Communications in Applied Analysis **14** (2010), no. 4, 373–392

STOCHASTIC LAPLACE TRANSFORM WITH APPLICATIONS

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ABSTRACT. In this work, by reviewing the integration by parts method, we present the development of a theory of Laplace transforms in the context of the Itô–Doob type of stochastic calculus. The resulting table of transforms has been initiated.

1. INTRODUCTION

The usage of the Laplace transform [2, 4] plays a very significant role in solving linear deterministic equations. In this work, we attempt to develop a theory of Laplace transform for a suitable class of stochastic process in the framework of Itô–Doob type of stochastic calculus. It is obvious that approach provides an algebraic approach for: (a) finding Itô–Doob type stochastic integrals and (b) solving the stochastic linear differential equations of the Itô–Doob type. In fact, by employing the Laplace transform approach, a few well known differential equations (Langevin equation [3, 5, 6, 7, 8, 9] and Chandrasekhar equation [1, 3, 5]) are solved in a closed form.

2. METHOD OF INTEGRATION BY PARTS

In the following, we present a procedure to compute an Itô–Doob integral of the following type:

$$I(t, w(t)) = \int f(t, w(t))\, dw(t), \tag{2.1}$$

where $f \in C[J \times R, R]$, and where it is continuously differentiable in both variables (t, x) as many times as desired.

Received November 30, 2009

PROCEDURE.

Step 1. By applying the Itô–Doob differential formula [5] to

$$V(t, w(t) = w(t)\, f(t, w(t)), \tag{2.2}$$

and using product and chain rules, we get the following expression for the Itô–Doob differential of $V(t, w(t))$

$$\begin{aligned} dV(t, w(t)) &= d(w(t)\, f(t, w(t))) \\ &= w(t) f_t(t, w(t))\, dt + f(t, w(t))\, dw(t) + w(t) f_w(t, w(t))\, dw(t) \\ &\quad + \frac{1}{2}\left[2 f_w(t, w(t)) + w(t) f_{ww}(t, w(t))\right] (dw(t))^2 \\ &= \left[w(t) f_t(t, w(t)) + f_w(t, w(t)) + \frac{1}{2} w(t) f_{ww}(t, w(t))\right] dt \\ &\quad + f(t, w(t))\, dw(t) + w(t) f_w(t, w(t))\, dw(t). \end{aligned} \tag{2.3}$$

Step 2. Now, by applying the stochastic integral formula to (2.3), we obtain

$$\begin{aligned} V(t, w(t)) = c_1 &+ \int f(t, w(t))\, dw(t) + \int w(t) f_w(t, w(t))\, dw(t) \\ &+ \int \left[w(t) f_t(t, w(t)) + f_w(t, w(t)) + \frac{1}{2} w(t) f_{ww}(t, w(t))\right] dt. \end{aligned}$$

From this, and the definition of $V(t, w(t))$ in (2.2), we have the following expression

$$\begin{aligned} w(t) f(t, w(t)) = c_1 &+ \int f(t, w(t))\, dw(t) + \int w(t) f_w(t, w(t))\, dw(t) \\ &+ \int \left[w(t) f_t(t, w(t)) + f_w(t, w(t)) + \frac{1}{2} w(t) f_{ww}(t, w(t))\right] dt. \end{aligned} \tag{2.4}$$

Step 3. By solving the second term in the right-hand side of (2.4), we obtain an expression for the integral in the RHS of (2.1):

$$\begin{aligned} \int f(t, w(t))\, dw(t) = w(t) f(t, w(t)) &- c_1 - \int w(t) f_w(t, w(t))\, dw(t) \\ &- \int \left[w(t) f_t(t, w(t)) + f_w(t, w(t)) + \frac{1}{2} w(t) f_{ww}(t, w(t))\right] dt. \end{aligned} \tag{2.5}$$

This expression is analogous to the expression obtained in the "usual" method of integration by parts learned in a deterministic calculus course.

Step 4. Now, by substituting the RHS expression in (2.5) into the RHS of (2.1), we have the following

$$\begin{aligned} I(t, w(t)) = c + w(t) f(t, w(t)) &- \int w(t) f_w(t, w(t))\, dw(t) \\ &- \int \left[w(t) f_t(t, w(t)) + f_w(t, w(t)) + \frac{1}{2} w(t) f_{ww}(t, w(t))\right] dt. \end{aligned} \tag{2.6}$$

Step 5. Again, by repeating the above Steps 1–4 with regard to the first integral in (2.6) and by defining as

$$V(t,w(t)) = \frac{1}{2}w^2(t)f_w(t,w(t)), \tag{2.7}$$

the expressions analogous to expressions in (2.3), (2.4) and (2.5), are as follows:

$$\begin{aligned} dV(t,w(t)) &= d\left(\frac{1}{2}w^2(t)f_w(t,w(t))\right) \\ &= \frac{1}{2}w^2(t)f_{tw}(t,w(t))\,dt + w(t)f_w(t,w(t))\,dw(t) + \frac{1}{2}w^2(t)f_{ww}(t,w(t))\,dw(t) \\ &\quad + \frac{1}{2}\left[f_w(t,w(t)) + 2w(t)f_{ww}(t,w(t)) + \frac{1}{2}w^2(t)f_{www}(t,w(t))\right](dw(t))^2 \\ &= \frac{1}{2}\Bigg[w^2(t)f_{tw}(t,w(t))\,dt + f_w(t,w(t)) + 2w(t)f_{ww}(t,w(t)) \\ &\quad + \frac{1}{2}w^2(t)f_{www}(t,w(t))\Bigg]\,dt \\ &\quad + w(t)f_w(t,w(t))\,dw(t) + \frac{1}{2}w^2(t)f_{ww}(t,w(t))\,dw(t), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{1}{2}w^2(t)f_w(t,w(t)) &= c_1 + \int w(t)f_w(t,w(t))\,dw(t) + \frac{1}{2}\int w^2(t)f_{ww}(t,w(t))\,dw(t) \\ &\quad + \frac{1}{2}\int\Bigg[w^2(t)f_{tw}(t,w(t)) + f_w(t,w(t)) + 2w(t)f_{ww}(t,w(t)) \\ &\quad + \frac{1}{2}w^2(t)f_{www}(t,w(t))\Bigg]\,dt, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \int w(t)f_w(t,w(t))\,dw(t) &= \frac{1}{2}w^2(t)f_w(t,w(t)) - c_1 - \frac{1}{2}\int w^2(t)f_{ww}(t,w(t))\,dw(t) \\ &\quad - \frac{1}{2}\int\Bigg[w^2(t)f_{tw}(t,w(t)) + f_w(t,w(t)) + 2w(t)f_{ww}(t,w(t)) \\ &\quad + \frac{1}{2}w^2(t)f_{www}(t,w(t))\Bigg]\,dt, \end{aligned} \tag{2.10}$$

respectively.

