of $P_\gamma^B(t)$ with respect to any ortho-normal basis of the Hilbert space X, it follows from simple computations that

$$Tr(P_\gamma^B(t)) = \sum_{j=1}^{\infty} k_j \left(\| \int_0^t U_B(t,s)\mu_B^j(ds)\|_X^2 \right) \leq b^2 \sum k_j |\mu_B^j|_v^2.$$

Hence,

$$Tr(P_\gamma^B(t)) \leq (b\eta C_\Gamma)^2 \, Tr(K) < \infty, \; \forall \, t \in I, B \in \Gamma. \tag{4.12}$$

Since K is a positive nuclear operator, this shows that $P_\gamma^B \in B_\infty(I, \mathcal{L}_1^+(X))$. Summarizing the above results we have $P^B \in B_\infty(I, \mathcal{L}_1^+(X))$ for each $B \in \Gamma$. Since the inequalities (4.5),(4.6),(4.7) and (4.12) hold uniformly with respect to $B \in \Gamma$, there exists a finite positive number $\hat{\pi}$ that

$$\left\{ Tr(P^B(t)), t \in I, B \in \Gamma \right\} \leq \hat{\pi} < \infty. \tag{4.13}$$

That is, the set $\{P^B, B \in \Gamma\}$ is a bounded subset of $B_\infty(I, \mathcal{L}_1^+(X))$. □

The following corollary readily follows from Lemma 4.2.

Corollary 4.3 Under the assumptions of Lemma 4.2, for each $B \in \Gamma$, the process $x^B \equiv x^B(t), t \in I$, given by the expression (3.5) is a second order X valued $\mathcal{F}_t$-adapted random process. In particular, $x^B \in B_\infty(I, L_2(\Omega, X))$ with the mean given by $\bar{x}^B(t) = U_B(t,0)\bar{x}_0, t \in I$, and the covariance $P^B(t), t \in I$, given by the expression (4.2) and so for each $t \in I$, the attainable set of measures $\mathcal{A}(t)$ have finite second moments.

Now we are prepared to prove compactness of the attainable sets $\{\mathcal{A}(t), t \in I\}$. For each $n \in N$, let Π^n denote the projector in the Hilbert space X given by

$$\Pi^n X \equiv \{\zeta \in X : \zeta \equiv \sum_{i>n} (\zeta, x_i) x_i\},$$

where $\{x_i\}$ is any complete orthonormal basis of the Hilbert space X.

Theorem 4.4 Let $\{x_i\}$ be any complete orthonormal basis of the Hilbert space X and suppose the assumptions of Lemma 4.2 hold. Further assume that the evolution operator $U_B(t,s), 0 \leq s \leq t \leq T$, is compact for $s < t$ satisfying the following conditions:

$$(\alpha): \qquad \lim_{n\to\infty} \sum_{j=1}^{\infty} p_j^o \|\Pi^n U_B(t,0) x_j\|_X^2 \longrightarrow 0,$$

$$(\beta): \qquad \lim_{n\to\infty} \sum_{j=1}^{\infty} \int_0^t q_j(s) \|\Pi^n U_B(t,s) x_j\|_X^2 ds \longrightarrow 0,$$

$$(\gamma): \qquad \lim_{n\to\infty} \sum_{j=1}^{\infty} k_j \|\Pi^n \left(\int_0^t U_B(t,s)\mu_B^j(ds) \right)\|_X^2 \longrightarrow 0$$

uniformly with respect to $B \in \Gamma$. Then, for each $t \in I$, the attainable set $\mathcal{A}(t)$ given by (3.6) is a conditionally weakly compact subset of $\mathcal{M}_1(X)$.

Proof. It is well known [8, Theorem 2, p. 377], that for a subset $M_0 \subset \mathcal{M}_1(X)$ to be conditionally weakly compact (called weakly compact in the Russian literature), it is necessary and sufficient that the following conditions hold:

(1): for every $\varepsilon > 0$ there exists a number $c > 0$ such that $\mu\{x \in X : |x|_X > c\} < \varepsilon$ for all $\mu \in M_0$.

