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ON THE SOLVABILITY OF SOME OPERATOR EQUATIONS AND INCLUSIONS IN BANACH SPACES WITH THE WEAK TOPOLOGY

SMAÏL DJEBALI[1], DONAL O'REGAN[2], AND ZAHIRA SAHNOUN[3]

[1]Lab. EDP-HM, École Normale Supérieure, Kouba. PB 92.
Algiers, 16050. ALGERIA
E-mail: djebali@ens-kouba.dz

[2]Department of Mathematics, National University of Ireland,
Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

[3]Department of Mathematics, E.N.S.,Kouba. PB 92.
Algiers, 16050. ALGERIA
E-mail: sahnoun_z@yahoo.fr

ABSTRACT. The aim of this paper is to present some existence results regarding operator equation and operator inclusions in some Banach spaces endowed with the weak topology. The results complement those obtained in [19, 20, 21, 22]. The Schauder-Tychonov fixed point is used with the weak measure of noncompactness. Applications to Volterra integral equations and inclusions are also provided.

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1. Introduction

In this paper, we consider the general operator equation

$$u(t) = Fu(t), \quad t \in [0, T] \tag{1.1}$$

in Banach spaces endowed with their weak topologies. Some general properties of operator equations are studied in [14] and [15]. As a particular case, we will study the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t, s)f(s, y(s))ds, \quad t \in [0, T], \tag{1.2}$$

where the integral is understood as the Pettis integral ([23], page 77–78). More precisely, we are interested in solution in the topological vector space $C([0, T], B_w)$ to be defined hereafter. This is the content of Sections 2 and 3 respectively. The operator and differential inclusions associated to (1.1) and (1.2) respectively are considered in

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Section 4, see also [1], [20] and the references therein. In the remainder of this section, we collect some material which will be used throughout this paper.

1.1 Preliminaries. Let $(B, \|\cdot\|)$ be a reflexive Banach space and B^* its topological dual. Consider the topology generated by the family of semi-norms T:

$$\{\rho_\varphi(x) = |<\varphi, x>| : \ \varphi \in B^* \ \text{ and } \ \|\varphi\|_{B^*} \leq 1\}.$$

This topology denoted by $\sigma(B, B^*)$ or B_w for short is called the weak topology (see e.g., [6, 7]). A map $y : B \to B$ is said to be weakly continuous if for every $\varphi \in B^*$, the map $\varphi \circ y : B \to \mathbb{R}$ is continuous. Also y is weakly Riemann integrable on $[a, b]$ if for any partition $\{t_0, \ldots, t_n\}$ of $[0, T]$ and any choice of points ξ_i, $t_{i-1} \leq \xi_i \leq t_i$, $i = 1, \ldots, n$, the sum $\Sigma_{i=1}^n y(\xi_i)\Delta t_i$ converges weakly to some element y_0 provided that

$$\max_{i=1,\ldots,n} \Delta t_i = \max_{i=1,\ldots,n} |t_i - t_{i-1}| \longrightarrow 0, \ \text{ as } n \to \infty.$$

In other words, y is weakly Riemann integrable if there exists $y_0 \in B$ such that for every ϕ, $\varphi(y)$ is Riemann integrable and $\int_0^T \varphi(y(s))ds = \varphi(y_0)$. y is said scalarly measurable if for any $\varphi \in B^*$, the function $\varphi(y)$ is measurable on $[0, T]$. Two scalarly measurable functions $y, z : [0, T] \longrightarrow B$ are said to be weakly equivalent if and only if for all $\varphi \in B^*$, $\varphi(y) = \varphi(z)$.

Denote by $C([0, T], B_w)$ the space of weakly continuous functions on $[0, T]$ with the topology of weak uniform convergence. This topology is also generated by the family of semi-norms $\{\eta_\varphi\}$ defined by

$$\eta_\varphi(y) = \sup_{t\in[0,T]} \varrho_\varphi(y(t)) = \sup_{t\in[0,T]} |\varphi(y(t))|, \ y \in C[0, T].$$

This topology is of course determined by the basis

$$V_y(\varphi_1, \ldots, \varphi_m, \varepsilon) = \bigcap_{i=1}^m \{g \in C([0, T], B_w) : \\ \sup_{t\in[0,T]} |\varphi_i(g(t) - y(t))| < \varepsilon, \ y \in C([0, T], B_w)\}$$

where $\varphi_1, \ldots, \varphi_m \in B^*$ and $\varepsilon > 0$.

Definition 1.1. A family $F = \{f_i, \ i \in J\}$, where J is some index set, is said to be weakly equicontinuous if, given $\varepsilon > 0$ and $\varphi \in B^*$, there exists $\delta > 0$ such that, for $s, t \in [0, T]$, if $|t - s| < \delta$, then

$$|\varphi(f_i(t) - f_i(s))| < \varepsilon, \quad \forall\, i \in J,$$

i.e. $\varphi(F)$ is equicontinuous for all $\varphi \in B^*$.

Clearly, we have

Proposition 1.2. $F = \{f_i, \ i \in J\}$ *is equicontinuous implies that* F *is weakly equicontinuous.*

Definition 1.3. A map $F : Q \subseteq C([0,T],B_\omega) \longrightarrow C([0,T],B_w)$ is said to be $w-$continuous if for every net $(y_\alpha)_\alpha \subseteq Q$ with $y_\alpha \longrightarrow y$ in $C([0,T],B_w)$, we have $(Fy_\alpha)_\alpha \longrightarrow Fy$ in $C([0,T],B_w)$.

