Communications in Applied Analysis **15** (2011), no. 2, 3 and 4, 273–282

BRANCHES OF HARMONIC SOLUTIONS FOR A CLASS OF PERIODIC DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Dedicated to Prof. J.R.L. Webb on the occasion of his retirement.

ABSTRACT. We study a class of T-periodic parametrized differential-algebraic equations, which are equivalent to suitable ordinary differential equations on manifolds. By combining a recent result on the degree of tangent vector fields, due to Spadini, with an argument on periodic solutions of ODEs on manifolds, we get a global continuation result for T-periodic solutions of our equations.

AMS (MOS) Subject Classification. 34A09, 34C25, 34C40.

1. INTRODUCTION AND PRELIMINARIES

In this paper we study the set of periodic solutions of a particular class of periodic differential-algebraic equations (DAEs) by means of topological methods. Namely, given $T > 0$, we consider the T-periodic solutions of

$$\begin{cases} \dot{x} = \lambda f(t,x,y), \\ g(x,y) = 0, \end{cases} \tag{1.1}$$

where λ is a nonnegative real parameter, $f : \mathbb{R} \times U \to \mathbb{R}^k$ is a continuous map which is T-periodic in the first variable, U being an open connected subset of $\mathbb{R}^k \times \mathbb{R}^s$, and the map $g : U \to \mathbb{R}^s$ is assumed to be of class C^∞ with $\partial_2 g(p,q)$, the partial derivative of g with respect to the second variable, invertible for each $(p,q) \in U$. The latter assumption implies that $0 \in \mathbb{R}^s$ is a regular value of g. Thus $M := g^{-1}(0)$ is a C^∞ submanifold of $\mathbb{R}^k \times \mathbb{R}^s$, and one can show that equation (1.1) gives rise to an ordinary differential equation on M (see [5, 7]). Moreover the manifold M can be locally represented as a graph of some map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^s$. Hence equation (1.1) can be in principle locally decoupled. However, this might not be true globally (see e.g. Example 2.7 below). Observe also that even when M is a graph of some map η, it might happen that the expression of η is complicated (or even impossible to determine analytically), so that the decoupled version of (1.1)

Received July 21, 2010

may be impractical. For instance, consider the case in which $k = s = 1$, $U = \mathbb{R} \times \mathbb{R}$, $g(p, q) = q^7 + q - p^2$ and $f(p, q) = q$.

We tackle our problem from another point of view following [7]. The crucial idea is to take advantage of the equivalence between the given DAEs and suitable ODEs on M, without requiring an explicit knowledge of the manifold M itself. In this way, by means of topological methods, based on the fixed point index, as well as results about periodic solutions of ordinary differential equations on differentiable manifolds, we are able to prove a global continuation result for T-periodic solutions of (1.1). In our main result (Theorem 2.2 below) we will give conditions ensuring the existence of a connected component of elements $(\lambda; x, y)$, with $\lambda > 0$ and (x, y) a T-periodic solution to (1.1), whose closure is not compact and emanates from the set of constant functions. This kind of results may be regarded as a useful tool to study the existence of T-periodic solutions of the equation

$$\begin{cases} \dot{x} = f(t, x, y), \\ g(x, y) = 0 \end{cases}$$

which corresponds to the choice $\lambda = 1$ in (1.1) (see Corollary 2.4 below).

As pointed out, quite a similar approach has been used in a recent paper [7] by Spadini. More precisely, in that paper he considers the differential-algebraic equation, depending on $\lambda \geq 0$,

$$\begin{cases} \dot{x} = h(x, y) + \lambda f(t, x, y), \\ g(x, y) = 0, \end{cases} \tag{1.2}$$

where $h : U \to \mathbb{R}^k$ is continuous and the maps f and g satisfy the same assumptions as above. The main result in [7] is a global continuation theorem for T-periodic solutions of (1.2).

We stress that it is not possible to deduce directly our results from those in [7], although the equation we consider here is a particular case of (1.2), corresponding to h identically zero. In fact it is not difficult to see that the degree-theoretical assumption needed there cannot be satisfied when $h \equiv 0$. Therefore, equation (1.1) requires a different approach.