Step 6. Again, we repeat the procedure as described in Step 4, meaning that we are now substituting the RHS expression in (2.10) for the first integral term in the RHS of (2.6). This gives us

$$\begin{aligned}I(t,w(t)) &= c+f(t,w(t))w(t)-\frac{1}{2}w^2(t)f_w(t,w(t))\\&\quad+\frac{1}{2}\int w^2(t)f_{ww}(t,w(t))\,dw(t)+\frac{1}{2}\int\Big[w^2(t)f_{tw}(t,w(t))\\&\quad+f_w(t,w(t))+2w(t)f_{ww}(t,w(t))+\frac{1}{2}w^2(t)f_{www}(t,w(t))\Big]\,dt\\&\quad-\int\Big[w(t)f_t(t,w(t))+f_w(t,w(t))+\frac{1}{2}w(t)f_{ww}(t,w(t))\Big]\,dt.\end{aligned}\tag{2.11}$$

Step 7. We continue this integration procedure (Steps 1–4) (the integration with respect w term in the RHS in (2.11)), until either the integral term is repeated or terminated. This completes the procedure of computing the integral (2.1).

Example 2.1. Find: $I(t)=\int t^4w^5(t)\,dw(s)$.

Here, $f(t,w(t))=t^4w^5(t)$. We set $V(t,w(t))=t^4w^6(t)$, and by imitating Step 1, we have

$$\begin{aligned}dV(t,w(t)) &= d\left(t^4w^6(t)\right)\\&=4t^3w^6(t)\,dt+6t^4w^5(t)\,dw(t)+15t^4w^4(t)\,(dw(t))^2\,.\end{aligned}\tag{2.12}$$

After integration, we obtain

$$c+t^4w^6(t)=\int 4t^3w^6(t)\,dt+\int 6t^4w^5(t)\,dw(t)+\int 15t^4w^4(t)\,dt$$

and hence $I(t)$ is given by:

$$\int t^4w^5(t)\,dw(t)=c+\frac{1}{6}t^4w^6(t)-\frac{2}{3}\int t^3w^6(t)\,dt-\frac{5}{2}\int t^4w^4(t)\,dt.\tag{2.13}$$

Example 2.2. Find $\int\limits_a^b g(t)\,dw(t)$, where g is a continuously differentiable function on $[a,b]$. Here $f(t,w(t))=g(t)$. We set $V(t,w(t))=g(t)w(t)$, and imitate Step 1, we have

$$dV(t,w(t))=d(g(t)w(t))=g'(t)w(t)+g(t)\,dw(t).\tag{2.14}$$

After integration, we get

$$(g(t)w(t)+c)\mid_a^b=\int\limits_a^b g'(t)w(t)\,dt+\int\limits_a^b g(t)\,dw(t),$$

which implies

$$\int\limits_a^b g(t)\,dw(t)=g(b)w(b)-g(a)w(a)-\int\limits_a^b g'(t)w(t)\,dt.\tag{2.15}$$

3. THE LAPLACE TRANSFORM

In this section, we present the concept of the Laplace transform. Also, its usage to solve the higher order linear nonhomogeneous differential equations with constant coefficients are outlined.

Definition 3.1. Let f be a real valued function of two variables $(t, w(t))$ defined for all real numbers $t \geq 0$ and $w(t)$ is the Wiener process. The Laplace transform of f in the sense of Cauchy–Riemann integral, is defined by:

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t, w(t))\, dt = \lim_{T\to\infty} \left[\int_0^T e^{-st} f(t, w(t))\, dt \right], \tag{3.1}$$

for all values of s for which this improper integral exists. It is denoted by $F(s) = \mathcal{L}(f)(s)$. Moreover, the Laplace transform of f in the sense of Itô–Doob integral, denoted by $F^w(s) = \mathcal{L}^w(f)(s)$, is defined by

$$F^w(s) = \mathcal{L}^w(f)(s) = \int_0^\infty e^{-st} f(t, w(t))\, dw(t) = \lim_{T\to\infty} \left[\int_0^T e^{-st} f(t, w(t))\, dw(t) \right], \tag{3.2}$$

for all values of s for which this improper integral exists.

Example 3.2. Find the Laplace transform of $f(t, w(t)) = c$ in the sense of the Itô–Doob integral, for $c \neq 0$.

Solution Process. Applying the Itô–Doob differential formula to $ce^{-st}w(t)$, we obtain

$$d\left(ce^{-st} w(t)\right) = -cse^{-st} w(t)\, dt + ce^{-st} dw(t).$$

Now using the Itô–Doob improper integrals in Definition 3.1, we have

$$c \lim_{T\to\infty} \left[e^{-st} w(t)\right|_0^T = -cs \lim_{T\to\infty} \left[\int_0^T e^{-st} w(t)\, dt \right] + c \lim_{T\to\infty} \left[\int_0^T e^{-st} dw(t) \right].$$

This together with the properties of Wiener process, we get

$$c \lim_{T\to\infty} \left[\int_0^T e^{-st} dw(t) \right] = cs \lim_{T\to\infty} \left[\int_0^T e^{-st} w(t)\, dt \right]$$

Hence

$$\mathcal{L}^w(c)(s) = s\mathcal{L}(cw)(s).$$

Thus for $s > 0$ and $c \neq 0$,

$$\mathcal{L}^w(c)(s) = s\mathcal{L}(cw)(s) \text{ if and only if } \mathcal{L}(w)(s) = \frac{\mathcal{L}^w(1)(s)}{s}. \tag{3.3}$$

Example 3.3. Find the Laplace transform of $f(t, w(t)) = w(t)$ in the sense of the Itô–Doob integral.

