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# A NEW THEME IN NONLINEAR ANALYSIS: CONTINUATION AND BIFURCATION OF THE UNIT EIGENVECTORS OF A PERTURBED LINEAR OPERATOR

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**ABSTRACT.** We review some recent results concerning nonlinear eigenvalue problems of the form $(*)\, Au + \epsilon B(u) = \delta u$, where $A$ is a linear Fredholm operator of index zero (with nontrivial kernel $\operatorname{Ker} A$) acting in a real Banach space $X$, and $B : X \to X$ is a (possibly) nonlinear perturbation term. We seek solutions $u$ of $(*)$ in the unit sphere $S$ of $X$, and the emphasis is put on the existence - under appropriate conditions on $B$ - of points $u_0 \in S \cap \operatorname{Ker} A$ (thus satisfying $(*)$ for $\epsilon = \delta = 0$) which either can be continued as solutions of $(*)$ for $\epsilon \neq 0$ or - more generally - are bifurcation points for solutions of that kind.

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $T$ be a bounded linear operator acting in a real Banach space $X$ and let $S$ be the unit sphere in $X$. Suppose that $u_0$ is a *unit eigenvector* of $T$, that is $u_0 \in S$ and $Tu_0 = \lambda_0 u_0$ for some $\lambda_0 \in \mathbb{R}$; we say in this case that $u_0$ is a *unit $\lambda_0$-eigenvector* of $T$. Also let $B : U \to X$ be a (possibly nonlinear) continuous operator defined on a neighborhood $U$ of $S$ and for $\epsilon$ small consider the perturbed "eigenvalue" problem

$$Tu + \epsilon B(u) = \lambda u, \qquad u \in S. \tag{1.1}$$

**Definition 1.1.** Let $u_0$ be a unit $\lambda_0$-eigenvector of $T$. We say that $u_0$ is continuable as a unit eigenvector of $T + \epsilon B$ ($\epsilon \neq 0$) *if there exists a continuous function* $\epsilon \mapsto (\lambda(\epsilon), u(\epsilon))$ *of an interval* $(-\epsilon_0, \epsilon_0)$ *into* $\mathbb{R} \times S$ *such that* $Tu(\epsilon) + \epsilon B(u(\epsilon)) = \lambda(\epsilon)u(\epsilon)$ *for* $|\epsilon| < \epsilon_0$ *and* $(\lambda(0), u(0)) = (\lambda_0, u_0)$.

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For example, $u_0$ is continuable if it is an "eigenvector" of $B$ too: for if $B(u_0) = \mu u_0$ for some $\mu \in \mathbb{R}$, then putting $(\lambda(\epsilon), u(\epsilon)) = (\lambda_0 + \epsilon\mu, u_0)$ for $\epsilon \in \mathbb{R}$ yields the required continuous family. On the other hand, putting $X = \mathbb{R}^2$, $T$ the zero operator, $B(x, y) = (-y, x)$ for $(x, y) \in \mathbb{R}^2$, we see that no unit 0-eigenvector of $T$ is continuable, for the perturbed linear operator $T + \epsilon B$ has no (real) eigenvalue for $\epsilon \neq 0$.

Assuming that $\lambda_0$ be an isolated eigenvalue of finite (geometric and algebraic) multiplicity, we have discussed in [2] and [4] conditions for the continuability of a unit $\lambda_0$-eigenvector of $T$. In particular, in [2] (see also [3]) it was essentially shown that when $\lambda_0$ is a simple eigenvalue - that is, $\lambda_0$ has geometric and algebraic multiplicity equal to one - then if $B$ is Lipschitz continuous, each of the two unit $\lambda_0$-eigenvectors is continuable (in a Lipschitz continuous fashion): see Theorem 2 and Remark 2.1 of [2]. While in [4], we have considered the case in which $\lambda_0$ has geometric multiplicity greater than one, and have given - for $B$ of class $C^2$ - necessary as well as sufficient conditions for continuability of a given unit eigenvector in the $C^1$ sense: see Theorem 2.2 and Remark 3.6 of [4].