Definition 1.4. A sequence $\{y_n\}_n$ of weakly continuous functions on $[0,T]$ into B is said to be weakly uniformly convergent on $[0,T]$ to a function y if for all $\varepsilon > 0$ and $\varphi \in B^*$, there exits an integer N such that

$$n > N \Longrightarrow (|\varphi(y_n(t) - y(t))| < \varepsilon,\ \forall\, t \in [0,T]).$$

Definition 1.5. A map $f : [0,T] \times B \longrightarrow B$ is said to be weakly-weakly continuous at (t_0, u_0) if for any $\varepsilon > 0$ and $\phi \in B^*$, there exist $\delta > 0$ and a weakly open set $U \subset B$ containing u_0 such that

$$|t - t_0| < \delta \Longrightarrow |\phi(f(t,u) - f(t_0,u_0))| < \varepsilon, \quad \forall\, u \in U.$$

Definition 1.6 ([6], page 26, [11], page 65, [13], page 10, [16], page 144)**.** (a) A partially ordered space $(D, \leq)$ is said to be directed, if every finite subset of D had an upper bound. Equivalently, for each $\alpha, \alpha' \in J$, there is $\alpha'' \in D$ such that $\alpha \leq \alpha''$ and $\alpha' \leq \alpha''$.
(b) Let D be a directed set and X a topological vector space. We call a net $(x_\alpha)_{\alpha\in D}$ a map $x : D \longrightarrow X :\ \alpha \mapsto x_\alpha$.
(c) The net (x_α) is said to be convergent to x if for each neighborhood V of x, there exists $\alpha_0 \in D$, such that $\alpha_0 \leq \alpha$ implies $x_\alpha \in V$.

Lemma 1.7 ([6], Lemma 2, [11], Theorem 2, [13], page 11, [16], Proposition 2.1.18)**.** *Let M be a set in a vector topological space. Then M is closed if and only if it contains the limits of all the convergent nets of elements of M.*

1.2 Auxiliary results. Next, we recall some classical results from functional analysis (see [6, 7, 11, 17, 25]). Detailed properties of weakly convex sets may be found in [8].

Proposition 1.8 ([21], Proposition 2.1)**.** *If $f : [0,T] \times B \longrightarrow B$ is weakly-weakly continuous and B is reflexive, then f is bounded in the sense that for any* $r > 0$, *there exists $M_r > 0$ such that*

$$|f(t,y)| \leq M_r,\ \forall\, t \in [0,T] \ \text{ and } \forall\, y \in B \ \text{ with } \|y\| \leq r. \tag{1.3}$$

Theorem 1.9 (Arzéla-Ascoli Theorem, [11], Theorem 7.17, page 233)**.** *Let F be a weakly equicontinuous family of functions from $I = [a,b]$ into B, and let $\{u_n\}$ be a sequence in F such that for each $t \in I$, the set $\{u_n(t),\ n \geq 1\}$ is weakly relatively compact in B. Then there exists a subsequence $\{u_{n_k}\}$ which converges uniformly on I to a weakly continuous function (i.e. $\{u_{n_k}\}$ is sequentially relatively compact in $C([0,T],B_w)$).*

Theorem 1.10 (Mitchell-Smith Theorem, [17]). *Let F be an equicontinuous family in $C([0,T],B)$ and let $\{y_\alpha\}_\alpha$, $y \in F$. Then*

$$y_\alpha \rightharpoonup y \Longleftrightarrow y_\alpha(t) \rightharpoonup y(t),\ \forall\, t \in [0,T].$$

Theorem 1.11. *A convex subset of a normed space is closed if and only if it is weakly closed.*

Theorem 1.12. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Theorem 1.13 (Eberlein-Šmulian Theorem). *Let K be a weakly closed subset of a Banach space B. Then the following are equivalent:*

(i) *K is weakly compact.*

(ii) *K is weakly sequentially compact.*

Theorem 1.14 (Schauder-Tychonoff Theorem). *Let K be a closed convex subset of a locally convex Hausdorff space X. Assume that $f : K \longrightarrow K$ is continuous and $f(K)$ is relatively compact in X. Then f has at least one fixed point in K.*

Finally, we present a direct consequence of the Hahn-Banach theorem.

Theorem 1.15. *Let X be a normed space with $0 \neq x_0 \in X$. Then there exists a $\phi \in X^*$ such that $\|\phi\| = 1$ and $\phi(x_0) = \|x_0\|$.*

1.3 The weak MNC.

Throughout this section, X denotes a Banach space, $B(X)$ is the collection of all nonempty bounded subsets of X and $W(X)$ is the subset of $B(X)$ consisting of all weakly compact subsets of X. Let B_r denotes the closed ball in X centered at 0 with radius $r > 0$. In [3], De Blasi introduced the map $\omega :\ B(X) \to [0,+\infty)$ defined, for all $M \in B(X)$ by

$$\omega(M) = \inf\{r > 0,\ \exists\, N \in W(X) :\ M \subseteq N + B_r\}.$$

We recall for the sake of completeness, some important properties of ω needed hereafter; for further details and proofs, we refer the reader to [3].

Lemma 1.16. *Let $M_1, M_2 \in B(X)$. Then*

(a) $\omega(M_1) \leq \omega(M_2)$ *whenever* $M_1 \subseteq M_2$.

(b) $\omega(M) = 0$ *if and only if M is relatively weakly compact.*

(c) $\omega(\overline{M}^w) = \omega(M)$ *where* $\overline{M}^w$ *is the weak closure of* M.

(d) $\omega(co(M)) = \omega(M)$ *where $co(M)$ refers to the convex hull of M.*

(e) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.

(f) $\omega(M_1 \cup M_2) = \max(\omega(M_1), \omega(M_2))$.

(g) *(Cantor intersection condition) If* $\{X_n\}_1^\infty$ *is a sequence of nonempty, weakly closed subsets of* E *with* X_1 *bounded and*

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots \supseteq X_n \supseteq \ldots$$

with $\lim_{n\to\infty} \omega(X_n) = 0$, *then the set* $\bigcap_{n=1}^\infty X_n$ *is nonempty and weakly compact.*

Definition 1.17. A map $f : M \subset X \to X$ is said to be $\omega-$contractive (or an $\omega-$contraction) if it maps bounded sets into bounded sets, and there exists some $\beta \in [0,1)$ such that $\omega(f(V)) \leq \beta\omega(V)$ for all bounded subsets $V \subseteq M$.

Next, we present a theorem of Ambrosetti type (for the proof, see [14], [15], or [22], Thm. 10.1.1).

Theorem 1.18. *Let* H *be a bounded subset of* $C([0,T], B)$. *Then*

(a) $\sup_{t\in[0,T]} \omega(H(t)) \leq \omega(H)$.