Solution Process. Again, applying the Itô–Doob differential formula to $e^{-st}w^2(t)$, we obtain

$$d\left(e^{-st}w^2(t)\right) = -se^{-st}w^2(t)\,dt + 2e^{-st}w(t)\,dw(t) + e^{-st}dt.$$

Now using Itô–Doob improper integral in Definition 3.1, we have

$$\lim_{T\to\infty}\left[e^{-st}w^2(t)\right|_0^T = -s\lim_{T\to\infty}\left[\int_0^T e^{-st}w^2(t)\,dt\right] + 2\lim_{T\to\infty}\left[\int_0^T e^{-st}w(t)\,dw(t)\right] + \lim_{T\to\infty}\left[\int_0^T e^{-st}dt\right].$$

By following the argument used in Example 3.2, we arrive at

$$2\lim_{T\to\infty}\left[\int_0^T e^{-st}w(t)\,dw(t)\right] = s\lim_{T\to\infty}\left[\int_0^T e^{-st}w^2(t)\,dt\right] - \lim_{T\to\infty}\left[\int_0^T e^{-st}dt\right],$$

and hence

$$2\mathcal{L}^w(w)(s) = s\mathcal{L}\left(w^2\right)(s) - \frac{1}{s}.$$

Thus for $s > 0$,

$$\frac{\mathcal{L}^w(w)(s)}{s} = \frac{1}{2}\left(\mathcal{L}\left(w^2\right)(s) - \frac{1}{s^2}\right), \text{ if and only if, } \mathcal{L}\left(w^2\right))(s) = \frac{2\mathcal{L}^w(w)(s)}{s} + \frac{1}{s^2}. \tag{3.4}$$

To use the Laplace transform, we need to know under what condition(s) the Laplace transform is defined. For this purpose, we will present the following to define a class of functions.

Definition 3.4. Let $L[f]$ be a class of smooth functions (random) that satisfy the following exponential growth condition

$$|f(t, w(t))| \leq Ke^{Mt+\Lambda w(t)}, s > M, \tag{3.5}$$

for some $M > 0$, $\Lambda > 0$ and $K \geq 0$.

From Definition 3.1 and the properties of both deterministic and the Itô–Doob type integrals [2, 4, 5], it is clear that the Laplace transform obeys the following property.

Theorem 3.5. *Let $f_1, f_2 \in L[f]$, let c_1 and c_2 be arbitrary given constants. Then,*

$$\mathcal{L}\left(c_1f_1 + c_2f_2\right)(s) = c_1\mathcal{L}\left(f_1\right)(s) + c_2\mathcal{L}\left(f_2\right)(s), \tag{3.6}$$

and

$$\mathcal{L}^w\left(c_1f_1 + c_2f_2\right)(s) = c_1\mathcal{L}^w\left(f_1\right)(s) + c_2\mathcal{L}^w\left(f_2\right)(s). \tag{3.7}$$

Example 3.6. Find the Laplace transform of $f(t,w(t)) = e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}$ in the sense of the Itô–Doob integral for $\sigma \neq 0$ and $\sigma, a \in R$.

Solution Process. Applying the Itô–Doob differential formula to $e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}$, we obtain

$$d\left(e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right) = -\left(s-a+\frac{\sigma^2}{2}\right)e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt$$
$$+\sigma e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}dw(t) + \frac{1}{2}\sigma^2 e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt.$$

Now using the strong law of large numbers and the Itô–Doob improper integral in Definition 3.1, we have

$$\lim_{T\to\infty}\left[e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}\Big|_0^T\right] = \lim_{T\to\infty}\left[\sigma\int_0^T e^{-st+\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dw(t)\right]$$
$$-(s-a)\lim_{T\to\infty}\left[\int_0^T e^{-\left(s-a+\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt\right],$$

which yields

$$(s-a)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) - 1 = \mathcal{L}^w\left(\sigma e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s).$$

Thus for $s > \left(a-\frac{\sigma^2}{2}\right)$,

$$\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) = \frac{1+\sigma\mathcal{L}^w\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{(s-a)}$$
$$= \frac{1}{(s-a)} + \frac{\sigma\mathcal{L}^w\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{(s-a)}. \tag{3.8}$$

Remark 3.7. From Example 3.6, we observe that for $s > \left(a+\frac{\sigma^2}{2}\right)$

$$\mathcal{L}\left(e^{at+\sigma w(t)}\right)(s) = \frac{1+\sigma\mathcal{L}^w\left(e^{at+\sigma w(t)}\right)(s)}{\left(s-a-\frac{\sigma^2}{2}\right)} = \frac{1}{\left(s-a-\frac{\sigma^2}{2}\right)} + \frac{\sigma\mathcal{L}^w\left(e^{at+\sigma w(t)}\right)(s)}{\left(s-a-\frac{\sigma^2}{2}\right)}; \tag{3.9}$$

and $\sigma = 0$, (3.8) or (3.9) reduces to the deterministic version of the Laplace transform

$$\mathcal{L}\left(e^{at}\right) = \frac{1}{(s-a)} \tag{3.10}$$

as a special case.

Example 3.8. Find the Laplace transform of $f(t,w(t)) = \cos(at+\sigma w(t))$ in the sense of the Itô–Doob integral for $\sigma \neq 0$ and $\sigma \in R$.

Solution Process. The Itô–Doob differential of $e^{-st}\sin(at+\sigma w(t))$ is

$$\begin{aligned} d\left(e^{-st}\sin(at+\sigma w(t))\right) = &-\left(s+\frac{1}{2}\sigma^2\right)e^{-st}\sin(at+\sigma w(t))\,dt \\ &+ ae^{-st}\cos(at+\sigma w(t))\,dt \\ &+ \sigma e^{-st}\cos(at+\sigma w(t))\,dw(t), \end{aligned}$$

which implies

$$\begin{aligned} &\lim_{T\to\infty}\left(e^{-sT}\sin(aT+\sigma w(T))\right) - e^{-s0}\sin(a0+\sigma w(0)) \\ &= -\left(s+\frac{1}{2}\sigma^2\right)\lim_{T\to\infty}\int_0^T e^{-st}\sin(at+\sigma w(t))\,dt + a\lim_{T\to\infty}\int_0^T e^{-st}\cos(at+\sigma w(t))\,dt \\ &\qquad + \sigma\lim_{T\to\infty}\int_0^T e^{-st}\cos(at+\sigma w(t))\,dw(t) \end{aligned}$$

and hence

$$\begin{aligned} \left(s+\frac{1}{2}\sigma^2\right)\mathcal{L}(\sin(at+\sigma w(t)))(s) - a\mathcal{L}(\cos(at+\sigma w(t)))(s) \\ = \sigma\mathcal{L}^w(\cos(at+\sigma w(t)))(s) \end{aligned} \tag{3.11}$$

which is equivalent to

$$\mathcal{L}(\sin(at+\sigma w(t)))(s) = \frac{a\mathcal{L}(\cos(at+\sigma w(t)))(s) + \sigma\mathcal{L}^w(\cos(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)}. \tag{3.12}$$