To obtain further information about the solutions of (1.1) it is useful to introduce a second concept, which relaxes the requirements in Definition 1.1.

**Definition 1.2.** Let $u_0$ be a unit $\lambda_0$-eigenvector of $T$. We say that $u_0$ is a bifurcation point for the unit eigenvectors of $T + \epsilon B$ $(\epsilon \neq 0)$ - *or simply a* bifurcation point for (1.1) - *if any neighborhood of* $(0, \lambda_0, u_0)$ *in* $\mathbb{R} \times \mathbb{R} \times X$ *contains a solution* $(\epsilon, \lambda, u)$ *of* (1.1) *with* $\epsilon \neq 0$.

Definition 1.2 expresses the property for a unit eigenvector of $T$ of being *persistent* under sufficiently small perturbations of $T$, and can be equivalently formulated as follows: *there exists a sequence* $\{(\epsilon_n, \lambda_n, u_n)\}$ *in* $\mathbb{R} \setminus \{0\} \times \mathbb{R} \times S$ *which converges to* $(0, \lambda_0, u_0)$ *and such that* $Tu_n + \epsilon_n B(u_n) = \lambda_n u_n$, $\forall n \in \mathbb{N}$. To appreciate better this Definition, it is useful to adopt as in [4] the general point of view in bifurcation theory introduced in [9]. A solution of (1.1) is a point $p = (\epsilon, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times X$ such that $F(p) = 0$, where $F$ is the map of $\mathbb{R} \times \mathbb{R} \times X$ into $X \times \mathbb{R}$ defined via

$$F(\epsilon, \lambda, u) = (Tu + \epsilon B(u) - \lambda u, \|u\|^2 - 1) \tag{1.2}$$

($\|.\|$ is the norm in $X$). Put

$$S_0 \equiv S \cap \mathrm{Ker}(T - \lambda_0 I) \tag{1.3}$$

where $I$ denotes the identity in $X$, and consider the subset

$$M \equiv \{0\} \times \{\lambda_0\} \times S_0 \tag{1.4}$$

of $\mathbb{R} \times \mathbb{R} \times X$ as the set of *trivial solutions* of (1.1), or the trivial zeroes of $F$. Assuming that $\lambda_0$ be an isolated eigenvalue, and considering solutions of (1.1) with $\lambda$ near $\lambda_0$, we see that $M$ is precisely the set of triples $(\epsilon, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times X$ solving (1.1) for $\epsilon = 0$. Solutions $(\epsilon, \lambda, u)$ with $\epsilon \neq 0$ are therefore the *nontrivial* solutions of (1.1), and

Definition 1.2 expresses - identifying $u_0$ with $p_0 \equiv (0, \lambda_0, u_0)$ and using the terminology of [9] - that $p_0 \in M$ is a *bifurcation point* (from $M$) for the equation $F(p) = 0$.

Very recently, we have proved the existence of at least one bifurcation point for the unit eigenvectors of $T + \epsilon B$ under the assumptions that $T$ is a self-adjoint operator in a Hilbert space (in which case the algebraic and geometric multiplicity of $\lambda_0$ coincide, and therefore we merely speak of *multiplicity*), that $B$ is of class $C^1$ and that one of the following conditions is satisfied:

- the multiplicity of $\lambda_0$ is *odd*;
- $B$ is a *gradient* operator.

Our aim in the present paper is to explain these results - proved in [5] and [6] respectively - also in connection with the older ones [4], and in particular to make available the main idea followed in the (yet unpublished) paper [6] to deal with the variational case.

We first set our problem in the context of perturbations of (linear) Fredholm operators of index zero: this turns out to be a sufficiently general framework in order to state our results on a common ground, compare their strength and appreciate the different assumptions. We also indicate the main points of the proofs. This is done in Section 2, while Section 3 is addressed to exhibit some simple examples of our problem in the Euclidean space $\mathbb{R}^3$. Working in this context - and even with a linear $B$ - gives some concrete evidence of the conditions involved on $T$ and $B$, and may thus help for a better understanding of the ideas previously expressed in infinite-dimensional Banach spaces.

