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EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH FOUR-POINT BOUNDARY CONDITIONS

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Dedicated to Professor J.R.L. Webb on the occasion of his retirement

ABSTRACT. This paper investigates the existence of solutions for a boundary value problem of fractional differential inclusions of order $q \in (1,2]$ with four-point nonlocal boundary conditions involving convex and non-convex multivalued maps. The main tools of our study are the nonlinear alternative of Leray Schauder type and some suitable fixed point theorems.

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1. INTRODUCTION

In recent years, there has been a significant progress in the investigation of initial and boundary value problems of fractional differential equations. Such problems arise in a variety of areas of applied mathematics, physics, variational problems of control theory, chemistry, biology, economics, biophysics, fitting of experimental data, etc. [1, 2]. It is found that the differential equations of arbitrary order provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. With these features, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. The advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials. In view of the recent interest in fractional calculus, the subject of differential inclusions of fractional order has attracted the attention of many researchers. For some recent work on the subject, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

In this paper, we study the existence of solutions for a nonlocal four-point boundary value problem of differential inclusions of order $q \in (1,2]$ involving convex and

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non-convex multivalued maps. Our existence results are based on the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory. Precisely, we consider the following problem

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)),\ t \in [0,1],\ 1 < q \le 2, \\ x'(0) + a x(\eta_1) = 0, \quad b x'(1) + x(\eta_2) = 0,\ 0 < \eta_1 \le \eta_2 < 1,\ a, b \in (0,1) \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$.

Here we remark that multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located at intermediate points [13].

2. PRELIMINARIES

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1], \mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Now we recall some basic definitions on multi-valued maps [14, 15, 16].

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(**i**) $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

(iii) for each $\nu > 0$, there exists $\varphi_\nu \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\nu(t) \text{ for all } \|x\|_\infty \leq \nu \text{ and for a. e. } t \in [0,1].$$

Note that the multivalued map F is said to be Carathéodory if the conditions (i) and (ii) hold in Definition 2.1.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ is a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 2.3. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X,d) be a metric space induced from the normed space $(X;\|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [17]).

Definition 2.4. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y) \text{ for each } x, y \in X;$$

(**b**) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.5. [18] *Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$, then the operator*

$$\Theta \circ S_F : C([0,1],X) \to P_{cp,c}(C([0,1],X)),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Lemma 2.6. [19] *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.7. [20] *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let us recall some definitions on fractional calculus [21].

Definition 2.8. For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds,\ n-1 < q < n, n = [q]+1,\ q > 0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.9. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds,\ q > 0,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

In order to define the solution of (1.1), we recall the following lemma [22].

Lemma 2.10. *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t),\ 0 < t < 1,\ 1 < q \leq 2, \\ x'(0) + ax(\eta_1) = 0, \quad bx'(1) + x(\eta_2) = 0,\ 0 < \eta_1 \leq \eta_2 < 1, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + a\mu_3(\mu_1 - t) \int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds \\ &\quad + \mu_3(at - \mu_2)\Big[b \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \int_0^{\eta_2} \frac{(\eta_2 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds\Big], \end{aligned} \tag{2.2}$$

where $\mu_1 = b + \eta_2,\ \mu_2 = 1 + a\eta_1,\ \mu_3 = [1 + a(\eta_1 - \eta_2 - b)]^{-1}$.

Definition 2.11. A function $x \in C^2([0,1])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f(s)ds\Big]. \quad (2.3) \end{aligned}$$

3. MAIN RESULTS

Theorem 3.1. *Assume that*

($\mathbf{H_1}$) $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact convex values;*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \text{ for each}(t,x) \in [0,1]\times\mathbb{R};$$

($\mathbf{H_3}$) *there exists a number* $M > 0$ *such that*

$$\frac{\Gamma(q)M}{\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_1^{q-1}\Big)\Big]\psi(M)\|p\|_{L^1}} > 1,$$

where

$$\lambda_1 = \sup_{t\in[0,1]} |a\mu_3(\mu_1 - t)|,\ \lambda_2 = \sup_{t\in[0,1]} |\mu_3(at-\mu_2)|. \quad (3.1)$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Define an operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle + a\mu_3(\mu_1 - t)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f(s)ds\Big], \end{array}\right. \end{array}\right\}$$

for $f \in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray- Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ *is convex for each* $x \in C([0,1],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ &\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds + \int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f_i(s)ds\Big],\ i = 1,2. \end{aligned}$$

Let $0 \leq \omega \leq 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
[\omega h_1 + (1-\omega)h_2](t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)](s)ds \\
&+ a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)](s)ds \\
&+ \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s) + (1-\omega)f_2(s)](s)ds \\
&+ \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)](s)ds\Big].
\end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.