Example 3.9. Find the Laplace transform of $f(t,w(t)) = \sin(at+\sigma w(t))$ in the sense of the Itô–Doob integral for $\sigma \neq 0$ and $\sigma \in R$.

Solution Process. By following the prior procedure we have

$$\begin{aligned} &d\left(e^{-st}\cos(at+\sigma w(t))\right) \\ &\quad = -\left(s+\frac{1}{2}\sigma^2\right)e^{-st}\cos(at+\sigma w(t))\,dt - ae^{-st}\sin(at+\sigma w(t))\,dt \\ &\qquad - \sigma e^{-st}\sin(at+\sigma w(t))\,dw(t). \end{aligned}$$

This implies

$$\begin{aligned} \sigma\mathcal{L}^w(\sin(at+\sigma w(t)))(s) = 1 - \left(s+\frac{1}{2}\sigma^2\right)\mathcal{L}(\cos(at+\sigma w(t)))(s) \\ - a\mathcal{L}(\sin(at+\sigma w(t)))(s) \end{aligned} \tag{3.13}$$

which is equivalent to

$$\mathcal{L}(\cos(at+\sigma w(t)))(s) = \frac{1 - a\mathcal{L}(\sin(at+\sigma w(t)))(s) - \sigma\mathcal{L}^w(\sin(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)}. \tag{3.14}$$

Remark 3.10. (i) From (3.11) and (3.13), we can easily obtain

$$\mathcal{L}(\cos(at+\sigma w(t)))(s) = \frac{\left(s+\frac{1}{2}\sigma^2\right) - a\sigma\mathcal{L}^w(\cos(at+\sigma w(t)))(s) - \sigma\left(s+\frac{1}{2}\sigma^2\right)\mathcal{L}^w(\sin(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)^2 + a^2}, \tag{3.15}$$

and

$$\mathcal{L}(\sin(at+\sigma w(t)))(s) = \frac{a - a\sigma\mathcal{L}^w(\sin(at+\sigma w(t)))(s) + \sigma\left(s+\frac{1}{2}\sigma^2\right)\mathcal{L}^w(\cos(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)^2 + a^2}. \tag{3.16}$$

(ii) For $a = 0$, (3.15) and (3.16) reduce to

$$\mathcal{L}(\cos(\sigma w(t)))(s) = \frac{1 - \sigma\mathcal{L}^w(\sin(\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)}, \tag{3.17}$$

and

$$\mathcal{L}(\sin(\sigma w(t)))(s) = \frac{\sigma\mathcal{L}^w(\cos(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}\sigma^2\right)}. \tag{3.18}$$

(iii) Furthermore, for $\sigma = 0$, (3.15) and (3.16) reduce to the deterministic version of the Laplace transform as special cases:

$$\mathcal{L}(\cos(at))(s) = \frac{s}{s^2+a^2}, \tag{3.19}$$

and

$$\mathcal{L}(\sin(at))(s) = \frac{a}{s^2+a^2}. \tag{3.20}$$

Now we will present a known result [2, 4] concerning the Laplace transforms of a derivative of a process.

Theorem 3.11 (Laplace Transform of Derivative)**.** *Let us suppose that f has $n-1$ continuous derivatives on $[0,\infty)$, and for each i, $0 \le i \le n-1$, let $f^{(i)} \in L[f]$.*

Further assume that $f^{(n)}$ is piecewise continuous in every subinterval $0 \le t < b$. Then $f^{(n)} \in L[f]$ and

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \tag{3.21}$$

In the following, we present a result concerning the Laplace transforms of an indefinite integral of the Itô–Doob type.

Theorem 3.12 (Laplace Transform of Indefinite Integral)**.** *Let us assume that $f \in L[f]$. Let I be Itô–Doob indefinite integrals of f. Then $I \in L[f]$,*

$$\mathcal{L}(I)(s) = \mathcal{L}\left(\int_0^t f(u,w(u))\,dw(u)\right)(s) = \frac{\mathcal{L}^w(f)(s)}{s}. \tag{3.22}$$

Proof. From the smoothness (for example piecewise continuity) of function f and the properties of indefinite integral of the Itô–Doob indefinite integral we have $I \in L[f]$. Moreover, to prove (3.22), we compute the Itô–Doob differential of $e^{-st}I(t)$ as

$$d\left(e^{-st}I(t)\right) = -se^{-st}I(t)\,dt + e^{-st}f(t, wt(t))\,dw(t) \tag{3.23}$$

which implies

$$\lim_{T\to\infty}\left[e^{-st}I(t)\,|_0^T\right] = -s\lim_{T\to\infty}\left[\int_0^T e^{-st}I(t)\,dt\right] + \lim_{T\to\infty}\left[\int_0^T e^{-st}f(t, w(t))\,dw(t)\right].$$

Hence,

$$\mathcal{L}(I)(s) = \frac{1}{s}\int_0^\infty e^{-st}f(t, w(t))\,dw(t)\Bigg] = \frac{\mathcal{L}^w(f)(s)}{s}.$$

This completes the proof of the theorem. □

Example 3.13. Find the Laplace transform of $(bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}$ in the sense of the Itô–Doob integral for σ, a, b and $c \in R$.