## 2. FINITE-DIMENSIONAL REDUCTION. NECESSARY CONDITIONS AND SUFFICIENT CONDITIONS FOR BIFURCATION

Consider equation (1.1) for a bounded linear operator $T : X \to X$, $X$ a real Banach space. We suppose in the sequel that:

- $\lambda_0$ is an isolated eigenvalue of $T$.

As already said, this ensures that for $\epsilon = 0$ and $\lambda$ near $\lambda_0$, the only solutions of (1.1) are those with $\lambda = \lambda_0$, that is the trivial ones. Now set

$$A = T - \lambda_0 I, \qquad \delta = \lambda - \lambda_0$$

and write the equation in (1.1) as

$$Au + \epsilon B(u) = \delta u. \tag{2.1}$$

We assume the following hypotheses upon $A$.

HA1) $A$ is a Fredholm operator of index zero, that is,

- $\operatorname{Ker} A = \{u \in X : Au = 0\}$ is of finite dimension; in words, $\lambda_0$ is an eigenvalue of finite geometric multiplicity;

- $\mathrm{Im}\,A = \{Au : u \in X\}$ is closed and of finite codimension;
- $\dim \mathrm{Ker}\,A = \mathrm{codim}\,\mathrm{Im}\,A$.

HA2) $\mathrm{Ker}\,A \cap \mathrm{Im}\,A = \{0\}$.

It follows from HA1) and HA2) that

$$E = \mathrm{Ker}\,A \oplus \mathrm{Im}\,A \tag{2.2}$$

and that the corresponding projections $P$ and $Q = I - P$ onto $\mathrm{Ker}\,A$ and $\mathrm{Im}\,A$ are bounded.

It is useful to recall two typical situations in which the above assumptions are satisfied:

- $T : X \to X$ is compact, $\lambda_0 \neq 0$ (ensuring HA1)) and $\mathrm{Ker}\,A = \mathrm{Ker}\,A^2$ (ensuring HA2). The last condition also implies that $\mathrm{Ker}\,A^n = \mathrm{Ker}\,A^{n+1}$ for all $n \in \mathbb{N}$, and therefore that the geometric multiplicity of $\lambda_0$ equals its *algebraic* multiplicity $\dim \bigcup_{n=1}^{\infty} \mathrm{Ker}\,A^n$.
- $X = H$, a Hilbert space, $T : H \to H$ is self-adjoint (that is, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, $\langle .,.\rangle$ denoting the scalar product in $H$) and $\dim \mathrm{Ker}\,A < \infty$. Indeed self-adjointness of $T$ implies that $\mathrm{Ker}\,A = \mathrm{Im}\,A^{\perp} \equiv \{x \in H : \langle x, y\rangle = 0,\ \forall y \in \mathrm{Im}\,A\}$, and it follows that $H = \mathrm{Ker}\,A \oplus \overline{\mathrm{Im}\,A}$, where the sum is orthogonal. However as $\lambda_0$ is isolated by assumption, $\mathrm{Im}\,A$ is closed (see e.g. [8, pg. 1395]) and therefore $H = \mathrm{Ker}\,A \oplus \mathrm{Im}\,A$. Self-adjointness also implies that $\mathrm{Ker}\,A = \mathrm{Ker}\,A^2$, so that the geometric and algebraic multiplicity of $\lambda_0$ always coincide in this case.

Writing $u = Pu + Qu \equiv v + w$ according to (2.2) and applying in turn $P, Q$ to both members of (2.1), we see that the latter equation is equivalent to the following two:

$$\epsilon PB(v + w) = \delta v \tag{2.3}$$

$$Aw + \epsilon QB(v + w) = \delta w. \tag{2.4}$$

This decomposition (the so-called *Lyapounov-Schmidt method*) reveals easily a necessary condition for bifurcation, provided that $B$ satisfies the following "minimal" regularity assumption:

HB0) $B$ is continuous in a neighborhood of $S$.