(2): The series $\sum_{i \geq 1}(Q_\mu x_i, x_i)$ is convergent uniformly in $\mu \in M_0$, where Q_μ is given by

$$(Q_\mu \xi, \xi) \equiv \int_X (x, \xi)_X^2 \, \mu(dx), \quad \xi \in X.$$

Here we are interested in sufficient conditions for conditional weak compactness of the attainable sets $\mathcal{A}(t), t \in I$. Therefore, we shall verify that for $M_0 \equiv \mathcal{A}(t), t \in I$, the conditions $(\alpha), (\beta)$ and (γ) imply conditions (1) and (2) as stated above. First note that for every $B \in M_{cabv}(\Sigma, \mathcal{L}(Y, X))$, the solution x^B, given by the expression (3.5), satisfies the relation

$$E|x^B(t)|_X^2 = Tr(P^B(t)) + |\bar{x}^B(t)|_X^2, \quad t \in I, \tag{4.14}$$

and hence it follows from (4.5) and (4.13) that

$$E|x^B(t)|_X^2 \leq \hat{\pi} + b^2|\bar{x}_0|_X^2 < \infty, \quad \forall\, t \in I, \quad B \in \Gamma,$$

and consequently by Chebyshev's inequality, for any $c > 0$, and $t \in I$,

$$\mu_t^B\{x \in X : |x| > c\} \equiv Prob.\{|x^B(t)|_X > c\} \leq (1/c^2)(\hat{\pi} + b^2|\bar{x}_0|^2)$$

uniformly with respect to $B \in \Gamma$. Thus, by definition of the attainable set given by the expression (3.6), it follows from this that

$$\mu\{x \in X : |x| > c\} \leq (1/c^2)(\hat{\pi} + b^2|\bar{x}_0|^2)$$

uniformly with respect to $\mu \in \mathcal{A}(t)$. Hence, for each $\varepsilon > 0$ there exists a $c > 0$ finite such that

$$\mu\{x \in X : |x| > c\} < \varepsilon, \quad \forall\, \mu \in \mathcal{A}(t),$$

verifying condition (1). Next, we verify condition (2). In view of the expression (4.14), it suffices to verify that the series $\sum_{i \geq 1}(P^B(t)x_i, x_i)$ is convergent uniformly with respect to $B \in \Gamma$. Using any basis $\{x_i\}$ of X, we compute $\sum_{i > n}(P^B(t)x_i, x_i)$ and verify that it converges to zero uniformly with respect to $B \in \Gamma$ as $n \to \infty$. We consider term by term the expression (4.2). Since $P^0 \in \mathcal{L}_1^+(X)$ there exists $p^0 \equiv (p_j^0) \in \ell_1^+$ such that P^0 has the representation $P^0 \equiv \sum p_j^o \, x_j \otimes x_j$. Hence, it follows from a simple computation that

$$\sum_{i > n}(P_\alpha^B(t)x_i, x_i) = \sum_{j=1}^{\infty} p_j^0 \, |\Pi^n U_B(t, 0)x_j|_X^2. \tag{4.15}$$

By our assumption, $\hat{Q} \in L_1(I, \mathcal{L}_1^+(X))$ and hence there exists $q \in L_1(I, \ell_1^+)$ such that

$$\hat{Q}(t) = \sum_{j=1}^{\infty} q_j(t)\; x_j \otimes x_j.$$

Recalling the expression for P_β^B given by the second term of equation (4.2) and using the representation of $\hat{Q}$ it follows from simple computation that

$$\sum_{i>n} (P_\beta^B(t)x_i, x_i) = \sum_{j\geq 1} \int_0^t q_j(s) |\Pi^n U_B(t,s) x_j|_X^2 ds. \tag{4.16}$$