Solution Process. We apply Itô–Doob differential formula to $(bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}$, and we obtain

$$\begin{aligned} d&\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right) \\ &= \left(a-\frac{\sigma^2}{2}\right)(bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt + ce^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt \\ &\quad + [\sigma(bw(t)+ct)+b]e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dw(t) \\ &\quad + \frac{1}{2}[\sigma[(\sigma(bw(t)+ct)+b]+b\sigma]e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dt \end{aligned}$$

which implies

$$\begin{aligned} (bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)} &= \int_0^t [a(bw(u)+cu)+c+b\sigma]e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}du \\ &\quad + \int_0^t [\sigma(bw(u)+cu)+b]e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}dw(u). \end{aligned}$$

We note that all of the terms in the above expression are continuous functions and belong to the class $L[f]$ defined in Definition 3.4. By applying the Laplace transform and Theorem 3.12, we get

$$\mathcal{L}\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)$$

$$= \frac{a\mathcal{L}\left(bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s} + \frac{(c+b\sigma)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s}$$
$$+\frac{\sigma\mathcal{L}^w\left(bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s} + \frac{b\mathcal{L}^w\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s}.$$

By simplifying the above expression, we have

$$\begin{aligned}(s-a)\mathcal{L}\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) - (c+b\sigma)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)\\ = \sigma\mathcal{L}^w\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) + b\mathcal{L}^w\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s).\end{aligned} \tag{3.24}$$

From this, $\sigma \neq 0$, and Example 3.6

$$\begin{aligned}&\sigma^2\mathcal{L}^w\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)\\ &= (s-a)\sigma\mathcal{L}\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) - \sigma(c+b\sigma)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)\\ &\quad -(s-a)b\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) + b\\ &= (s-a)\left[\sigma\mathcal{L}\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) - b\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)\right] + b\\ &\quad -\sigma(c+b\sigma)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s).\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\sigma\mathcal{L}\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) - b\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)\\ &= \frac{\sigma^2\mathcal{L}^w\left((bw(t)+ct)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{(s-a)} - \frac{b}{(s-a)} + \frac{\sigma(c+b\sigma)\mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{(s-a)}.\end{aligned} \tag{3.25}$$

Remark 3.14. (i) For $\sigma = 0 = c$, $b = 1$ and a, (3.24) reduces to:

$$\begin{aligned}(s-a)\mathcal{L}\left(w(t)e^{at}\right)(s) = \mathcal{L}^w\left(e^{at}\right)(s) \text{ if and only if}\\ \mathcal{L}\left(w(t)e^{at}\right)(s) = \frac{a\mathcal{L}\left(w(t)e^{at}\right)(s)}{s} + \frac{\mathcal{L}^w\left(e^{at}\right)(s)}{s}.\end{aligned} \tag{3.26}$$

(ii) For $b = 0$, $a, \sigma \neq 0$ and $c = 1$, (3.24) reduces to:

$$\mathcal{L}\left(te^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) = \frac{\sigma\mathcal{L}^w\left(te^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s) + \mathcal{L}\left(e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{(s-a)} \tag{3.27}$$

if and only if

$$\mathcal{L}\left(te^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)=\frac{\mathcal{L}\left((at+1)e^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s}+\frac{\sigma\mathcal{L}^w\left(te^{\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)}\right)(s)}{s}. \tag{3.28}$$

(iii) for σ, a, b and $c\in R$,

$$\begin{aligned}&\sigma\mathcal{L}\left((bw(t)+ct)e^{at+\sigma w(t)}\right)(s)-b\mathcal{L}\left(e^{at+\sigma w(t)}\right)(s)\\ =&\frac{\sigma^2\mathcal{L}^2\left((bw(t)+ct)e^{at+\sigma w(t)}\right)(s)}{\left(s-a-\frac{\sigma^2}{2}\right)}-\frac{b}{\left(s-a-\frac{\sigma^2}{2}\right)}+\frac{\sigma(c+b\sigma)\mathcal{L}\left(e^{\sigma w(t)}\right)(s)}{\left(s-a-\frac{\sigma^2}{2}\right)}.\end{aligned} \tag{3.29}$$

(iv) for $b=0=a$, $\sigma\neq 0$ and $c=1$, (3.29) reduces to

$$\mathcal{L}\left(te^{\sigma w(t)}\right)(s)=\frac{\sigma\mathcal{L}^w\left(te^{\sigma w(t)}\right)(s)}{\left(s-\frac{\sigma^2}{2}\right)}+\frac{\mathcal{L}\left(e^{\sigma w(t)}\right)(s)}{\left(s-\frac{\sigma^2}{2}\right)}\text{ if and only if} \tag{3.30}$$

$$\mathcal{L}\left(te^{\sigma w(t)}\right)(s)=\frac{\sigma\mathcal{L}^w\left(te^{\sigma w(t)}\right)(s)}{s}+\frac{\mathcal{L}\left(\left(1+\frac{\sigma^2}{2}\right)e^{\sigma w(t)}\right)(s)}{s}. \tag{3.31}$$

Definition 3.15. A real valued function $\kappa_{R_+}\equiv\kappa$ is called a characteristic function with respect to a set $[0,\infty)$, if:

$$\kappa_{R_+}(t)\equiv\kappa(t)=\begin{cases}0, & \text{if } t<0\\ 1, & \text{if } t\geq 0\end{cases},\quad \kappa(t-c)=\begin{cases}0, & \text{if } t<c\\ 1, & \text{if } t\geq c\end{cases} \tag{3.32}$$

for any $c\in R$.

Theorem 3.16. *If $\mathcal{L}^w(g)(s)$ exists, $0\leq c$ and $g(t)=0$ for $-c\leq t<0$. Then,*

$$\mathcal{L}^w(g(t-c,w(t-c))\kappa(t-c))(s)=e^{-sc}\mathcal{L}^w(g)(s). \tag{3.33}$$

Proof. From Definition 3.1, the conclusion of the theorem remains valid. □

Example 3.17. Given $g(t,w(t))=\begin{cases}3, & \text{if } 0<t<1\\ w(t), & \text{if } t\geq 1.\end{cases}$ Find $\mathcal{L}(g)(s)$.

Solution Process. First, we rewrite $g(t,w(t))$ in terms of a unit step function as follows:

$$g(t,w(t))=3-3\kappa(t-1)+(w(t-1+\kappa(t-1))\kappa(t-1).$$

This is in the form of $g(t-1,w(t-1))$. Hence, we can now apply Theorem3.16, and we have

$$\begin{aligned}\mathcal{L}(g,(t,w(t)))(s)&=\frac{3}{s}-\frac{3}{s}e^{-s}+e^{-s}\mathcal{L}(w(t))(s)\\ &=\frac{3}{s}-\frac{3}{s}e^{-s}+\frac{e^{-s}\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

Now we will introduce the concept of a convolution integral of two functions. Moreover, we will obtain an expression for the Laplace transform for the convolution integral of two functions.