**Proposition 2.1.** *Suppose that* HA1), HA2) *and* HB0) *are satisfied. If* $v_0 \in S_0 = S \cap \mathrm{Ker}(T - \lambda_0 I)$ *is a bifurcation point for (1.1), then there exists* $\mu_0 \in \mathbb{R}$ *such that*

$$PB(v_0) = \mu_0 v_0. \tag{2.5}$$

**Proof**. If $v_0 \in S_0$ is a bifurcation point, there exists by definition a sequence $(\delta_n, \epsilon_n, u_n) \in \mathbb{R} \times \mathbb{R} \times S$, with $\epsilon_n \neq 0$ for each $n \in \mathbb{N}$, such that $(\delta_n, \epsilon_n, u_n) \to (0, 0, v_0)$ as $n \to \infty$ and

$$Au_n + \epsilon_n B(u_n) = \delta_n u_n, \qquad \forall n \in \mathbb{N}. \tag{2.6}$$

Then putting $v_n = Pu_n, w_n = Qu_n$ we have $v_n \to Pv_0 = v_0$, $w_n \to Qv_0 = 0$ and moreover from (2.3)

$$PB(v_n + w_n) = \frac{\delta_n}{\epsilon_n} v_n.$$

We claim that the sequence $(\delta_n/\epsilon_n)$ is bounded. For otherwise, since $\|v_n\| \to \|v_0\| = 1$, it would follow (passing if necessary to a subsequence) that $\|\frac{\delta_n}{\epsilon_n} v_n\| \to +\infty$, contradicting the boundedness of the sequence $PB(v_n + w_n)$ which in fact converges to $PB(v_0)$. Hence we can assume (again through a subsequence) that $(\delta_n/\epsilon_n)$ converges to some $\mu_0$, so that in the limit we obtain (2.5).

**Remark 2.2.** For $B$ of class $C^1$, the above condition was proved in [4].

It is not difficult to show (see, for instance, Example 3.5) that this necessary condition is not sufficient for bifurcation. In order to discuss sufficient conditions, we shall henceforth strengthen HB0) as follows:

HB1) $B$ is of class $C^1$ in a neighborhood of $S$.

Indeed put

$$N = \operatorname{Ker} A, \qquad W = \operatorname{Im} A$$

and identify $X$ with $N \times W$. Then HB1) guarantees, via the Implicit Function Theorem, that given any $v_0 \in S_0 \subset N$, equation (2.4) - the so-called *complementary equation* - can be solved uniquely w.r.t. $w$ for each given $(\delta, \epsilon, v)$ in a neighborhood $U_0 \subset \mathbb{R} \times \mathbb{R} \times N$ of $(0, 0, v_0)$. Moreover if $w(\delta, \epsilon, v)$ denotes the solution corresponding to $(\delta, \epsilon, v) \in U_0$, then $w(0, 0, v) = 0$ for any $v$ and the mapping $(\delta, \epsilon, v) \to w(\delta, \epsilon, v)$ of $U_0$ into $W$ is of class $C^1$ in $U_0$. Therefore by definition

$$Aw(\delta, \epsilon, v) + \epsilon QB(v + w(\delta, \epsilon, v)) = \delta w(\delta, \epsilon, v) \tag{2.7}$$

for any $(\delta, \epsilon, v) \in U_0$; and we see from (2.3) that in order to solve our problem (1.1), it is enough to find $(\delta, \epsilon, v) \in U_0$ satisfying the finite-dimensional equation (the *bifurcation equation*)