The plan of the paper is the following. Our first observation will be that, as $\partial_2 g(p, q)$ is invertible for all $(p, q) \in U$, equation (1.1) induces the T-periodic tangent vector field $(t, p, q) \mapsto \Psi(t, p, q)$ on $M = g^{-1}(0)$ given by

$$\Psi(t, p, q) = \big(f(t, p, q), -[\partial_2 g(p, q)]^{-1} \partial_1 g(p, q) f(t, p, q)\big),$$

and leads to an ordinary differential equation on M. In Theorem 2.2 we will get information on the set of T-periodic solutions of (1.1). For this purpose we will use an argument of Furi and Pera (see [1]), recalled in Theorem 1.3, about periodic solutions of ordinary differential equations on a differentiable manifold. However, to apply Theorem 1.3 one has to evaluate the degree of the following *mean value* tangent vector field on M:

$$\Phi(p,q) = \frac{1}{T}\int_0^T \Psi(t,p,q)dt.$$

In this context it will be crucial to deduce a formula (Theorem 2.1 below) relating the degree of the vector field Φ to that of $F : U \to \mathbb{R}^k \times \mathbb{R}^s$, given by

$$(p,q) \mapsto \left(\frac{1}{T}\int_0^T f(t,p,q)dt\,, g(p,q)\right).$$

In this way, combining Theorems 1.3 and 2.1, we shall give conditions only on the degree of F instead of Φ.

We point out that, since in Euclidean spaces vector fields can be regarded as maps and vice versa, the degree of the vector field F is essentially the well known Brouwer degree, with respect to 0, of F (seen as a map). Thus the degree of F has a simpler nature than that of Φ and, as a consequence, it is also potentially easier to compute. Notice that, in addition, the manifold M is known only implicitly and the form of Φ may not be simple.

We conclude the paper with some illustrating examples.

1.1. ASSOCIATED VECTOR FIELDS.

Our first step, following [5, 7], is to associate to (1.1) an ordinary differential equation on the manifold $M = g^{-1}(0)$.

Given $\lambda \geq 0$, a solution of equation (1.1) is a pair of functions, $x : I \to \mathbb{R}^k$ of class C^1 and $y : I \to \mathbb{R}^s$ continuous, both of them defined on some interval I, with the property that $(x(t), y(t)) \in U$ and

$$\begin{cases} \dot{x}(t) = \lambda\, f\big(t, x(t), y(t)\big), \\ g\big(x(t), y(t)\big) = 0, \end{cases}$$

for each $t \in I$. Notice that, as a consequence of the regularity assumptions on g, the Implicit Function Theorem implies that y is actually a C^1 function. In fact, in what follows, it will be convenient to consider a solution of (1.1) as a C^1 function $\zeta := (x, y)$ defined on I with values in $\mathbb{R}^k \times \mathbb{R}^s$.

Let (x, y) be a solution of (1.1) for a given $\lambda \geq 0$, defined on $I \subseteq \mathbb{R}$. Differentiating the identity $g\big(x(t), y(t)\big) = 0$ we get

$$\partial_1 g\big(x(t), y(t)\big)\dot{x}(t) + \partial_2 g\big(x(t), y(t)\big)\dot{y}(t) = 0,$$

which yields

$$\dot{y}(t) = -\lambda \left[\partial_2 g\big(x(t), y(t)\big)\right]^{-1} \partial_1 g\big(x(t), y(t)\big)\, f\big(t, x(t), y(t)\big) \tag{1.3}$$

for all $t \in I$.

Now, consider the map $\Psi : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ given by

$$\Psi(t,p,q) = \big(f(t,p,q), -[\partial_2 g(p,q)]^{-1}\partial_1 g(p,q) f(t,p,q)\big). \tag{1.4}$$

It is not hard to prove that Ψ is tangent to M in the sense that, for any $(t,p,q) \in \mathbb{R} \times M$, $\Psi(t,p,q)$ belongs to the tangent space $T_{(p,q)}M$ to M at (p,q). Taking (1.3) into account, one can see that (1.1) is equivalent to the following ODE on M:

$$\dot{\zeta} = \lambda\, \Psi(t,\zeta), \qquad \lambda \geq 0, \tag{1.5}$$

where $\zeta = (x,y)$.

Observe that, when f is of class C^1, so is the vector field Ψ. Thus, the local results on existence, uniqueness and continuous dependence of local solutions of the initial value problems translate to (1.1) from the theory of ordinary differential equations on manifolds by virtue of the equivalence of (1.1) with (1.5).