(b) *If* H *is bounded and equicontinuous, then*

$$\omega(H) = \sup_{t\in[0,T]} \omega(H(t)) = \omega(H([0,T])),$$

where $H([0,T]) = \bigcup_{t\in[0,T]}\{\phi(t),\ \phi \in H\}$.

2. Existence principles

We first recall the following result

Theorem 2.1 ([19], Thm. 2.1 or [22], Thm. 10.2.1)**.** *Let* B *be a Banach space and* $Q \subseteq C([0,T], B_w)$ *a nonempty, closed, convex subset. Let* $F : Q \longrightarrow Q$ *be* $w-$*continuous and the family* FQ *is weakly equicontinuous. Assume, in addition that* $F(Q(t))$ *is weakly relatively compact in* B *for each* $t \in [0,T]$. *Then (1.1) has a solution in* Q.

Remark 2.2. If we wish to guarantee a solution of (1.1) in $C([0,T], B)$, we add $Q \subset C([0,T], B)$.

Now, we restate Thm. 2.2 in [18] (see also Thm. 2.2 in [19] or Thm. 10.2.2 in [22]). Note we need to add the equicontinuity of Q.

Theorem 2.3. *Let* B *be a Banach space and* $Q \subseteq C([0,T], B_w)$ *a nonempty, closed, convex, and bounded, equicontinuous subset of* $C([0,T], B)$. *Let* $F : Q \longrightarrow Q$ *be* $w-$*continuous and there exists* $0 \leq \alpha < 1$ *with* $\omega(F(X)) \leq \alpha\omega(X)$, *for all bounded* $X \subseteq Q$. *Then* (1.1) *has a solution in* Q.

Remark 2.4. Note we drop the condition that FQ is weakly equicontinuous here from Theorem 10.2.2 in [22]. Indeed, this follows vacuously from $FQ \subseteq Q$, the equicontinuity of Q, and Proposition 1.2.

Proof. The proof is split into four parts.

(a) Let $S_1 = Q$ and $S_{n+1} = \overline{co}(F(S_n)),\ n \in \mathbb{N}^*$. Notice that

$$\omega(S_2) = \omega(F(S_1)) \leq \alpha\omega(S_1) \text{ and } S_2 \subseteq \overline{co}(Q) = Q = S_1$$

and more generally

$$S_{n+1} \subseteq S_n \text{ with } \omega(S_{n+1}) \leq \alpha^n \omega(S_1), \text{ for } n = 1, 2, \dots$$

$0 \leq \alpha < 1$ implies $\lim_{n\to\infty} \omega(S_n) = 0$. Also, since S_n is a weakly closed (and closed, convex) subset of $C([0,T], B)$ for each n, then $S_\infty = \bigcap_1^\infty S_n$ is nonempty, convex and weakly closed. Moreover, S_∞ is weakly compact in $C([0,T], B)$ by the Cantor intersection condition for the weak measure of noncompactness (Lemma 1.16, (g)).

(b) Next, we prove that S_∞ is closed in $C([0,T], B_w)$ using the equicontinuity of Q. Indeed, let $(y_\alpha) \subset S_\infty$ be a net such that $y_\alpha \to y$ in $C([0,T], B_w)$. Then for all $\phi \in B^*$, $\sup_{t\in[0,T]} |\phi(y_\alpha(t)) - \phi(y(t))| \to 0$. In particular

$$\phi(y_\alpha(t)) \longrightarrow \phi(y(t)), \quad \forall\, t \in [0,T].$$

Using Theorem 1.10, the fact that $S_\infty \subset Q$ and the equicontinuity of Q, we obtain that $y_\alpha \rightharpoonup y$ in $C([0,T], B)$. Since S_∞ is weakly closed, we have that $y \in S_\infty$ and thus, by Lemma 1.7, S_∞ is closed in $C([0,T], B_w)$, as claimed.

(c) Since $F(S_n) \subset F(S_{n-1}) \subseteq \overline{co}(F(S_{n-1})) = S_n$ for all n, we have that F maps S_∞ into itself.

(d) We claim that $F(S_\infty)$ is relatively compact in $C([0,T], B_w)$. For this, we will make use of Theorem 1.9. First, notice that $F(S_\infty) \subset S_\infty \subset Q$ and the equicontinuity of Q implies that $F(S_\infty)$ is weakly equicontinuous. So, it remains to show that for each $t \in [0,T]$, the set $F(S_\infty)(t) = \{Fy(t) : y \in S_\infty\}$ is weakly relatively compact in B. To see this, notice that since $\omega(S_\infty) = 0$ and $F(S_\infty) \subset S_\infty$, we have $\omega(F(S_\infty)) = 0$. This together with Theorem 1.18(a) imply that $\omega(FS_\infty(t)) = 0$ for each $t \in [0,T]$. Thus, for each $t \in [0,T]$, we have that $FS_\infty(t)$ is weakly relatively compact in B. Now Theorem 1.9 implies that FS_∞ is relatively compact in $C([0,T], B_w)$. By Theorem 1.14, $F : S_\infty \longrightarrow S_\infty$ has a fixed point in $C([0,T], B_w)$, proving the theorem. □

Remark 2.5. In Theorem 2.3, the equicontinuity of Q is needed to prove the following implication:

$$(S_\infty \text{ weakly closed in } C([0,T]; B)) \Longrightarrow (S_\infty \text{ closed in } C([0,T]; B_w)).$$

3. Application to a Volterra integral equation

3.1 A special case. Consider the integral equation

$$y(t) = y_0 + \int_0^t k(s) f(s, y(s))ds, \quad t \in [0,T], \tag{3.1}$$

where $y_0 \in B$, B is reflexive, and k, f satisfy:
$(A1)$ $k \in L^1[0,1]$.
$(A2)$ The function $f : [0,T] \times B \longrightarrow B$ is weakly-weakly continuous.
$(A3)$ (Nagumo-type condition) The exists $\psi : [0,+\infty) \longrightarrow (0,+\infty)$ a nondecreasing continuous function such that

$$\begin{cases} |f(s,u)| \leq \psi(|u|), \text{ for a.e. } s \in [0,T] \text{ and all } u \in B \\ \text{with } \int_0^T |k(s)|ds < \int_{|y_0|}^{\infty} \frac{du}{\psi(u)}. \end{cases}$$

Theorem 3.1. *Under Assumptions* (A1)-(A3), *Equation* (1.2) *has a solution in* $C([0,T],B)$.