Definition 3.18. Let f and g be piecewise continuous functions defined on $t \geq 0$. The Cauchy–Riemann and the Itô–Doob convolution integrals of f and g are defined by:

$$(f * g)(t) = \int_0^t g(t-u) f(u)\, dw(u). \tag{3.34}$$

Remark 3.19. From (3.34), we observe that:

$$\begin{aligned}(f * g)(t) &= \int_0^t g(t-u) f(u)\, dw(u) \\ &= \int_0^t g(z) f(t-z)\, dw(z) = (g * f)(t). \end{aligned} \tag{3.35}$$

This shows that the convolution integral defined in (3.34) satisfies the commutative law.

Theorem 3.20 (Laplace Transform of Convolution Integral)**.** *Let us assume that* $f, g \in L[f]$*. Then* $(f * g) \in L[f]$*,*

$$\mathcal{L}(f * g)(s) = \mathcal{L}\left(\int_0^t g(t-u, w(t-u)) f(u, w(u)) dw(u)\right)(s) = \mathcal{L}(f)(s).\mathcal{L}^w(g)(s). \tag{3.36}$$

Proof. We can easily verify the accuracy of the conclusion. From Definition 3.4 one can conclude that $(f * g) \in L[f]$, and from Definitions 3.1 and 3.18 we have

$$\mathcal{L}(f * g)(s) = \int_0^\infty e^{-st} \left[\int_0^t g(t-u, w(t-u)) f(u, w(u)) dw(u)\right] dt$$

$$= \int_0^\infty \int_0^t e^{-st} g(t-u, w(t-u)) f(u, w(u))\, dw(u) dt \quad \text{(by Fubini's theorem)}$$

$$= \int_0^\infty \int_u^\infty e^{-st} g(t-u, w(t-u)) f(u, w(u))\, dt dw(u)$$

$$= \int_0^\infty \int_0^\infty e^{-st} g(t-u, w(t-u)) \kappa(t-u) f(u, w(u))\, dt dw(u)$$

(by the definition of $\kappa(t-u)$)

$$= \int_0^\infty e^{-su} \mathcal{L}(g)(s) f(u, w(u))\, dw(u)$$

$$= \mathcal{L}(g)(s) \int_0^\infty e^{-su} f(u, w(u))\, dw(u)$$

$$= \mathcal{L}(g)(s) \mathcal{L}^w(f)(s).$$

This completes the proof of the theorem. □

Example 3.21. Solve the given equation: $g(t) = w(t) + \int_0^t \sin(t-u) g(u)\, du$.

Solution Process. Let $\mathcal{L}(g)(s)$. The $\mathcal{L}(\sin t)(s) = \frac{1}{1+s^2}$. Now applying Theorem 3.20, we have,

$$\mathcal{L}(g)(s) = \mathcal{L}(f * g)(s) = \mathcal{L}(w(t)) + \mathcal{L}(g)(s)\mathcal{L}(f)(s) = \frac{\mathcal{L}^w(1)(s)}{s} + \frac{\mathcal{L}(g)(s)}{s^2+1}.$$

By solving for $\mathcal{L}(g)(s)$, we obtain

$$\mathcal{L}(g)(s) = \frac{(1+s^2)\,\mathcal{L}^w(1)(s)}{s^3} = \frac{(1+s^2)}{s^2} \frac{\mathcal{L}^w(1)(s)}{s} = \left(1 + \frac{1}{s^2}\right) \frac{\mathcal{L}^w(1)(s)}{s}.$$

The inverse Laplace transform of this is given by

$$g(t) = \mathcal{L}^{-1}\left[\left(1 + \frac{1}{s^2}\right) \frac{\mathcal{L}^w(1)(s)}{s}\right] = \int_0^1 (1 + (t-u)) \left(\int_0^u dw(v)\right) du$$

$$= \int_0^t (1 + (t-u)) w(u)\, du.$$

The computation of the Laplace transforms of

a) $\sin(at + \sigma w(t)) e^{-\left(\alpha + \frac{\sigma^2}{2}\right)t} + \sigma w(t)$
b) $\cos(at + \sigma w(t)) e^{-\left(\alpha + \frac{\sigma^2}{2}\right)t} + \sigma w(t)$
c) $(at + \sigma w(t)) \cos(bt + \sigma w(t))$
d) $(at + \sigma w(t)) \sin(bt + \sigma w(t))$

are left as exercises.

4. APPLICATIONS OF LAPLACE TRANSFORM

The Laplace transform will be used to solve the initial value problems (IVP). The Laplace transform transforms a linear differential equation with constant coefficients into an algebraic equation. The techniques for solving the algebraic equations are easier than the method of solving the initial value problems and the higher order linear nonhomogeneous differential equations with constant coefficients.