We stress that if we drop our assumption on $\partial_2 g$, equation (1.1) may fail to induce a (continuous) tangent vector field Ψ on M and, if this happens, (1.1) might not be equivalent to (1.5). The following simple examples illustrates these possibilities.

Example 1.1. Let $k = s = 1$ and $U = \mathbb{R}^2$. Consider the following DAE:

$$\dot{x} = y, \qquad x - y^3 = 0. \tag{1.6}$$

Obviously $M = \{(p,q) \in \mathbb{R}^2 : p = q^3\}$ is a C^∞ submanifold of $\mathbb{R}^2$. Equation (1.6) induces the vector field

$$(p,q) \mapsto \left(q, \frac{1}{3q}\right)$$

on $M \setminus \{(0,0)\}$ and, clearly, this vector field cannot be extended to a continuous tangent vector field on the whole of M. However, this difficulty can be overcome by taking $U = \mathbb{R}^2 \setminus \{(0,0)\}$, so that the manifold $M = \{(p,q) \in U : p = q^3\}$ consists of two connected sets. In fact, the vector field $\Psi(p,q) = \left(q, \frac{1}{3q}\right)$ is tangent to M and (1.6) turns out to be equivalent to (1.5) on M.

Example 1.2. Take again $k = s = 1$ and $U = \mathbb{R}^2$. Consider the DAE:

$$\dot{x} = y, \qquad x^2 + y^2 = 1. \tag{1.7}$$

In this case, the manifold M is the unit circle S^1 of $\mathbb{R}^2$ centered at the origin. Clearly, (1.7) induces on $S^1 \setminus \{(\pm 1, 0)\}$ the vector field $(p,q) \mapsto (q,-p)$ that can be extended uniquely to a vector field Ψ defined on the whole of S^1 by $\Psi(p,q) = (q,-p)$. Notice, however, that (1.7) is not equivalent to (1.5) on S^1. In fact, the constant maps $t \mapsto (\pm 1, 0)$ are solutions of (1.7), but not of (1.5). On the other hand, if we let $U = \mathbb{R}^2 \setminus \{(\pm 1, 0)\}$, one has that M consists of two connected components and the equivalence between (1.7) and (1.5) holds true on M.

In order to investigate the set of T-periodic solutions of (1.1) we will study the T-periodic solutions of the equivalent equation (1.5). For this purpose it will be crucial to determine a formula for the computation of the degree (sometimes called characteristic or rotation) of the mean value tangent vector field $\Phi : M \to \mathbb{R}^k \times \mathbb{R}^s$ given by formula (2.2) below. Before doing that, however, we will recall some basic facts about the notion of the degree of a tangent vector field.

1.2. THE DEGREE OF A TANGENT VECTOR FIELD.

Let $M \subseteq \mathbb{R}^n$ be a manifold. Given any $p \in M$, we denote by $T_pM \subseteq \mathbb{R}^n$ the tangent space of M at p. Let w be a tangent vector field on M, and let V be an open subset of M. We say that the pair (w, V) is *admissible* (or, equivalently, that w is admissible on V) if the set of zeros of w in V is compact. In this case one can associate to the pair (w, V) an integer, $\deg(w, V)$, called the *degree* (or *characteristic*) of the tangent vector field w on V which, roughly speaking, counts algebraically the number of zeros of w in V (for general references see e.g. [3, 4, 6]).

We recall that, when w is a C^1 tangent vector field on M, a zero $p \in M$ of w is said to be *nondegenerate* if $w'(p) : T_pM \to \mathbb{R}^n$ is one-to-one. Since the condition $w(p) = 0$ implies that $w'(p)$ maps T_pM into itself (see e.g. [6]), then $w'(p)$ is actually an isomorphism of T_pM. Thus, the determinant $\det(w'(p))$ is nonzero and its sign is called the *index* of w at p. In the particular case when an admissible pair (w, V) is *regular* (i.e. w is smooth with only nondegenerate zeros), one can show that $\deg(w, V)$ coincides with the sum of the indices at the zeros of w in V. This makes sense, since $w^{-1}(0) \cap V$ is compact (w being admissible in V) and discrete; therefore, the sum is finite.

When $M = \mathbb{R}^n$, that is, V is an open subset of $\mathbb{R}^n$, $\deg(w, V)$ is just the classical Brouwer degree, $\deg(w, V, 0)$, of the map w on V with respect to zero.