Proof. Let

$$Q = \{y \in C([0,T],B), \ |y(t)| \leq b(t), \ \forall\, t \in [0,T] \text{ and } \\ |y(t) - y(s)| \leq b(t) - b(s), \ \forall\, t,s \in [0,T]\},$$

where

$$b(t) := I^{-1}\left(\int_0^t |k(s)|ds\right) \text{ and } I(z) := \int_{|y_0|}^{z} \frac{du}{\psi(u)}.$$

Clearly, Q is convex, bounded, closed, hence weakly closed in $C([0,T],B)$ by Theorem 1.11. In addition, arguing as in the proof of Theorem 2.3, Part (b), we can see, from Theorem 1.10 and the equicontinuity of Q, that Q is also closed in $C([0,T],B_w)$. Let the operator F be defined on B by

$$Fy(t) = y_0 + \int_0^t k(s)f(s,y(s))ds, \quad t \in [0,T].$$

The proof is divided into three parts.

(a) $FQ \subseteq Q$:

• For any $y \in Q$, Fy is norm-continuous. Let $t, x \in [0,T]$ with $t > x$. Without loss of generality, assume that $Fy(t) - Fy(x) \neq 0$. Then, again by Theorem 1.15, there exists $\phi \in B^*$ with $\|\phi\|_B^* = 1$ and $|y(t) - Fy(x)| = \phi(Fy(t) - Fy(x))$. Thus

$$\begin{aligned} |Fy(t) - Fy(x)| &= \phi(\textstyle\int_x^t k(s)f(s,y(s))ds) \\ &\leq M_r \textstyle\int_x^t |k(s)|ds, \end{aligned}$$

where r is such that $|y|_0 = \sup_{t\in[0,T]} |y(t)| \leq r$. The existence of M_r is ensured by Proposition 1.8. Thus Fy is continuous and so $F : C([0,T],B) \longrightarrow C([0,T],B)$.

• Let $y \in Q$. Without loss of generality, assume that $Fy(s) \neq 0$ for all $s \in [0,T]$. By Theorem 1.15, there exists $\phi \in B^*$ with $\|\phi\|_{B^*} = 1$ and $\phi(Fy(s)) = |Fy(s)|$. Consequently, for each $t \in [0,T]$, we have

$$\begin{aligned} |Fy(t)| &= \phi\left(y_0 + \textstyle\int_0^t k(s)f(s,y(s))ds\right) \\ &\leq |y_0| + \textstyle\int_0^t |k(s)|\psi(|y(s)|)ds \\ &\leq |y_0| + \textstyle\int_0^t |k(s)|\psi(b(s))ds \\ &= |y_0| + \textstyle\int_0^t b'(s)ds = b(t), \end{aligned}$$

since $\int_{|y_0|}^{b(s)} \frac{du}{\psi(u)} = \int_0^s |k(u)|du$. Let $y \in Q$, $t, s \in [0,T]$ with $t > s$. Without loss of generality, assume that $F(y(t)) - F(y(s)) \neq 0$. Then, there exists a $\phi \in B^*$ with $\|\phi\|_{B^*} = 1$ and $\phi(Fy(t) - Fy(s)) = |Fy(t) - Fy(s)|$. Then

$$\begin{aligned} |Fy(t) - Fy(s)| &= \phi\left(\int_s^t k(x) f(x, y(x)) dx\right) \\ &\leq \int_s^t |k(x)| \psi(b(x)) dx \\ &= \int_t^s b'(x) dx = b(t) - b(s). \end{aligned}$$

Thus, $Fy \in Q$, for all $y \in Q$.

(b) F is $w-$continuous. By (H_2), for any $\phi \in B^*$, $\varepsilon > 0$, and $y \in C([0,T], B_w)$, there exists a weak neighborhood U of zero in B such that

$$|\phi(f(t, y(t)) - f(t, x(t)))| \leq \varepsilon/k_0, \quad \forall\, t \in [0,T] \text{ and } \forall\, x \in C([0,T], B_w) \\ \text{with } \; y(s) - x(s) \in U, \; \text{ for all } \; s \in [0,T],$$

where $k_0 := \int_0^T |k(s)| ds$. Thus, for each $x, y \in Q$ such that $(y(s) - x(s) \in U, \ \forall\, s \in [0,T])$, we have

$$\int_0^t |\phi\left(k(s)[f(s, y(s)) - f(s, x(s))]\right)| \, ds \leq \varepsilon.$$

It follows that $F : Q \longrightarrow Q$ is $w-$continuous.

(c) FQ is relatively compact in $C([0,T], B_w)$. This will follow from Theorem 1.9 and Theorem 1.13. Since B is reflexive, Q is bounded, and $FQ \subset Q$, Theorem 1.12 implies that $FQ([0,T])$ is weakly relatively compact. Finally, we have that FQ is equicontinuous since $FQ \subset Q$ and Q is equicontinuous, whence claim (c).

Therefore Theorem 2.1 implies that Equation (3.1) has a solution in Q. □

3.2 The general case. To discuss the solvability of Equation (1.2), we make the following assumptions where B is a reflexive Banach space.

$(H1)$ The function $h : [0,T] \longrightarrow B$ is continuous.

$(H2)$ The function $f : [0,T] \times B \longrightarrow B$ is weakly-weakly continuous.

$(H3)$ $k_t(s) = k(t,s) \in L^1([0,T], \mathbb{R})$ for each $t \in [0,T]$ and there exist $v \in L^1[0,T]$ and positive constants α, β such that for $x, t \in [0,T]$ $(x < t)$, we have

$$\int_x^t |k(t,s)| ds \leq \beta \left(\int_x^t v(s) ds\right)^{\alpha}.$$

$(H4)$ $\int_0^{t^*} |k_t(s) - k_{t'}(s)| ds \longrightarrow 0$, as $t \to t'$, where $t^* = \min(t, t')$.