Considering the last term and using the spectral representation of the correlation operator K given by equation (4.9), it follows from straight forward computation that

$$\begin{aligned}\sum_{i>n} \big(P_\gamma^B(t)x_i, x_i\big) &= \sum_{i>n} \sum_{j=1}^{\infty} k_j \big(\int_0^t U_B(t,s) B(ds) \psi_j(s), x_i)_X^2 \\ &= \sum_{j=1}^{\infty} k_j |\Pi^n \big(\int_0^t U_B(t,s) B(ds) \psi_j(s) \big)|_X^2 \\ &= \sum_{j=1}^{\infty} k_j |\Pi^n \big(\int_0^t U_B(t,s) \mu_B^j(ds) \big)|_X^2,\end{aligned} \tag{4.17}$$

where μ_B^j is a Y valued vector measure given by

$$\mu_B^j(\Delta) \equiv \int_\Delta B(ds)\psi_j(s), \quad \Delta \in \Sigma.$$

In view of the assumptions (α), (β), (γ), it follows from the expressions (4.15), (4.16) and (4.17) that for every $\varepsilon > 0$, there exists an integer n_ε such that

$$\sup\{\sum_{i>n} (P^B(t)x_i, x_i), B \in \Gamma\} < \varepsilon, \tag{4.18}$$

for all $n \geq n_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows from this that the series $\sum_{i\geq 1}(P^B(t)x_i, x_i)$ is convergent uniformly with respect to $B \in \Gamma$. Thus, we have condition (2) and hence, for each $t \in I$, $\mathcal{A}(t)$ is conditionally weakly compact. This completes the proof. □

Remark. Under the assumptions of Lemma 4.2 and compactness of the evolution operator $U_B(t,s)$ for $0 \leq s < t \leq T$, the conditions $(\alpha) - (\gamma)$ are also necessary. This follows readily from the equivalence of the two expressions

$$\{\sum_{i>n} (P^B(t)x_i, x_i), B \in \Gamma\} \text{ and } \{\sum_{i>n} (Q_\mu x_i, x_i), \mu \in \mathcal{A}(t)\},$$

where Q_μ is the covariance operator associated with any measure $\mu \in \mathcal{A}(t)$.

5. EXISTENCE OF OPTIMAL POLICIES

Now we are prepared to consider the control problems stated in Section 3. First we consider problem P1.

Theorem 5.1 Consider the control problem P1 subject to the system (3.1)-(3.2) or equivalently equation (3.4) and suppose the assumptions of Theorem 4.4 hold and further the admissible set of structural controls Γ is a weakly sequentially compact subset of $M_\varrho \subset M_{cabv}(\Sigma, \mathcal{L}(Y,X))$. Then the problem P1 has a solution.