All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in the more general context of differentiable manifolds (see e.g. [2]).

1.2. A CONTINUATION RESULT OF FURI AND PERA.

We conclude the preliminary part of the paper with a continuation result for ordinary differential equations on manifolds due to Furi and Pera (see [1]).

Let $M \subseteq \mathbb{R}^n$ be a boundaryless smooth manifold and $\psi : \mathbb{R} \times M \to \mathbb{R}^n$ a T-periodic continuous tangent vector field on M. Consider the parametrized equation

$$\dot{\zeta} = \lambda\,\psi(t, \zeta), \qquad \lambda \geq 0. \tag{1.8}$$

By $C_T(M)$ we denote the metric subspace of the Banach space $C_T(\mathbb{R}^n)$ of all the continuous T-periodic functions with values in M. We say that $(\lambda, \zeta) \in [0, \infty) \times$

$C_T(M)$ is a *solution pair of* (1.8) if ζ is a T-periodic solution of (1.8) corresponding to λ. Given any $p \in M$, it is convenient to denote by $\hat{p} \in C_T(M)$ the map which is constantly equal to p. A solution pair of the form $(0, \hat{p})$ is called *trivial*. Given an open subset $\mathcal{O}$ of $[0,\infty) \times C_T(M)$, we will denote by $V_{\mathcal{O}}$ the following open subset of M:

$$V_{\mathcal{O}} = \{p \in M : (0, \hat{p}) \in \mathcal{O}\}.$$

The following result is an immediate consequence of Theorem 2.2 of [1].

Theorem 1.3. *Let $M \subseteq \mathbb{R}^n$ be a boundaryless smooth manifold, $\psi : \mathbb{R} \times M \to \mathbb{R}^n$ a T-periodic continuous tangent vector field on M, and $\phi : M \to \mathbb{R}^n$ the mean value autonomous vector field given by*

$$\phi(p) = \frac{1}{T} \int_0^T \psi(t,p)dt.$$

Let $\mathcal{O}$ be an open subset of $[0,\infty) \times C_T(M)$, and assume that $\deg(\phi, V_{\mathcal{O}})$ is defined and nonzero. Then, the equation (1.8) *admits in $\mathcal{O}$ a connected set Σ of nontrivial solution pairs whose closure in $\mathcal{O}$ is not compact and meets the set $\big\{(0,\hat{p}) \in \mathcal{O} : \phi(p) = 0\big\}$.*

2. MAIN RESULTS

This section is devoted to the study of the set of T-periodic solutions of equation (1.1). Recall that $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is open and connected, $f : \mathbb{R} \times U \to \mathbb{R}^k$ and $g : U \to \mathbb{R}^s$ are given, and we assume that f is continuous and T-periodic in the first variable. We also assume that g is C^∞ and such that $\det \partial_2 g(p,q) \neq 0$ for all $(p,q) \in U$. Define $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ by

$$F(p,q) = \left(\frac{1}{T} \int_0^T f(t,p,q)dt \, , g(p,q) \right). \tag{2.1}$$

Let $M = g^{-1}(0)$. Define the tangent vector field Φ on M by

$$\begin{aligned} \Phi(p,q) =& \frac{1}{T} \int_0^T \Psi(t,p,q)dt \\ =& \left(\frac{1}{T} \int_0^T f(t,p,q)dt, -\frac{1}{T}[\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) \int_0^T f(t,p,q)dt \right). \end{aligned} \tag{2.2}$$

Theorem 2.1 below gives a simple formula for the computation of the degree of Φ that does not require the explicit expression of the manifold M.

We will omit the proof since it is an immediate consequence of Theorem 4.1 of [7].

Theorem 2.1. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $g : U \to \mathbb{R}^s$ and $f : \mathbb{R} \times U \to \mathbb{R}^k$ be such that f is continuous and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$. Let $M = g^{-1}(0)$. Let also $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Phi : M \to \mathbb{R}^k \times \mathbb{R}^s$ be* *given by* (2.1) *and* (2.2)*, respectively. Given* $W \subseteq U$ *open, if either* $\deg(\Phi, M \cap W)$ *or* $\deg(F, W)$ *is well defined, so is the other, and*

$$\big|\deg(\Phi, M \cap W)\big| = \big|\deg(F, W)\big|.$$

As mentioned in the introduction, the main result of this section, Theorem 2.2 below, follows from a combination of Theorems 1.3 and 2.1 above.