Table 1. A Short Table of Laplace Transforms

$f(t,w(t))$	$\mathcal{L}(f(t))(s)$	$f(t,w(t))$	$\mathcal{L}^w(f(t))(s)$
c	$\frac{c}{s}$, $s>0$	c	$\mathcal{L}(cw)(s)=\frac{c\mathcal{L}^w(1)(s)}{s}$
e^{at}	$\frac{1}{s-a}$, $s>a$	$e^{at+\sigma w(t)}$	$\mathcal{L}\left(\sigma e^{at+\sigma w(t)}\right)(s)=\frac{1+\sigma\mathcal{L}^w\left(e^{at+\sigma w(t)}\right)(s)}{\left(s-a-\frac{1}{2}\sigma^2\right)}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sin\sigma w(t)$	$\mathcal{L}(\sin(\sigma w(t)))(s)=\frac{\sigma\mathcal{L}^w(\cos(at+\sigma w(t)))(s)}{\left(s+\frac{1}{2}a^2\right)}$
$\cos at$	$\frac{s}{s^2+a^2}$	$\cos\sigma w(t)$	$\mathcal{L}(\cos\sigma w(t))(s)=1-\frac{\frac{1}{2}\sigma^2\mathcal{L}(\cos\sigma w(t))(s)}{s}-\frac{\sigma\mathcal{L}^w(\sin\sigma w(t))(s)}{s}$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$	$w^n(t)$	$\mathcal{L}\left(w^n\right)(s)=\frac{n\mathcal{L}^w\left(w^{n-1}\right)(s)}{s}+\frac{n(n-1)\mathcal{L}\left(w^{n-2}\right)(s)}{2s}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n-1}}$, $s>0$	$w(t)e^{at}$	$\mathcal{L}\left(w(t)e^{at}\right)(s)=\frac{a\mathcal{L}\left(w(t)e^{at}\right)(s)}{s}+\frac{\mathcal{L}^w\left(e^{at}\right)(s)}{s}$
		$te^{\sigma w(t)}$	$\mathcal{L}\left(te^{\sigma w(t)}\right)(s)=\frac{\sigma\mathcal{L}^w\left(te^{\sigma w(t)}\right)(s)}{\left(s-\frac{\sigma^2}{2}\right)}+\frac{\mathcal{L}\left(e^{\sigma w(t)}\right)(s)}{\left(s-\frac{\sigma^2}{2}\right)}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$		
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$		
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$		

Example 4.1. Use the Laplace transform to solve the given initial value problem:

$$dy'+ydt=\sigma dw(t),\quad y(0)=0,\quad y'(0)=1,\text{ for }\sigma\neq 0.$$

Solution Process. We note that the Itô–Doob differential equation is equivalent to the following integral equation

$$y'(t)=y'(0)-\int_0^t y(u)\,du+\sigma\int_0^t dw(u).$$

Now, by applying the Laplace transform to both sides and using Theorem 3.12, we obtain

$$\begin{aligned}\mathcal{L}\left(y'(t)\right)&=\mathcal{L}\left(e^{-st}\left[y'(0)-\int_0^t y(u)\,du+\sigma\int_0^t dw(u)\right]\right)\\&=\mathcal{L}\left(e^{-st}y'(0)\right)-\mathcal{L}\left(\int_0^t y(u)\,du\right)+\sigma\mathcal{L}\left(\int_0^t dw(u)\right)\\&=\frac{1}{s}-\frac{\mathcal{L}(y(t))}{s}+\frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

Moreover, from Theorem 3.11, we have: $\mathcal{L}\left(y'(t)\right)=s\mathcal{L}(y(t))-y(0)$. From this and using the initial conditions, the above expression reduces to:

$$s\mathcal{L}(y(t))=\frac{1}{s}-\frac{\mathcal{L}(y(t))}{s}+\frac{\sigma\mathcal{L}^w(1)(s)}{s}.$$

Now, we solve for $\mathcal{L}(y(t))$, and get

$$\begin{aligned}\mathcal{L}(y(t)) &= \frac{s}{1+s^2}\left(\frac{1}{s} + \frac{\sigma\mathcal{L}^w(1)(s)}{s}\right)\\ &= \frac{1}{1+s^2} + \frac{s}{1+s^2}\,\frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

By applying the inverse Laplace transform to both sides and using Theorem 3.20, we have

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(\frac{1}{1+s^2} + \frac{s}{1+s^2}\,\frac{\sigma\mathcal{L}^w(1)(s)}{s}\right)\\ &= \mathcal{L}^{-1}\left(\frac{1}{1+s^2}\right) + \mathcal{L}^{-1}\left(\frac{s}{1+s^2}\,\frac{\sigma\mathcal{L}^w(1)(s)}{s}\right)\\ &= \sin t + \sigma\int_0^t \cos(t-u)w(u)\,du.\end{aligned}$$

Thus, the solution of the initial value problem is given by

$$y(t) = \sin t + \sigma\int_0^t \cos(t-u)w(u)\,du.$$

Example 4.2. [Langevin equation [6, 7]] Use the Laplace transform to solve the IVP:

$$dy' + \beta y' dt = \sigma\,dw(t), \quad y(0) = y_0, \quad y'(0) = v_0, \text{ for } \sigma \neq 0 \text{ and } \beta > 0.$$

Solution Process. We note that the Itô–Doob differential equation is equivalent to the following integral equation

$$y'(t) = y'(0) - \beta\int_0^t y'(u)\,du + \sigma\int_0^t dw(u).$$

Now, by applying the Laplace transform to both sides and using Theorem 3.12, we obtain

$$\begin{aligned}\mathcal{L}\left(y'(t)\right) &= \mathcal{L}\left(e^{-st}\left[y'(0) - \beta\int_0^t y'(u)\,du + \sigma\int_0^t dw(u)\right]\right)\\ &= \mathcal{L}\left(e^{-st}y'(0)\right) - \beta\mathcal{L}\left(\int_0^t y'(u)\,du\right) + \sigma\mathcal{L}\left(\int_0^t dw(u)\right)\\ &= \frac{v_0}{s} - \frac{\beta\mathcal{L}\left(y'(t)\right)}{s} + \frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

Moreover, from Theorem 3.11, we have: $\mathcal{L}\left(y'(t)\right) = s\mathcal{L}(y(t)) - y(0)$. From this and using the initial conditions, the above expression reduces to:

$$s\mathcal{L}(y(t)) - y_0 = \frac{v_0}{s} - \frac{s\beta\mathcal{L}(y(t)) - \beta y_0}{s} + \frac{\sigma\mathcal{L}^w(1)(s)}{s}.$$

Now, we solve for $\mathcal{L}(y(t))$, and have

$$\begin{aligned}\mathcal{L}(y(t)) &= \frac{v_0}{\beta s + s^2} + \frac{(\beta+s)y_0}{\beta s + s^2} + \frac{\sigma\mathcal{L}^w(1)(s)}{\beta s + s^2} \\ &= \frac{v_0}{s(\beta+s)} + \frac{y_0}{s} + \frac{\sigma\mathcal{L}^w(1)(s)}{s(\beta+s)} \\ &= \frac{u_0}{\beta}\left[\frac{1}{s} + \frac{1}{\beta+s}\right] + \frac{y_0}{s} + \frac{1}{\beta+s}\,\frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

By applying the inverse Laplace transform to both sides, we get

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(\frac{v_0}{\beta}\left[\frac{1}{s} - \frac{1}{\beta+s}\right] + \frac{y_0}{s} + \frac{1}{\beta+s}\,\frac{\sigma\mathcal{L}^w(1)}{s}\right) \\ &= y_0 + \frac{v_0}{\beta}\left(1 - e^{-\beta t}\right) + \sigma\int_0^t e^{-\beta(t-u)} w(u)\,du.\end{aligned}$$

Thus the solution of the initial value problem is given by

$$y(t) = y_0 + \frac{v_0}{\beta}\left(1 - e^{-\beta t}\right) + \sigma\int_0^t e^{-\beta(t-u)} w(u)\,du.$$

From this solution process, we can determine the mean, covariance and the variance of the solution process. In fact, the mean of $y(t)$ is

$$E[y(t)] = E\left[y_0\right] + \frac{E\left[v_0\right]}{\beta}\left(1 - e^{-\beta t}\right).$$

The covariance and the variance of the solution process can be determined in a similar manner.