Proof. Define the functional $\tilde{J}_1$ on $\mathcal{M}_1(X)$ by $\tilde{J}_1(\mu) \equiv \mu(D)$ where D is the target set as described in the problem P1. Thus, the stated problem is equivalent to the following problem: find $\mu^o \in \mathcal{A}(T)$ that maximizes the functional $\tilde{J}_1$ on the attainable set $\mathcal{A}(T)$. We prove the existence of such a μ^o by verifying that $\mathcal{A}(T)$ is weakly sequentially compact and that $\tilde{J}_1$ is weakly upper semicontinuous (w.u.s.c). By Theorem 4.4, $\mathcal{A}(T)$ is conditionally weakly compact and so we prove that it is weakly closed. By a direct computation using the expression (4.2), we can show that $Tr(P_n(T)) \longrightarrow Tr(P_o(T))$ whenever $B^n \xrightarrow{w} B^o$ in $\Gamma \subset M_\varrho$. This means that $\|x^n(T)\|_{L_2(\Omega,X)} \longrightarrow \|x^o(T)\|_{L_2(\Omega,X)}$ where x^n and x^o are the (mild) solutions of equation (3.4) corresponding to B^n and B^o, respectively. Using equation (3.5), we show that $x^n(T) \xrightarrow{w} x^o(T)$ in $L_2(\Omega,X)$ whenever $B^n \xrightarrow{w} B^o$ in $\Gamma \subset M_\varrho$. Since $L_2(\Omega,X)$ is a Hilbert space, it is a locally uniformly convex space and therefore it follows from well known Radon-Riesz theorem that $x^n(T) \xrightarrow{s} x^o(T)$ in $L_2(\Omega,X)$. Thus, the map $B \longrightarrow x^B(T)$ from $\Gamma \subset M_\varrho$ to $L_2(\Omega,X)$ is continuous with respect to the weak topology on M_ϱ and norm topology of $L_2(\Omega,X)$. Clearly, norm convergence in $L_2(\Omega,X)$ implies weak convergence in $\mathcal{M}_1(X)$. Hence, the map $B \longrightarrow \mu_T^B$ from M_ϱ to $\mathcal{M}_1(X)$ is weak-weak continuous. It follows from this that every weakly convergent sequence from $\mathcal{A}(T)$ has its limit in $\mathcal{A}(T)$. Thus $\mathcal{A}(T)$ is also weakly closed and hence it is weakly compact. For the proof of weak u.s.c of $\tilde{J}_1$, let $\{\mu_\alpha\}$ be any net (generalized sequence) from $\mathcal{A}(T)$ and suppose it converges weakly to μ. Since D is a closed subset of X, and X is a Hilbert space and so a complete metric space with respect its norm topology, it follows from a well known result [12, Theorem 6.1, p. 40] that

$$\overline{\lim}\, \mu_\alpha(D) \leq \mu(D).$$

Thus, by the definition of $\tilde{J}_1$, we have $\overline{\lim}\, \tilde{J}_1(\mu_\alpha) \leq \tilde{J}_1(\mu)$ proving weak u.s.c. Hence, there exists a $\mu^o \in \mathcal{A}(T)$ at which $\tilde{J}_1$ attains its supremum. Therefore, there exists a $B^o \in \Gamma$ such that $\mu_T^{B^o} = \mu^o$. This proves the existence of an optimal (structural) control and hence problem P1 has a solution. □

Now we consider problem P2.

Theorem 5.2 Consider the control problem P2 subject to the system (3.4) and suppose the assumptions of Theorem 4.4 hold. Further assume that the admissible set of structural controls Γ is a weakly sequentially compact subset of M_ϱ. Then the problem P2 has a solution.

Proof. The proof is identical to that of the preceding theorem only with minor changes. In this case the functional $\tilde{J}_2$ related to J_2 is weakly lower semicontinuous (w.l.s.c) and this follows again from the fact that for open sets

$$\mu(\mathcal{O}) \leq \underline{\lim}\, \mu_\alpha(\mathcal{O})$$

whenever $\mu_\alpha \xrightarrow{w} \mu$ [12, Theorem 6.1, p. 40]. Thus, $\tilde{J}_2$ is weakly lower semicontinuous and hence it attains its minimum on $\mathcal{A}(T)$. This in turn implies the existence of an optimal policy. Hence, the problem P2 has a solution. □

Now we consider problem P3. Let $BC(X)$ denote the space of bounded real valued continuous functions defined on X furnished with the standard sup norm topology. A probability measure valued function $\mu : I \longrightarrow \mathcal{M}_1(X)$ is said to be weakly measurable if for every $f \in BC(X)$, the function $t \longrightarrow \mu_t(f) \equiv \int_X f(x)\mu_t(dx)$ is a measurable scalar valued function taking values from the real number system. Recall that $\mathcal{M}_1(X)$ furnished with the weak topology is a Hausdorff topological space. Thus, the function space $\mathcal{M}_1(X)^I$ is a Tychonoff space in the product topology.