Let us introduce some further notation. By $C_T(U)$ we mean the metric subspace of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions with values in U. We say that $(\lambda; x, y) \in [0, \infty) \times C_T(U)$ is a *solution pair of* (1.1) if (x, y) is a T-periodic solution of (1.1) which corresponds to λ. It is convenient, given any $(p, q) \in \mathbb{R}^k \times \mathbb{R}^s$, to denote by $(\hat{p}, \hat{q})$ the element of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ that is constantly equal to (p, q). A solution pair of the form $(0; \hat{p}, \hat{q})$ is called *trivial*. Observe that $(\hat{p}, \hat{q}) \in C_T(U)$ is a constant solution of (1.1) corresponding to $\lambda = 0$ if and only if $g(p, q) = 0$.

Given an open subset Ω of $[0, \infty) \times C_T(U)$, we will denote by W_Ω the open subset of U given by

$$W_\Omega = \{(p, q) \in U : (0; \hat{p}, \hat{q}) \in \Omega\}.$$

We are now ready to state and prove our main result concerning the T-periodic solutions of (1.1).

Theorem 2.2. *Let* $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ *be open and connected, and let* $f : \mathbb{R} \times U \to \mathbb{R}^k$ *be continuous and* T*-periodic in the first variable. Assume that* $g : U \to \mathbb{R}^s$ *is* C^∞ *with* $\partial_2 g(p, q)$ *invertible for all* $(p, q) \in U$*. Define* $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ *by* (2.1)*. Given an open set* $\Omega \subseteq [0, \infty) \times C_T(U)$*, suppose that* $\deg(F, W_\Omega)$ *is well-defined and nonzero. Then, the equation* (1.1) *admits in* Ω *a connected set* Γ *of nontrivial solution pairs whose closure in* Ω *is not compact and meets the set*

$$\{(0; \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}.$$

Proof. Define the T-periodic tangent vector field Ψ on M as in (1.4). Then, as we already pointed out, equation (1.1) is equivalent to (1.5) on $M = g^{-1}(0)$. Moreover, define Φ as in (2.2). Since, by assumption, $\deg(F, W_\Omega)$ is well defined and nonzero, from Theorem 2.1 above we get

$$|\deg(\Phi, M \cap W_\Omega)| = |\deg(F, W_\Omega)| \neq 0.$$

Denote by $\mathcal{O}$ the open subset of $[0, \infty) \times C_T(M)$ given by

$$\mathcal{O} = \Omega \cap ([0, \infty) \times C_T(M)).$$

Recalling the notation introduced in Subsection 1.2, we have $V_{\mathcal{O}} = M \cap W_\Omega$. Therefore, $\deg(\Phi, V_{\mathcal{O}}) = \deg(\Phi, M \cap W_\Omega) \neq 0$. Hence, Theorem 1.3 implies the existence of a connected subset Σ of $\mathcal{O}$ of nontrivial solution pairs of (1.5), whose closure in $\mathcal{O}$ is noncompact and meets the set

$$\big\{(0;\hat{p},\hat{q}) \in \mathcal{O} : \Phi(p,q) = (0,0)\big\}.$$

Now, since $[0,\infty) \times C_T(M)$ is contained in $[0,\infty) \times C_T(U)$, the equivalence of equations (1.1) and (1.5) on M implies that each pair $(\lambda; x, y) \in \Sigma$ can be also thought as a nontrivial solution pair of (1.1). Moreover, since M is closed in U, it is not difficult to prove that $[0,\infty) \times C_T(M)$ is closed in $[0,\infty) \times C_T(U)$. Consequently, the closure of Σ in Ω coincides with the closure of Σ in $\mathcal{O}$. Thus, it is not compact. Finally, observe that $\big\{(p,q) \in M : \Phi(p,q) = (0,0)\big\}$ coincides with $\{(p,q) \in U : F(p,q) = (0,0)\}$. Therefore, the set $\big\{(0;\hat{p},\hat{q}) \in \mathcal{O} : \Phi(p,q) = (0,0)\big\}$ coincides with $\{(0;\hat{p},\hat{q}) \in \Omega : F(p,q) = (0,0)\}$. Hence, the assertion holds with $\Gamma = \Sigma$. □

Remark 2.3. Let $\Gamma \subseteq \Omega$ be a connected set as in the statement of Theorem 2.2. Then, clearly, the assertion "the closure of Γ in Ω is not compact" is equivalent to the following: "the closure $\overline{\Gamma}$ of Γ in the metric space $[0,\infty) \times C_T(U)$ is not contained in any compact subset of Ω".