$(H5)$ (Nagumo-type condition) The exist $\alpha \in L^1[0,T]$ and $\psi : [0, +\infty) \longrightarrow (0, +\infty)$ a nondecreasing continuous function such that

$$\begin{aligned} |k(t,s) f(s,u)| &\leq \alpha(s)\psi(|u|), \text{ for a.e. } s, t \in [0,T] \ (s < t), \text{ all } u \in B \\ \text{and } \int_0^T \alpha(s) ds &< \int_{|h|_0}^{\infty} \frac{du}{\psi(u)}. \end{aligned}$$

Theorem 3.2. *Under Assumptions* (H1)-(H5), *Equation* (1.2) *has a solution in* $C([0,T], B_w)$ *(and of course in* $C([0,T], B)$.

Remark 3.3. A solution y is in $C([0,T],B)$ but satisfies Equation (1.2) relatively to the topology of $C([0,T],B_w)$, i.e. $y\in C([0,T],B_w)\bigcap C([0,T],B)=C([0,T],B)$ and

$$\phi(y(t))=\phi(h(t))+\phi\left(\int_0^t k(t,s)f(s,y(s))ds\right),\quad \forall\,\phi\in B^*.$$

Proof. Let

$$Q=\{y\in C([0,T],B),\ |y(t)|\le b(t),\ \forall\,t\in[0,T]\},$$

where

$$b(t):=I^{-1}\left(\int_0^t \alpha(s)ds\right)\ \text{ and }\ I(z):=\int_{|h|_0}^{z}\frac{du}{\psi(u)}.$$

Clearly, Q is convex, bounded, closed subset of $C([0,T],B)$. Note the condition $|y(t)-y(s)|\le b(t)-b(s)$ is removed from the definition of Q which may be shown to be closed in $C([0,T],B_w)$ without equicontinuity.

(a) Q is closed for the topology of $C([0,T],B_w)$. Indeed, let $(y_\alpha)\subset Q$ be a net such that $y_\alpha\longrightarrow y$ in $C([0,T],B_w)$. Then,

$$\sup_{t\in[0,T]}|\phi(y_\alpha(t)-y(t))|\to 0,\ \forall\,\phi\in B^*. \tag{3.2}$$

We show that y is strongly continuous. Let $\varepsilon>0$, $t_0\in[0,T]$ be fixed and let $t\in[0,T]$. Without loss of generality, assume that $y(t)-y(t_0)\neq 0$. By Theorem 1.15, there exists $\phi=\phi_{t,t_0,y}\in B^*$ such that

$$|y(t)-y(t_0)|=\phi(y(t)-y(t_0)).$$

Notice (3.2) implies that for $\varepsilon/3>0$,

$$|\phi(y_\alpha(t))-\phi(y(t))|\le\varepsilon/3,\ \forall\,t\in[0,T].$$

Also, we know that $\phi\circ y_\alpha$ is continuous. Hence for $\varepsilon/3$, there exists $\delta>0$ such that

$$|t-t_0|\le\delta\Longrightarrow|\phi(y_{\alpha_0}(t))-\phi(y_{\alpha_0}(t_0))|\le\varepsilon/3.$$

It follows that

$$\begin{aligned}|y(t)-y(t_0)| &\le |\phi(y(t))-\phi(y_{\alpha_0}(t))|+|\phi(y_{\alpha_0}(t))-\phi(y_{\alpha_0}(t_0))|\\ &\quad +|\phi(y_{\alpha_0}(t_0))-\phi(y(t_0))|\le\varepsilon,\quad \forall\,t,\ |t-t_0|\le\delta.\end{aligned}$$

Next, we show that $|y(t)|\le b(t),\ \forall\,t\in[0,T]$. Let $t\in[0,T]$ be fixed. Without loss of generality, assume that $y(t)\neq 0$. By Theorem 1.15, there exists $\phi\in B^*$ such that $\|\phi\|_{B^*}=1$ and $|y(t)|=\phi(y(t))$ and (3.2) yields that for any $\varepsilon>0$, there exists α_0 such that $\sup\limits_{t\in[0,T]}|\phi(y_{\alpha_0}(t))-\phi(y(t))|\le\varepsilon$. Therefore, we have the estimates

$$\begin{aligned}|y(t)|=\phi(y(t)) &\le |\phi(y(t))-\phi(y_{\alpha_0}(t))|+|\phi(y_{\alpha_0}(t))|\\ &\le \varepsilon+\|\phi\|_{B^*}.|y_{\alpha_0}(t)|_B\\ &\le \varepsilon+1.b(t)\end{aligned}$$

since $y_{\alpha_0} \in Q$. Hence $\|y(t)\|_B \leq b(t),\ \forall\, t \in [0,T]$, proving that $y \in Q$. To sum up, we have proved that Q is closed in $C([0,T], B_w)$, proving our claim.

Let the operator F be defined by

$$Fy(t) = h(t) + \int_0^t k(t,s)f(s,y(s))ds, \quad t \in [0,T]. \tag{3.3}$$

(b) $FQ \subseteq Q$, for all $y \in Q$:

• For any $y \in Q$, Fy is norm-continuous. For this, let $t, x \in [0,T]$ with $t > x$. Without loss of generality, assume that $Fy(t) - Fy(x) \neq 0$. Then, by Theorem 1.15, there exists $\phi \in B^*$ with $\|\phi\|_{B^*} = 1$ and $|Fy(t) - Fy(x)| = \phi(Fy(t) - Fy(x))$. Thus

$$\begin{aligned} |Fy(t) - Fy(x)| &= \phi(h(t) - h(x)) + \phi(\textstyle\int_0^x [k(t,s) - k(x,s)]f(s,y(s))ds) \\ &\quad + \phi(\textstyle\int_x^t k(t,s)f(s,y(s))ds) \\ &\leq |h(t) - h(x)| + M_r \textstyle\int_0^x |k(t,s) - k(x,s)|ds \\ &\quad + M_r \textstyle\int_x^t |k(t,s)|ds, \end{aligned}$$

where r is such that $|y|_0 = \sup_{t\in[0,T]} |y(t)| \leq r$. The existence of M_r is ensured by Proposition 1.8. Thus Fy is norm-continuous by Assumptions (H_3) and (H_4).