Next, we show that $\Omega(x)$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + a\mu_3(\mu_1-t)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}f(s)ds \\
&+\mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s)ds + \int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}f(s)ds\Big]
\end{aligned}$$

and

$$\begin{aligned}
|h(t)| &\leq \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}|f(s)|ds + |a\mu_3(\mu_1-t)|\int_0^{\eta_1}\frac{|\eta_1-s|^{q-1}}{\Gamma(q)}|f(s)|ds \\
&+|\mu_3(at-\mu_2)|\Big[b\int_0^1 \frac{|1-s|^{q-2}}{\Gamma(q-1)}|f(s)|ds + \int_0^{\eta_2}\frac{|\eta_2-s|^{q-1}}{\Gamma(q)}|f(s)|ds\Big] \\
&\leq \frac{1}{\Gamma(q)}\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_2^{q-1}\Big)\Big]\psi(\|x\|_\infty)\int_0^1 p(s)ds,
\end{aligned}$$

where λ_1 and λ_2 are given by (3.1). Thus,

$$\|h\|_\infty \leq \frac{1}{\Gamma(q)}\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_2^{q-1}\Big)\Big]\psi(\|x\|_\infty)\int_0^1 p(s)ds.$$

Now we show that Ω *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}
&|h(t'') - h(t')| \\
= &\Big|\int_0^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds - \int_0^{t'}\frac{(t'-s)^{q-1}}{\Gamma(q)}f(s)ds \\
&+a\mu_3(t''-t')\Big[-\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}f(s)ds \\
&+b\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s)ds + \int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}f(s)ds\Big]\Big|
\end{aligned}$$

$$\begin{aligned}
\leq & \Big|\int_0^{t'} \frac{[(t''-s)^{q-1}-(t'-s)^{q-1}]}{\Gamma(q)} f(s)ds\Big| + \Big|\int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)ds\Big| \\
& + \Big| a\mu_3(t''-t')\Big[-\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f(s)ds \\
& + b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f(s)ds\Big]\Big|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t''-t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by Ascoli-Arzelá theorem that $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step,we show that Ω *has a closed graph* by employing a technique used in [23]. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f_n(s)ds \\
&\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds + \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f_n(s)ds\Big].
\end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\
&\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds + \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f_*(s)ds\Big].
\end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ so that

$$\begin{aligned}
f \mapsto \Theta(f) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} f(s)ds \\
&\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)} f(s)ds\Big].
\end{aligned}$$

Observe that

$$\begin{aligned}
\|h_n(t) - h_*(t)\| &= \Big\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\
&\quad + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\
&\quad + \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} (f_n(s) - f_*(s))ds \\
&\quad + \int_0^{\eta_2} \frac{(\eta_2-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds\Big]\Big\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, it follows by Lemma 2.5 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, it follows that

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &\quad + \mu_3(at - \mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds + \int_0^{\eta_2} \frac{(\eta_2 - s)^{q-1}}{\Gamma(q)} f_*(s)ds\Big], \end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, *we discuss a priori bounds on solutions*. Let x be a solution of (1.1). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} f(s)ds \\ &\quad + \mu_3(at - \mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \int_0^{\eta_2} \frac{(\eta_2 - s)^{q-1}}{\Gamma(q)} f(s)ds\Big]. \end{aligned}$$

In view of (H_2), for each $t \in [0,1]$, we obtain

$$|x(t)| \le \frac{1}{\Gamma(q)}\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_2^{q-1}\Big)\Big]\psi(\|x\|_\infty)\int_0^1 p(s)ds,$$

where λ_1 and λ_2 are given by (3.1). Consequently, we have

$$\frac{\Gamma(q)\|x\|_\infty}{\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_1^{q-1}\Big)\Big]\psi(\|x\|_\infty)\|p\|_{L^1}} \le 1,$$

In view of (H_3), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty < M + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [24], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [19] for lower semi-continuous maps with decomposable values.