As an immediate consequence of our main result we get the following *Continuation Principle*, namely, Corollary 2.4 below. We point out that in Corollary 2.4 the manifold $M = g^{-1}(0)$ is assumed to be a closed subset of $\mathbb{R}^k \times \mathbb{R}^s$ so that $C_T(M)$ is a complete metric space.

Corollary 2.4. *Let U, f, g, F and Ω be as in Theorem 2.2. Assume moreover that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\deg(F, W_\Omega)$ be nonzero. Then there exists a connected component of the set of solution pairs of (1.1) that meets $\{(0;\hat{p},\hat{q}) \in \Omega : F(p,q) = (0,0)\}$ and cannot be both bounded and contained in Ω.*

If, in particular, Ω is of the form $[0,\infty) \times A$, with $A \subseteq C_T(U)$ open and bounded, and such that there are no T-periodic solutions of (1.1) on the boundary $\operatorname{Fr} A$ of A for $\lambda \in [0,1]$, then equation

$$\begin{cases} \dot{x} = f(t,x,y), \\ g(x,y) = 0. \end{cases} \tag{2.3}$$

admits a T-periodic solution in A.

Proof. As pointed out in Remark 2.3, from Theorem 2.2 we get the existence of a connected set $\Gamma \subseteq \Omega$ of nontrivial solution pairs of (1.1) whose closure, $\overline{\Gamma}$, in the space $[0,\infty) \times C_T(U)$ is not contained in any compact subset of Ω and meets the set $\{(0;\hat{p},\hat{q}) \in \Omega : F(p,q) = (0,0)\}$

Denote by Z the connected component of the set of all solution pairs that contains $\overline{\Gamma}$. Since, by assumption, M is a closed subset of $\mathbb{R}^k \times \mathbb{R}^s$, the metric space $[0,\infty) \times C_T(M)$ is complete. Moreover, the Ascoli-Arzelà Theorem implies that any bounded set of T-periodic solutions of (1.5) is totally bounded. Thus, if Z is bounded, then it is also compact. If, in addition, Z is contained in Ω then so is $\overline{\Gamma} \subseteq Z$, which is impossible. This contradiction proves that Z cannot be both bounded and contained in Ω.

To prove the last part of the assertion, let $A \subseteq C_T(U)$ be open and bounded and let $\Omega = [0,\infty) \times A$. Consider the subset $[0,1] \times A$ of Ω. Since Z is connected, contains some nontrivial pair (as $\Gamma \subseteq Z$) and cannot be contained in $[0,1] \times A$, then Z necessarily meets the boundary of $(0,1] \times A$. Since there are no solution pairs of (1.2) in $[0,1] \times \operatorname{Fr} A$, the set Z intersects $\{1\} \times A$. This completes the proof. $\square$

Remark 2.5. One can deduce an existence result from Corollary 2.4 as follows. Let V be a relatively compact open subset of U, and assume that the following properties hold:

- the set $F^{-1}(0,0) \cap V$ is compact and $\deg(F,V) \neq 0$;
- there are no T-periodic solutions of (1.1) whose image intersects the boundary ∂V of V for $\lambda \in (0,1]$.

In this situation, taking $A = C_T(V) \subseteq C_T(U)$ and $\Omega = [0,\infty) \times A$, we have $W_\Omega = V$, and thus $\deg(F, W_\Omega) = \deg(F,V) \neq 0$. Hence, Corollary 2.4 yields a T-periodic solution of (2.3).

The following examples illustrate our results.