$$\epsilon PB(v + w(\delta, \epsilon, v)) = \delta v \tag{2.8}$$

and the additional normalization constraint

$$v + w(\delta, \epsilon, v) \in S. \tag{2.9}$$

At this stage, in order to prove that a given $v_0 \in S_0$ - satisfying (2.5) - is indeed a bifurcation point, we need find a sequence $(\delta_n, \epsilon_n, v_n)$ of solutions of the above system (2.8)–(2.9), with $\epsilon_n \neq 0$ for each $n \in \mathbb{N}$, such that $(\delta_n, \epsilon_n, v_n) \to (0, 0, v_0)$ as $n \to \infty$. While if for each sufficiently small $\epsilon$ we find $\delta(\epsilon), v(\epsilon)$ - depending continuously upon $\epsilon$ - such that $(\delta(0), v(0)) = (0, v_0)$ and $(\delta(\epsilon), \epsilon, v(\epsilon))$ solves (2.8)–(2.9), then so much the better as $v_0$ will be continuable by means of the equation

$$u(\epsilon) = v(\epsilon) + w(\delta(\epsilon), \epsilon, v(\epsilon)). \tag{2.10}$$

When $B$ and the space $X$ (that is, its norm) are sufficiently smooth, the Implicit Function Theorem can be further employed to perform such construction and yield a sufficient condition for continuation.

**Theorem 2.3.** *For $x \in X$, put $g(x) = \|x\|^2 - 1$. Suppose that $B$ and $g$ are* of class *$C^2$ in an open neighborhood of $S = g^{-1}(0)$ and that* HA1) *and* HA2) *are satisfied. Let $v_0 \in S_0$ be such that $PB(v_0) = \mu_0 v_0$, put $V = \{h \in X : g'(v_0)h = 0\}$ and let $\pi$ be a linear projection of $N$ onto $N \cap V$ such that $\pi(v_0) = 0$. If $v_0$ satisfies the condition*

$$h \in N \cap V, \qquad \pi PB'(v_0)h = \mu_0 h \Rightarrow h = 0, \tag{2.11}$$

*then $v_0$ is continuable.*

**Remark 2.4.** One has $g'(v)v = 2 \neq 0$ for all $v \in g^{-1}(0)$. This implies in particular that $0 \in \mathbb{R}$ is a regular value for the restriction of $g$ to $N$ and, consequently, $N \cap V$ is the tangent space to $S_0 = N \cap S$ at $v_0$. The condition (2.11) means that the operator $\pi PB'(v_0) - \mu_0 I$, restricted to $N \cap V$, is an isomorphism of $N \cap V$ onto iself. Thus, in the special and interesting case in which $B$ is linear, this condition indicates that $\mu_0$ is a simple eigenvalue of $PB$, regarded as an operator of $N$ into itself. In fact, in this linear case, (2.11) implies that, in the space $N$, $\mathrm{Ker}(PB - \mu_0 I)$ is one dimensional and $N = \mathrm{Ker}(PB - \mu_0 I) \oplus \mathrm{Im}(PB - \mu_0 I)$.

Theorem 2.1 ia a special case of Theorem 3.4 in [4], where it is shown that similar results hold when the operators involved act between different Banach spaces, and when the unit sphere $S$ is replaced by more general manifolds $M = g^{-1}(0)$ given as level sets of a $C^2$ functional $g$. In turn, Theorem 3.4 of [4] is an application to Banach space operator equations of results formulated in [9] in the context of general bifurcation theory. This considers a $C^1$ map $f$ defined in an open set $U$ of a Banach space $E$ and with values in a Banach space $F$. Given a differentiable manifold $M \subseteq f^{-1}(0)$, regard $M$ as the set of *trivial solutions* of the equation $f(u) = 0$, so that $f^{-1}(0) \setminus M$ represents the set of nontrivial solutions. An element $p \in M$ is a *bifurcation point (from $M$) of $f(u) = 0$* if any neighborhood of $p$ contains elements of $f^{-1}(0) \setminus M$. Necessary as well as sufficient conditions for bifurcation are proved in [9] in essentially geometrical terms, starting from the observation that the condition $M \subseteq f^{-1}(0)$ implies that, for any $u \in M$, the tangent space $T_u M$ of $M$ at $u$ is contained in the kernel of $f'(u)$.