• For $y \in Q$, assume, without loss of generality, that $Fy(t) \neq 0$ for all $t \in [0,T]$. By Theorem 1.15, there exists $\phi_t \in B^*$ with $\|\phi_t\|_{B^*} = 1$ and $\phi_t(Fy(t)) = |Fy(t)|$. Consequently, for each $t \in [0,T]$, we have

$$\begin{aligned} |Fy(t)| &= \phi_t\left(h(t) + \textstyle\int_0^t k(t,s)f(s,y(s))ds\right) \\ &\leq |h(t)| + \textstyle\int_0^t \alpha(s)\psi(|y(s)|)ds \\ &\leq |h|_0 + \textstyle\int_0^t \alpha(s)\psi(b(s))ds \\ &= |h|_0 + \textstyle\int_0^t b'(s)ds = b(t), \end{aligned}$$

since $\int_{|h|_0}^{b(s)} \frac{du}{\psi(u)} = \int_0^s \alpha(u)du$.

(b) F is $w-$continuous. The proof is similar to the one in Theorem 3.1 and is omitted.

(c) FQ is relatively compact in $C([0,T], B_w)$. It suffices to prove that FQ is weakly equicontinuous in which case the result follows from Theorem 1.9. Arguing as in (b), we have

$$|Fy(t) - Fy(x)| \leq |h(t) - h(x)| + M_r \int_0^x |k(t,s) - k(x,s)|ds + M_r \int_x^t |k(t,s)|ds,$$

and with Assumptions (H_3) and (H_4), we conclude part (c).

Theorem 2.1 then implies that Equation (1.2) has a solution in Q. □

Remark 3.4. When $F : Q \longrightarrow Q$ is $w-$continuous and $\omega-$contractive, another existence result for (1.2) is given by [22], Them. 10.2.4.

Remark 3.5. In order to apply Theorem 2.1 in Section 3, we need to show (among others) that:

(a) Q is closed in $C([0,T];B_w)$ which may be proved in two different ways:
(i) use Q equicontinuous: see part (a), proof of Theorem 3.1,
(ii) directly using the "particular" Q: see part (a), proof of Theorem 3.2.
(b) FQ is $w-$equicontinuous which may be proved in two different ways:
(i) use Q equicontinuous and $FQ \subset Q$: see part (c), proof of Theorem 3.1,
(ii) directly using Assumptions $(H_3)-(H_4)$: see part (c), proof of Theorem 3.2 (even if we do not know whether or nor Q is equicontinuous; in fact, it does not matter). Note this method could also be applied in the proof of Theorem 3.1 since $(H_3)-(H_4)$ are obviously satisfied for Equation (3.1).

4. Operator and Volterra integral inclusions

4.1 Preliminaries. In this section, we will consider the operator inclusion

$$u(t) \in Fu(t), \quad t \in [0,T], \tag{4.1}$$

where $F\colon\ Q \to C(Q)$ is a multi-map, $Q \subset C([0,T],B) \subset C([0,T],B_w)$, and $C(Q)$ stands for the family of nonempty, convex, closed subsets of Q in a topological vector space X. Also, we study the Volterra integral inclusion

$$y(t) \in h(t) + \int_0^t k(t,s)f(s,y(s))ds, \quad t \in [0,T], \tag{4.2}$$

where the multi-functions h, k, f satisfy some assumptions to be defined later on. Our aim is to prove existence of solutions in $C([0,T],B_w)$. First some auxiliary results regarding multi-valued analysis are recalled hereafter. More details may be found in [2], [4], [5], [9], [20], [24].

A single-valued map $f : B \to B$ is said to be a selection of F and we write $f \subset F$ whenever $f(u) \in F(u)$ for every $u \in B$. Denote by
(a) $C_c(Q)$ the family of nonempty, convex, closed (in $C([0,T],B)$) subsets of Q (here $Q \subset C([0,T],B)$).
(b) $C_w(Q)$ the family of nonempty, convex, closed (in $C([0,T],B)$) subsets of Q (here $Q \subset C([0,T],B_w)$).
(c) $C_{c,e}(Q)$ the family of nonempty, convex, closed (in $C([0,T],B)$), equicontinuous subsets of Q (here $Q \subset C([0,T],B)$).

Definition 4.1. (a) F is called upper semi-continuous (*u.s.c.* for short) if the set $F^{-1}(V) = \{x \in B,\ F(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Q. Equivalently, F is *u.s.c.* if the set $F^{+1}(V) = \{x \in B,\ F(x) \subset V\}$ is open for any open set V in Q.
(b) F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq B$, $F(A)$ is relatively compact, i.e. there exists a relatively compact set $K \subset X$ depending on A such that $F(A) = \bigcup\{F(x),\ x \in A\} \subset K$. The multimap F is compact if $F(X)$ is relatively compact.

4.2 An abstract result. Our main tool is a classical fixed point theorem:

Lemma 4.2. [10] *Let Q be a nonempty, convex, closed subset of a locally convex Hausdorff linear vector space X. Assume that $F : Q \longrightarrow C(Q)$ is u.s.c. and $F(Q)$ is relatively compact in X. Then F has a fixed point in C.*

The following result is an immediate consequence from Theorem 1.9 and Lemma 4.2.

Theorem 4.3. *Let B be a Banach space and Q a nonempty, closed, convex subset of $C([0,T], B_w)$. Assume*

$$F : Q \longrightarrow C_w(Q) \text{ is } w-u.s.c.$$

Assume further that the family $F(Q)$ is weakly equicontinuous and $FQ(t)$ is weakly relatively compact in B for each $t \in [0,T]$. Then (4.1) has a solution.