Example 2.6. Let $k = s = 1$ and let $U = \mathbb{R}^2$. Consider the following parametrized DAE:

$$\dot{x} = \lambda(x - y + \sin t), \qquad y^3 + y - x^2 = 0. \tag{2.4}$$

Clearly, $M = \{(p,q) \in \mathbb{R}^2 : q^3 + q - p^2 = 0\}$ is a C^∞ closed submanifold of $\mathbb{R}^2$. With the same notation as above, we have $f(t,x,y) = x - y + \sin t$, and

$$F(p,q) = (p - q, q^3 + q - p^2),$$

hence $F^{-1}(0,0) = \{(0,0)\}$ and $\deg(F,\mathbb{R}^2) = 1$, so that our results apply to (2.4). In particular, in view of Remark 2.5, we seek for a bounded open subset V of $\mathbb{R}^2$ containing $(0,0)$ such that $\deg(F,V) = 1$ and there are no T-periodic solutions of (2.4) whose image intersects the boundary ∂V of V for $\lambda \in (0,1]$. In this case we get that equation

$$\dot{x} = x - y + \sin t, \qquad y^3 + y - x^2 = 0 \tag{2.5}$$

admits a 2π-periodic solution. For this purpose, take $V = \{(x,y) \in \mathbb{R}^2 : |y - x| < 2, |y| < 3\}$. Then, clearly, $\deg(F,V) = 1$. Observe that $\partial V \cap M$ consists of two points, one in each segment $\{(x,y) \in \mathbb{R}^2 : y - x = -2, |y| < 3\}$ and $\{(x,y) \in \mathbb{R}^2 : y - x = 2, |y| < 3\}$. Moreover, we have $f(t,x,y) < 0$ for $x - y \leq -2$ and $f(t,x,y) > 0$ if $x - y \geq 2$. Thus for any $\lambda \in (0,1]$ no 2π-periodic solution of (2.4) intersects the boundary ∂V of V. Therefore, Corollary 2.4 applies yielding a 2π-periodic solution of (2.5).

Example 2.7. Let $k = 1$, $s = 2$ and let $U = \mathbb{R}^3$. Consider the DAE:

$$\dot{x} = \lambda(y_2 + \sin t), \qquad g(x, y) = 0, \tag{2.6}$$

where $y = (y_1, y_2)$ and $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$g(x; y_1, y_2) = \big(e^{y_1} \cos y_2 - x, e^{y_1} \sin y_2 + x - 1\big).$$

Clearly $\partial_2 g(x, y)$ is nonsingular for any $(x, y) \in \mathbb{R} \times \mathbb{R}^2$. Indeed

$$\det \partial_2 g(x, y) = \det \begin{pmatrix} e^{y_1} \cos y_2 & -e^{y_1} \sin y_2 \\ e^{y_1} \sin y_2 & e^{y_1} \cos y_2 \end{pmatrix} = e^{2y_1} > 0.$$

We have

$$F(x; y_1, y_2) = \big(y_2, e^{y_1} \cos(y_2) - x, e^{y_1} \sin(y_2) + x - 1\big).$$

Thus $F^{-1}(0, 0, 0) = \{(1, 0, 0)\}$ and a simple computation shows that $\deg(F, \mathbb{R}^3) = 1$, so Theorem 2.2 and Corollary 2.4 apply with $\Omega = [0, \infty) \times C_T(U)$, yielding the existence of an unbounded branch of nontrivial solution pairs of (2.6).

ACKNOWLEDGMENTS

The author thanks Marco Spadini for useful suggestions and discussions.

REFERENCES

[1] M. Furi and M.P. Pera, Carathéodory periodic perturbations of the zero vector field on manifolds, *Topological Meth. in Nonlin. Anal.* **10** (1997), 79–92.

[2] M. Furi, M.P. Pera and M. Spadini, The fixed point index of the Poincaré operator on differentiable manifolds, in: *Handbook of topological fixed point theory*, (Ed: R.F. Brown, M. Furi, L. Górniewicz, B. Jiang), Springer, 2005.

[3] V. Guillemin and A. Pollack, *Differential Topology,* Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[4] M.W. Hirsch, *Differential Topology*, Graduate Texts in Math. Vol. 33, Springer Verlag, Berlin, 1976.

[5] P. Kunkel and V. Mehrmann, *Differential Algebraic Equations, analysis and numerical solutions*, EMS textbooks in Math., European Mathematical Society, Zürich 2006.

[6] J.W. Milnor, *Topology from the differentiable viewpoint*, Univ. press of Virginia, Charlottesville, 1965.

[7] M. Spadini, A note on topological methods for a class of Differential-Algebraic Equations, *Nonlinear Anal.* **73** (2010), 1065–1076.