We introduce the following class of measure valued functions $\mathcal{T}_{ad}$ associated with the attainable sets $\mathcal{A}(t), t \in I$.

$$\mathcal{T}_{ad} \equiv \Big\{\mu : I \longrightarrow \mathcal{M}_1(X) \text{ such that it is weakly measurable,} \\ \text{and that } \mu_t \in \mathcal{A}(t)\ \forall\, t \in I\Big\}.$$

Clearly, this is a subcollection of the function space $(\mathcal{M}_1(X))^I$. This is given the topology of point wise convergence on I in the weak topology of $\mathcal{M}_1(X)$. Naturally, a sequence $\mu^n \in \mathcal{T}_{ad}$ is said to converge weakly to $\mu \in \mathcal{T}_{ad}$ if for every $t \in I$ and $\varphi \in BC(X)$, $\mu^n_t(\varphi) \longrightarrow \mu_t(\varphi)$. Note that the set $\mathcal{T}_{ad}$ furnished with the topology of point wise convergence on I in the weak topology of $\mathcal{M}_1(X)$ is a weakly compact subset of $(\mathcal{M}_1(X))^I$. This follows easily from [13, Theorem 42.2, p. 278].

Theorem 5.3 Consider the control problem P3 subject to the system (3.4) and suppose the assumptions of Theorem 4.4 hold. Further, assume that the admissible set of structural controls Γ is a weakly sequentially compact subset of $M_\varrho \subset M_{cabv}(\Sigma, \mathcal{L}(Y, X))$. Let $\psi : I \times X \longrightarrow R$ be a real valued function measurable in the first argument and continuous in the second, and $\nu \in M^+_{cabv}(\Sigma)$. Suppose there exist $c_1 \in L_1(I, \nu)$ and $c_2 \geq 0$ such that

$$|\psi(t,x)| \leq c_1(t) + c_2|x|^2_X, t \in I, x \in X.$$

Then the problem P3 has a solution.

Proof. First note that the functional J_3 defined on Γ is equivalent to the functional $\tilde{J}_3$ defined on $\mathcal{T}_{ad}$ given by

$$\tilde{J}_3(\mu) \equiv \int_I \int_X \psi(t,x)\mu_t(dx)\nu(dt),\ \mu \in \mathcal{T}_{ad}. \tag{5.1}$$

The functional $\tilde{J}_3$ is bounded for each $\mu \in \mathcal{T}_{ad}$. Indeed, for any $\mu \in \mathcal{T}_{ad}$, $\mu_t \in \mathcal{A}(t)$ and so by Corollary 4.3 it has finite second moment. Hence, it follows from the assumption on ψ and the identity (4.14) with the associated estimate following the identity that

$$\begin{aligned} |\tilde{J}_3(\mu)| &\le \int_I \int_X \{c_1(t) + c_2|x|_X^2\}\mu_t(dx)\nu(dt) \\ &\le \int_I c_1(t)\nu(dt) + c_2 \int_I \{Tr(P^\mu(t)) + |\bar{x}^\mu(t)|_X^2\}\nu(dt) \\ &\le \|c_1\|_{L_1(I,\nu)} + c_2(\hat{\pi} + b^2|\bar{x}_0|_X^2)|\nu|_v. \end{aligned} \tag{5.2}$$

Since ν has bounded variation and the expression on the righthand side of the above inequality is independent of the choice of $\mu \in \mathcal{T}_{ad}$, it follows from this that $\tilde{J}_3$ is uniformly bounded on $\mathcal{T}_{ad}$. Thus, $\inf\{\tilde{J}_3(\mu), \mu \in \mathcal{T}_{ad}\} > -\infty$. We show that the infimum is attained on $\mathcal{T}_{ad}$. Since $\mathcal{T}_{ad}$ is weakly compact, it suffices to verify that $\tilde{J}_3$ is weakly continuous. Let $\mu^n \xrightarrow{w} \mu^o$ in $\mathcal{T}_{ad}$. Then, for each $t \in I$ and $\varphi \in BC(X)$, $\mu_t^n(\varphi) \longrightarrow \mu_t^o(\varphi)$. For any finite positive number r, let $B_r \subset X$ denote the closed ball of radius r centered at the origin. Define

$$\psi_r(t,x) = \begin{cases} \psi(t,x), & \text{for } x \in B_r \\ \psi(t, rx/|x|_X), & \text{for } x \notin B_r. \end{cases}$$