Corollary 4.4. *Let B be a Banach space and Q a nonempty, closed, convex subset of $C([0,T], B_w)$ such that $FQ(t)$ is weakly relatively compact in B for each $t \in [0,T]$. Assume that either one of the following conditions hold:*

(a) $F : Q \longrightarrow C_c(Q)$ is $w-u.s.c.$ and $Q \subset C([0,T], B)$ is equicontinuous.
(b) $F : Q \longrightarrow C_{c,e}(Q)$ is $w-u.s.c.$, $Q \subset C([0,T], B)$ and the family $F(Q)$ is weakly equicontinuous.

Then (4.1) has a solution.

Proof. When (a) holds, using an argument similar to that in the proof of Theorem 2.3(b), we have that $F : Q \longrightarrow C_w(Q)$. Part (b) implies that for each $y \in Q$, Fy is a convex, closed, equicontinuous subset of Q. Since $Q \subset C([0,T], B)$, then Fy is closed in $C([0,T], B_w)$, and again $F : Q \longrightarrow C_w(Q)$ is $w-u.s.c.$ Theorem 4.3 concludes the proof. □

A more general version of Theorem 4.2 is given by:

Theorem 4.5. *Let B be a Banach space and a nonempty, closed, convex subset of $C([0,T], B_w)$. Assume further that Q is a closed, bounded, equicontinuous subset of $C([0,T], B)$. Let $F : Q \longrightarrow C_c(Q)$ be a $w-u.s.c.$ operator which is an $\alpha-$contraction, relatively to a weak MNC. Then (4.1) has a solution.*

Remark 4.6. Note we drop the condition FQ is weakly equicontinuous here from Theorem 2.9 in [20] and add the equicontinuity of Q.

Proof. Let $S_1 = Q$ and $S_{n+1} = \overline{co}(F(S_n))$, $n \in \mathbb{N}^*$. It is easy to see that

$$S_{n+1} \subseteq S_n \text{ and } \omega(S_{n+1}) \leq \alpha^n \omega(S_1), \text{ for } n = 1, 2, \ldots$$

Since $0 \leq \alpha < 1$, then $\lim_{n\to\infty} \omega(S_n) = 0$. Moreover S_n is a weakly closed subset of $C([0,T], B)$ for each n, then $S_\infty = \bigcap_1^\infty S_n$ is nonempty, convex and weakly closed.

Moreover, S_∞ is nonempty and weakly compact in $C([0,T],B)$ by the Cantor intersection condition for the weak measure of noncompactness (Lemma 1.16, (g)). Arguing as in the proof of Theorem 2.3 and using the fact that Q is equicontinuous and $S_\infty \subset S_1 \subset Q$, we find that S_∞ is closed in $C([0,T],B_w)$

(b) F is $w-u.s.c.$ Since

$$F(S_n) \subset F(S_{n-1}) \subseteq \overline{co}(F(S_{n-1})) = S_n, \text{ for all } n \in \mathbb{N}^*,$$

F maps S_∞ into $C_c(S_\infty)$. By the equicontinuity of S_∞, we deduce that F maps S_∞ into $C_w(S_\infty)$. This yields that $F : S_\infty \longrightarrow C_w(S_\infty)$ is $w-u.s.c.$

(c) $F(S_\infty)$ is relatively compact in $C([0,T],B_w)$. We shall appeal to Theorem 1.9. Since the equicontinuity of $F(S_\infty)$ follows from $F(S_\infty) \subset S_\infty \subset Q$ and the equicontinuity of Q, we only have to show that, for each $t \in [0,T]$, the set $F(S_\infty)(t) = \{Fy(t),\ y \in S_\infty\}$ is weakly relatively compact in B. Now, $F(S_\infty) \subseteq S_\infty$ and $\omega(S_\infty) = 0$ imply that $\omega(F(S_\infty)) = 0$. Then Theorem 1.18 yields that $\omega(S_\infty)(t) = 0$ for each $t \in [0,T]$, whence part (c). Finally, by Lemma 4.2, we conclude that F has a fixed point, solution of (4.1). □

4.3 Application to an integral inclusion. To discuss the solvability of the nonlinear Volterra inclusion (4.2), we first make the following assumptions (2.6)-(2.12) in [20]):

$(G1)$ $F : [0,T] \times B \longrightarrow B$ has nonempty, compact, convex values.

$(G2)$ $\begin{cases} \text{For each continuous } y : [0,T] \longrightarrow B, \text{ there exists a scalarly measurable} \\ \text{selection } f : [0,T] \longrightarrow B \text{ with } f(t) \in F(t, y(t)), \text{ a.e. on } [0,T]. \end{cases}$

$(G3)$ $\begin{cases} \text{For any } r > 0, \text{ there exists } M_r > 0 \text{ with } |F(t,y)| \le M_r, \text{ for all} \\ t \in [0,T] \text{ and all } y \in B \text{ with } |y| \le r. \end{cases}$

$(G4)$ $\begin{cases} \text{For each continuous } y : [0,T] \longrightarrow B, \text{ there exists a selection } f \\ \text{with either } f([0,T]) \text{ is relatively weakly compact or } f \text{ is Pettis} \\ \text{integrable and } \overline{co}(f([O,T]) \text{ has the Radon-Nikodym property.} \end{cases}$

$(G5)$ $h : [0,T] \longrightarrow B$ is a continuous single valued function.

$(G6)$ $\begin{cases} k_t(s) = k(t,s) \in L^1([0,t],\mathbb{R}) \text{ for each } t \in [0,T] \text{ and there exist} \\ v \in L^1[0,T] \text{ and positive constants } \alpha, \beta \text{ such that for } x,t \in [0,T]\ (x < t), \\ \text{we have } \int_x^t |k(t,s)|ds \le \beta \left(\int_x^t v(s)ds\right)^\alpha. \end{cases}$

$(G7)$ $\int_0^{t^*} |k_t(s) - k_{t'}(s)|ds \longrightarrow 0,$ as $t \to t'$, where $t^* = \min(t,t')$.