In particular when $f$ is a $C^2$ Fredholm map of index 1, and $p \in M$ is such that $\dim \mathrm{Ker}\, f'(p) = \dim T_p M + 1$, then a sufficient "transversality" condition for $p \in M$ to be a bifurcation point is provided in [9], which extends that contained in the Crandall–Rabinowitz Bifurcation Theorem [7], in which $\dim M = 1$. For these general conditions see, for instance, Theorem 2.2 of [4] and the comments accompanying it.

Moreover in [4], the results about (1.1) are applied to show the existence of $2\pi$-periodic solutions of the differential equation

$$x'' + x + \epsilon(tx + x^2) = \lambda x$$

normalized by

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2(t)\, dt = 1,$$

and in particular to study the continuability of a given *trivial* (i.e., obtained for $\epsilon = \lambda = 0$) normalized solution: that is, of a solution of the type $x(t) = c\sin t + d\cos t$, with $c^2 + d^2 = 1$.

Proposition 2.1 and Theorem 2.1 are results of *local* nature, as they give conditions upon an individual point $v_0 \in S_0$ to be a bifurcation point for (1.1). A related question is: under which conditions (on $A, B$, etc.) does $S_0$ possess at least one bifurcation point? We are able to give some partial answer to this problem in the special case that $X = H$, a Hilbert space, and $T : H \to H$ is self-adjoint.

Recall that in this case the assumptions HA1) and HA2) about the linear part $A = T - \lambda_0 I$ of our equation are satisfied - provided of course that $\lambda_0$ is isolated and of finite multiplicity, as we have always assumed. Here is our first result [5]:

**Theorem 2.5.** *Consider the problem* (1.1) *where* $T$ *is a bounded self-adjoint operator acting in a real Hilbert space and* $B$ *satisfies the assumption* HB1). *If* $\lambda_0$ *is an isolated eigenvalue of* $T$ *of* odd multiplicity, *then* $S_0 = S \cap \mathrm{Ker}(T - \lambda_0 I)$ *possesses at least one bifurcation point.*

**Sketch of the proof.** The proof of Theorem 2.5 relies on the fact that the Euler–Poincaré characteristic of the even dimensional sphere $S_0$ is nonzero, and this implies that any self-map of this sphere has a fixed point if it is homotopic to the identity: for this matter see, for instance, [1] or [10]. Therefore, the methods employed are of topological nature, and quite different from those used in [2] and [4], which rely almost entirely upon the Implicit Function Theorem.

Nevertheless, it is precisely with a strengthened version of this Theorem that we start our work in [5], to the aim of solving the complementary equation *globally* with respect to $S_0$. Indeed for $\eta > 0$, consider the (compact) neighborhood of $S_0$

$$M = \left\{ v \in N : \big| \|v\| - 1 \big| \leq \eta \right\}$$

where we recall that $N$ is $\mathrm{Ker}\, A$ (and $W$ is $\mathrm{Im}\, A$). Taking $\eta > 0$ small, we can assume that $B$ is of class $C^1$ in an open neighborhood of $M \times \{0\} \subset N \times W$, and then it follows from Lemma 2.2 of [5] that the function $w = w(\delta, \epsilon, v)$ obtained solving (2.4) is defined and of class $C^1$ in an open neighborhood $U_1$ of $\{0\} \times \{0\} \times M \subset \mathbb{R} \times \mathbb{R} \times N$.

Once this is done, a further reduction can be made on "eliminating $\delta$" from our equations. Indeed in the present Hilbert space context, taking the scalar product in (2.8) we get

$$\langle \epsilon PB(v + w(\delta, \epsilon, v)), v \rangle = \delta \|v\|^2. \tag{2.12}$$

Dividing both members of (2.12) by $\|v\|^2$ and applying again Lemma 2.2 of [5] to the resulting equation, we see that $\delta$ can be written as a $C^1$ function $\delta(\epsilon, v)$ of $(\epsilon, v)$, defined on an open subset $V$ of $\mathbb{R} \times (N \setminus \{0\})$ containing $\{0\} \times M$ and such that $\delta(0, v) = 0$ for any $v$, and $(\delta(\epsilon, v), \epsilon, v) \in U_1$ for $(\epsilon, v) \in V$.