Clearly, for ν almost all $t \in I$, $\psi_r(t,\cdot) \in BC(X)$ and

$$|\psi_r(t,x)| \le c_1(t) + c_2 r^2,\ \forall\, t \in I, \text{ and } x \in X,$$

and $\psi_r \longrightarrow \psi$ for each $(t,x) \in I \times X$. Thus, for ν almost all $t \in I$, and every finite number $r > 0$, we have, as $n \to \infty$,

$$g_{r,n}(t) \equiv \int_X \psi_r(t,x)\mu_t^n(dx) \longrightarrow \int_X \psi_r(t,x)\mu_t^o(dx) \equiv g_{r,o}(t). \tag{5.3}$$

Therefore, by Lebesgue dominated convergence theorem we have, as $n \to \infty$,

$$\int_I g_{r,n}(t)\nu(dt) \longrightarrow \int_I g_{r,o}(t)\nu(dt) \text{ for each finite } r > 0.$$

Clearly, this is equivalent to the following statement

$$\tilde{J}_{3,r}(\mu^n) \longrightarrow \tilde{J}_{3,r}(\mu^o) \tag{5.4}$$

as $n \to \infty$. Hence, $\mu \longrightarrow \tilde{J}_{3,r}(\mu)$ is weakly continuous for each finite positive number r. Since the elements of $\mathcal{T}_{ad}$ have bounded second moments, it follows from the estimate (5.2) that

$$\sup\{\tilde{J}_{3,r}(\mu), r > 0, \mu \in \mathcal{T}_{ad}\} < \infty.$$

Thus, we have $\lim_{r\to\infty}\lim_{n\to\infty}\tilde{J}_{3,r}(\mu^n) = \lim_{r\to\infty}\tilde{J}_{3,r}(\mu^o) = \tilde{J}_3(\mu^o)$ and we conclude that $\mu \longrightarrow \tilde{J}_3(\mu)$ is weakly continuous on $\mathcal{T}_{ad}$. And since $\mathcal{T}_{ad}$ is weakly compact it follows from the abstract Wierstrass theorem that $\tilde{J}_3$ attains its minimum on $\mathcal{T}_{ad}$. This in turn implies that there exists a control policy $B^o \in \Gamma$ at which $J_3(B)$ attains its minimum. Thus, problem P3 has a solution. □

Next we consider problem P4.

Theorem 5.4 Consider the control problem P4 subject to the feedback system (3.4) and suppose the assumptions of Theorem 4.4 hold. Further, assume that the admissible set of structural controls Γ is a weakly sequentially compact subset of M_ϱ. Let $\varphi_i \in BC(X), t_i \in I$, (distinct) $i = 1, 2, \ldots m$ and $F : R^m \longrightarrow R$ is a lower semicontinuous function satisfying

$$\inf\{F(\zeta), \zeta \in R^m\} > -\infty. \tag{5.5}$$

Then problem P4 has a solution.

Proof. Clearly the expression (3.10) is equivalent to

$$\tilde{J}_4(\mu) \equiv F\big(\mu_{t_1}(\varphi_1), \mu_{t_2}(\varphi_2), \ldots, \mu_{t_m}(\varphi_m)\big) \tag{5.6}$$

for $\mu \in \mathcal{T}_{ad}$. Let $\{\mu^n\} \in \mathcal{T}_{ad}$ be a minimizing sequence for $\tilde{J}_4$. Since $\mathcal{T}_{ad}$ is weakly sequentially compact, there exists a subsequence of the sequence $\{\mu^n\}$, relabeled as the original sequence, and an element $\mu^o \in \mathcal{T}_{ad}$ such that $\mu^n \stackrel{w}{\longrightarrow} \mu^o$. This, along with the assumption that $\varphi_i \in BC(X)$, implies that for each $t_i \in I$,

$$\mu^n_{t_i}(\varphi_i) \equiv \int_X \varphi_i(x)\mu^n_{t_i}(dx) \longrightarrow \int_X \varphi_i(x)\mu^o_{t_i}(dx) \equiv \mu^o_{t_i}(\varphi_i).$$