Recall that function y from a measure space (Ω, M) to a Banach space E is *scalarly measurable* if for any $\phi \in E^*$, the function $\phi(y)$ is measurable on (Ω, M). It is clear that all solutions of Equation (4.2) are fixed points of the multi-valued operator $N : C([0,T],B) \to C_c((C([0,T],B))) := Y$ defined by

$$N(y) := \left\{ g \in Y \mid g(t) = h(t) + \int_0^t k(t,s)f(s)ds,\ t \in [0,T] \right\},$$

where

$$f \in S_{F,y} := \{ f \text{ is scalarly measurable} \mid f(t) \in F(t,y(t)),\ t \in [0,T] \}.$$

Remark 4.7. $(G4)$ implies that $S_{F,y} \neq \emptyset$ and $(G4),(G6)$ ensure the integral in $N(y)$ is well defined. Since, for each $y \in B$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and therefore N has convex values. The proof that N has closed values in $C([0,T],B)$ is well detailed in [20].

Recall that F is said to be $w-$upper semi-continuous ($w-u.s.c.$ for short) if for any closed set $V \subset C([0,T],B_w)$, the set $F^{-1}(V)$ is closed in $C([0,T],B_w)$. Then our main existence result is

Theorem 4.8. *Let B be a Banach space and Q a nonempty, closed, convex subset of $C([0,T],B_w)$ with Q a bounded subset of $C([0,T],B)$. Also assume* (G1)-(G7), *and let the following conditions hold:*

$(G8)$ $N : C([0,T],B_w) \cap C([0,T],B) \longrightarrow C_c(C([0,T],B))$ *is* $w-u.s.c.$

$(G9)$ $\begin{cases} K(\{t\} \times [0,T] \times Q[0,t]) \text{ is weakly relatively compact in } B \text{ for each} \\ t \in [0,T], \text{ where } K(t,s,u) := k(t,s)F(s,u). \end{cases}$

$(G10)$ $N : Q \longrightarrow C_w(Q)$.

Then (4.2) *has a solution* Q.

Remark 4.9. Note if Q is convex, closed, and equicontinuous, then any convex, closed subset of Q is closed in $C([0,T],B_w)$, in which case $N : Q \longrightarrow C_c(Q)$ (that is condition (2.17) in [20]) implies $(G10)$.

Proof. The proof is split into three steps.

(a) The set $NQ(t) = \{Ny(t),\ y \in Q\}$ is weakly relatively compact in B, for each $t \in [0,T]$. Fix $t \in [0,T]$ and let $y \in Q$, and $g \in Ny$. By $(G3)$, there exists a scalarly measurable selection $f(\cdot) \in F(\cdot, y(\cdot))$ and $g(t) = h(t) + \int_0^t k(t,s)f(s)ds$. By [12], we have

$$\int_0^t k(t,s)f(s)ds \in t\overline{co}\{k(t,s)f(s),\ s \in [0,T]\}.$$

Hence

$$\begin{aligned} \omega(NQ(t)) &\leq \omega(t\overline{co}\{K(t,s,y(s)),\ y \in Q,\ s \in [0,t]\}) \\ &= T\omega(K(\{t\} \times [0,t] \times Q[0,t])) = 0, \end{aligned}$$

yielding our claim.

(b) By Assumption, we have $N : Q \longrightarrow C_w(Q)$.

(c) NQ is weakly equicontinuous. Now Q is bounded, so there exists $r > 0$ such that $|y_0| \leq r$ for all $y \in Q$. By $(G3)$, there exists $M_r > 0$ such that

$$|F(t, y(t))| \leq M_r \text{ for all } t \in [0, T] \text{ and all } y \in Q.$$

Let $g \in NQ$ and $t, x \in [0, T]$ with $t > x$. Without loss of generality, assume $g(t) - g(x) \neq 0$. By Theorem 1.15, there exists $\phi \in B^*$ with $\|\phi\|_{B^*} = 1$ and $|g(t) - g(x)| = \phi(g(t) - g(x))$. Hence

$$\begin{aligned}&|g(t) - g(x)| = \phi(g(t) - g(x)) \\ &\leq |h(t) - h(x) + M_r \textstyle\int_0^x |k(t, s) - k(x, s)|ds + M_r \int_x^t |k(t, s)|ds,\end{aligned}$$

proving (c). With parts (a)-(c), Theorem 4.3 concludes the proof. □

We can also prove a more general version of Theorem 4.8.

Theorem 4.10. *Let B be a Banach space and Q a nonempty, closed, convex, subset of $C([0, T], B)$ with Q a bounded, equicontinuous subset of $C([0, T], B)$. Assume* (G1)-(G8), (G10)*, and let the following condition hold:*

$$(G11) \quad \begin{cases} \textit{There exists a constant } 0 \leq \gamma T < 1 \textit{ such that} \\ \omega(K(\{t\}) \times [0, t] \times \Omega)) \leq \gamma\omega(\Omega), \quad \textit{for } t \in [0, T] \\ \textit{and for any bounded subset } \Omega \subset Q. \end{cases}$$

Then (4.2) *has a solution Q.*

Remark 4.11. Note the equicontinuity of Q with $(G11)$ implies $(G9)$.

Sketch of the proof. The proof is identical to that of Theorem 4.8 and uses Theorem 4.5. Thus, we only have to check $N : Q \longrightarrow C_w(Q)$ satisfies

$$\omega(N(\Omega)) \leq T\gamma\omega(\Omega), \text{ for any bounded subset } \Omega \subset Q.$$

For this, let $\Omega \subset Q$. For $t \in [0, T]$, we have

$$\begin{aligned} \omega(N\Omega(t)) &= \omega\left(\left\{h(t) + \textstyle\int_0^t k(t, s)f(s)ds, \ f \in \Omega\right\}\right) \\ &= \omega\left(\left\{\textstyle\int_0^t k(t, s)f(s)ds, \ f \in \Omega\right\}\right) \\ &\leq \omega\left(t\overline{co}K(t, s, f(s)), \ f \in \Omega, \ s \in [0, t]\right) \\ &= T\omega(K\{t\} \times [0, t] \times \Omega[0, t])) \\ &\leq T\gamma\omega(\Omega). \end{aligned}$$

Note the equicontinuity of Q allows us to use Theorem 1.18, part (b). Since $T\gamma < 1$, N is $\omega-$contractive, ending the proof of the theorem. □

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