Example 4.3. [Chandrasekhar Equation [1]] Use the Laplace transform to solve the IVP:

$$dy' + \left(\beta y' + \nu^2 y\right) dt = \sigma dw(t), \quad y(0 = y_0, \quad y'(0) = v_0, \text{ for } \sigma \neq 0 \text{ and } \beta > 0.$$

Solution Process. We note that the Itô–Doob differential equation is equivalent to the following integral equation

$$y'(t) = y'(0) - \beta\int_0^t y'(u)\,du - \nu^2\int_0^t y(u)\,du + \sigma\int_0^t dw(u).$$

Now, by applying the Laplace transform to both sides and using Theorem 3.12, we obtain

$$\begin{aligned}\mathcal{L}\left(y'(t)\right) &= \mathcal{L}\left(e^{-st}\left[y'(0)-\beta\int_0^t y'(u)\,du-\nu^2\int_0^t y(u)\,du+\sigma\int_0^t dw(u)\right]\right)\\ &= \mathcal{L}\left(e^{-st}y'(0)\right)-\beta\mathcal{L}\left(\int_0^t y'(u)\,du\right)-\nu^2\mathcal{L}\left(\int_0^t y(u)\,du\right)+\sigma\mathcal{L}\left(\int_0^t dw(u)\right)\\ &= \frac{v_0}{s}-\frac{\beta\mathcal{L}\left(y'(t)\right)}{s}-\frac{\nu^2\mathcal{L}(y(t))}{s}+\frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

Moreover, from Theorem 3.11, we have: $\mathcal{L}\left(y'(t)\right)=s\mathcal{L}(y(t))-y(0)$, and by following the argument used in Example 4.2, we have

$$\begin{aligned}s\mathcal{L}(y(t))-y_0 &= \frac{v_0-\beta(s\mathcal{L}(y(t))-y(0))-\nu^2\mathcal{L}(y(t))+\sigma\mathcal{L}^w(1)(s)}{s}\\ &= \frac{v_0-\left(\beta s+\nu^2\right)\mathcal{L}(y(t))+\beta y(0)+\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

After various algebraic manipulation and simplifications, we get

$$\begin{aligned}\mathcal{L}(y(t)) &= \frac{v_0+\beta y(0)+\sigma\mathcal{L}^w(1)(s)}{\nu^2+\beta s+s^2}+\frac{sy(0)}{\nu^2+\beta s+s^2}\\ &= \frac{v_0}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}+\frac{\left(\frac{1}{2}\beta+s\right)y_0}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}\\ &\quad+\frac{s}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}\,\frac{\sigma\mathcal{L}^w(1)(s)}{s}.\end{aligned}$$

By applying the inverse of the Laplace transform to both sides, we obtain

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\Bigg(&\frac{v_0+\frac{1}{2}\beta y(0)}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}+\frac{\left(\frac{1}{2}\beta+s\right)y_0}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}\\ &+\frac{s}{\left(s+\frac{1}{2}\beta\right)^2+\frac{(4\nu^2-\beta^2)}{4}}\,\frac{\sigma\mathcal{L}^w(1)}{s}\Bigg)\end{aligned}$$

and hence

$$y(t) = \begin{cases} e^{-\frac{1}{2}\beta t}\left[\frac{2v_0+\beta y_0}{b} + \sin\frac{1}{2}bt + y_0\cos\frac{1}{2}bt\right] \\ +\sigma\int_0^t \left[\cos\frac{1}{2}b(t-u) - \frac{\beta}{b}\sin\frac{1}{2}b(t-u)\right] e^{-\frac{1}{2}\beta(t-u)}w(u)\,du, \\ \quad \text{if } b^2 = (4\nu^2-\beta^2) > 0, \\ e^{-\frac{1}{2}\beta t}\left[\left(v_0+\frac{1}{2}\beta y_0\right)t + y_0\right] \\ +\sigma\int_0^t \left[e^{-\frac{1}{2}\beta(t-u)}1 - \beta(t-u)e^{-\frac{1}{2}\beta(t-u)}\right] w(u)\,du, \\ \quad \text{if } (4\nu^2-\beta^2) = 0, \\ \frac{(2v_0+\beta y_0)\left[e^{-\frac{1}{2}(\beta-b)t}+e^{-\frac{1}{2}(\beta+b)t}\right]}{4b} + \frac{y_0\left[e^{-\frac{1}{2}(\beta-b)t}-e^{-\frac{1}{2}(\beta+b)t}\right]}{2} \\ +\sigma\int_0^t \left[\frac{\left[e^{-\frac{1}{2}(\beta-b)(t-u)}+e^{-\frac{1}{2}(\beta+b)(t-u)}\right]}{2} - \frac{\beta\left[e^{-\frac{1}{2}(\beta-b)(t-u)}-e^{-\frac{1}{2}(\beta+b)(t-u)}\right]}{4b}\right] du, \\ \quad \text{if } b^2 = (\beta^2-4\nu^2) > 0. \end{cases}$$

Depending on the nature magnitudes of ν^2 and β^2 and sign of $(4\nu^2-\beta^2)$, the representation of the solution process of Chandrasekhar's equation is derived. From this solution process, one can determine the mean, covariance and variance of the solution process.

ACKNOWLEDGMENT

This research was supported by the Mathematical Sciences Division, US Army Research Office, Grant No. W911NF-07-1-0283.

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