Hence, it follows from the lower semicontinuity of F on R^m that the functional $\mu \longrightarrow \tilde{J}_4(\mu)$ given by (5.6) is weakly lower semicontinuous on $\mathcal{T}_{ad}$ and by virtue of the assumption (5.5),

$$\inf\{\tilde{J}_4(\mu), \mu \in \mathcal{T}_{ad}\} > -\infty.$$

Hence, $\tilde{J}_4$ attains its minimum on $\mathcal{T}_{ad}$. Thus, $J_4(B)$ attains its minimum on Γ proving that problem P4 has a solution. □

P5: Another problem of significant interest is: find $B^o \in \Gamma$ that minimizes the Prohorov distance of a target measure $\mu^o \in \mathcal{M}_1(X)$ from the attainable set $\mathcal{A}(T)$. The cost functional in this case is given by the Prohorov metric which we denote by ρ. Then the problem is to find a $\mu^* \in \mathcal{A}(T)$ that minimizes the functional

$$\tilde{J}_5(\mu) = \rho(\mu^o, \mu).$$

Since the Prohorov metric is equivalent to the topology of weak convergence, it is obvious that $\mu \longrightarrow \tilde{J}_5(\mu)$ is weakly continuous. Thus, $\tilde{J}_5$ attains its minimum on the attainable set $\mathcal{A}(T)$ which is weakly compact. Hence an optimal policy exists.

The reader may find many other interesting problems of this nature, like minimal time problems.

Open Problems: We mention here two open problems. (1): Development of necessary conditions of optimality. (2): Extension to nonlinear systems.

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REFERENCES

[1] G. Da Prato and Zabczyk, (1992), *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

[2] G. Da Prato and Zabczyk, (1996), *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press.

[3] V. Lakshmikantham (Ed), *Nonlinear Analysis: Hybrid systems*, Special Issue, Proceedings of the International conference on Hybrid Systems and Applications, Lafayette, LA, (2006), **2**, June 2006.

[4] N. U. Ahmed, Optimal Structural Feedback Control for Partially Observed Linear Stochastic Systems on Hilbert Space, *Nonlinear Analysis: Hybrid Systems*, (submitted).

[5] N. U. Ahmed, Existence of Optimal Controls for a General Class of Impulsive Systems on Banach Spaces, *SIAM J. Control. Optim.*, **42** (2003), 669–685.

[6] N. U. Ahmed, Dynamics of Hybrid Systems Induced by Operator Valued Measures,*Nonlinear Analysis, Hybrid Systems* **2**, (2008), 359–367.

[7] N. U. Ahmed, Vector and Operator Valued Measures as Controls for Infinite Dimensional Systems: Optimal Control; *Differential Inclusions, Control and Optimization*, **28** (2008), 95–131.

[8] I. I. Gihman and A. V. Skorohod, (1971), *The Theory of Stochastic Processes I*, Springer-Verlag New York Heidelberg Berlin.

[9] N. U. Ahmed, Some Remarks on the Dynamics of Impulsive Systems in Banach Spaces, *Dynamics of Continuous, Discrete and Impulsive Systems (DCDIS), Series A*, **8** (2001), 261–264.

[10] J. Diestel and J. J. Uhl Jr., (1977), *Vector Measures*, AMS, Providence, Rhode Island.

[11] N. Dunford and J. T. Schwartz, (1964), *Linear Operators, Part 1*, General Theory, Second Printing.

[12] K. R. Parathasarathy, (1967), *Probability Measures on Metric Spaces*, Academic Press, New York and London.

[13] S. Willard, (1970), *General Topology*, Addison Wesley Publishing Company, Reading, Massachusetts. Menlo park, California. London. Don Mills, Ontario.