Theorem 3.2. *Assume that* $(H_2) - (H_3)$ *and the following conditions hold:*

($\mathbf{H_4}$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
  (**a**) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
  (**b**) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$*;*

($\mathbf{H_5}$) *for each* $\sigma > 0$*, there exists* $\varphi_\sigma \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t) \text{ for all } \|x\|_\infty \le \sigma \text{ and for a.e.} t \in [0,1].$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 2.6, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)),\ t \in [0,1],\ 1 < q \le 2, \\ x'(0) + ax(\eta_1) = 0, \quad bx'(1) + x(\eta_2) = 0,\ 0 < \eta_1 \le \eta_2 < 1,\ a, b \in (0,1). \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0,1])$ is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &+ \mu_3(at - \mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds + \int_0^{\eta_2} \frac{(\eta_2 - s)^{q-1}}{\Gamma(q)} f(x(s))ds\Big]. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [20].

Theorem 3.3. *Assume that* (H_4) *and the following condition hold:*

(H$_6$) $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : [0,1] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

(H$_7$) $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$ *for almost all* $t \in [0,1]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in L^1([0,1],\mathbb{R})$ *and* $d(0, F(t,0)) \le m(t)$ *for almost all* $t \in [0,1]$.

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$ *if*

$$\frac{\|m\|_{L^1}}{\Gamma(q)}\Big[1 + \lambda_1\eta_1^{q-1} + \lambda_2\Big(b(q-1) + \eta_2^{q-1}\Big)\Big] < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_7), so F has a measurable selection (see Theorem III.6 [25]). Now we show that the operator Ω satisfies the assumptions of Lemma 2.7. To show that $\Omega(x) \in P_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds + a\mu_3(\mu_1 - t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} v_n(s)ds$$

$$+\mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds+\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}v_n(s)ds\Big].$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[0,1]$,

$$\begin{aligned} u_n(t)\to u(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds+a\mu_3(\mu_1-t)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}v(s)ds\\ &+ \mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}v(s)ds+\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}v(s)ds\Big]. \end{aligned}$$

Hence $u\in\Omega(x)$.

Next we show that there exists $\gamma<1$ such that

$$H_d(\Omega(x),\Omega(\bar{x}))\le\gamma\|x-\bar{x}\|_\infty \text{ for each } x,\bar{x}\in C([0,1],\mathbb{R}).$$

Let $x,\bar{x}\in C([0,1],\mathbb{R})$ and $h_1\in\Omega(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)ds+a\mu_3(\mu_1-t)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}v_1(s)ds\\ &+\mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}v_1(s)ds+\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}v_1(s)ds\Big]. \end{aligned}$$

By (H_4), we have

$$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w\in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|,\ t\in[0,1].$$

Define $U:[0,1]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [25]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,1]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,1]$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_2(s)ds+a\mu_3(\mu_1-t)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}v_2(s)ds\\ &+\mu_3(at-\mu_2)\Big[b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}v_2(s)ds+\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}v_2(s)ds\Big]. \end{aligned}$$

Thus

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds\\ &+|a\mu_3(\mu_1-t)|\int_0^{\eta_1}\frac{|\eta_1-s|^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds \end{aligned}$$

$$+|\mu_3(at-\mu_2)|\Big[b\int_0^1 \frac{|1-s|^{q-2}}{\Gamma(q-1)}|v_1(s)-v_2(s)|ds$$
$$+\int_0^{\eta_2} \frac{|\eta_2-s|^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds\Big]$$
$$\leq \frac{1}{\Gamma(q)}\Big[1+\lambda_1\eta_1^{q-1}+\lambda_2\Big(b(q-1)+\eta_2^{q-1}\Big)\Big]\int_0^1 m(s)\|x-\overline{x}\|ds.$$

Hence

$$\|h_1(t)-h_2(t)\|_\infty \leq \frac{\|m\|_{L^1}}{\Gamma(q)}\Big[1+\lambda_1\eta_1^{q-1}+\lambda_2\Big(b(q-1)+\eta_2^{q-1}\Big)\Big]\|x-\overline{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x),\Omega(\bar{x})) &\leq \gamma\|x-\bar{x}\|_\infty \\ &\leq \frac{\|m\|_{L^1}}{\Gamma(q)}\Big[1+\lambda_1\eta_1^{q-1}+\lambda_2\Big(b(q-1)+\eta_2^{q-1}\Big)\Big]\|x-\overline{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.7 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

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