For convenience put $\phi(\epsilon, v) \equiv w(\delta(\epsilon, v), \epsilon, v)$. Then we see - from (2.8) and the normalization condition (2.9) - that in order to solve (1.1) it is enough to find

$(\epsilon, v) \in V$ such that

$$\epsilon PB(v + \phi(\epsilon, v)) = \delta(\epsilon, v)v \tag{2.13}$$

and

$$\|v + \phi(\epsilon, v)\|^2 = \|v\|^2 + \|\phi(\epsilon, v)\|^2 = 1. \tag{2.14}$$

Under the assumptions of Theorem 2.2, we show that a stronger result holds: namely, for any sufficiently small $\epsilon$ there exists $v_\epsilon \in M$ such that $(\epsilon, v_\epsilon)$ satisfies (2.13) and (2.14). To this purpose, assume for simplicity that $\lambda_0 = 1$. Then adding $v$ to both sides of (2.13) and putting $h(\epsilon, v) = 1 + \delta(\epsilon, v)$ we get

$$v + \epsilon PB(v + \phi(\epsilon, v)) = h(\epsilon, v)v. \tag{2.15}$$

Fix $\epsilon \neq 0$ and let $\sigma$ be the radial projection of $N \setminus \{0\}$ onto its unit sphere $S_0$, defined putting $\sigma(v) = v/\|v\|$ for $v \in N, v \neq 0$. Then looking for solutions $v \in M$ of (2.15) is equivalent to finding $v \in M$ such that

$$\sigma(v + \epsilon PB(v + \phi(\epsilon, v))) = \frac{v}{\|v\|}. \tag{2.16}$$

On the other hand, using (2.14) this last equation becomes

$$f_\epsilon(v) \equiv \sqrt{1 - \|\phi(\epsilon, v)\|^2}\sigma(v + \epsilon PB(v + \phi(\epsilon, v))) = v, \tag{2.17}$$

which is a fixed point equation for the map $f_\epsilon : M \to M$. The Lefschetz number of $f_\epsilon$ equals the Euler–Poincaré characteristic of $S_0$ [5], and thus is not zero since $S_0$ is even dimensional. By the Lefschetz fixed point theorem [1], there exists $v_\epsilon \in M$ such that $f_\epsilon(v_\epsilon) = v_\epsilon$.

Now fix a sequence $(\epsilon_n)$ with $\epsilon_n \to 0$ and $\epsilon_n \neq 0$ for all $n \in \mathbb{N}$ and put $v_n \equiv v_{\epsilon_n}$; also let

$$\delta_n \equiv \delta(\epsilon_n, v_n), \qquad u_n \equiv v_n + \phi(\epsilon_n, v_n).$$

By the compactness of $M$ we can assume - passing if necessary to a subsequence - that $v_n \to v_0$. It follows that $\phi(\epsilon_n, v_n) \to \phi(0, v_0) = 0$, which implies by (2.14) that $\|v_n\| \to 1$ and therefore that $v_0 \in S$. Moreover since $(\delta_n, \epsilon_n, u_n)$ solves (1.1) for any $n$ and $u_n \to v_0$, it follows that $v_0 \in S_0$ and is a bifurcation point for (1.1).

Let us now come to our most recent results [6], dealing with the case in which (1.1) is a variational problem: we then prove that bifurcation from $S_0$ takes place irrespective of the multiplicity of $\lambda_0$. To be precise, assume that

HBG) $B$ is a gradient operator in neighborhood of $S$ which means that there exists a differentiable functional $b$ defined on a open neighborhood $U$ of $S$ such that

$$\langle B(x), y \rangle = b'(x)y \quad \text{for all} \quad x \in U, y \in H. \tag{2.18}$$

Here $b'(x)$ denotes the (Fréchet) derivative of $b$ at the point $x \in U$.

**Theorem 2.6.** *Suppose that $T : H \to H$ is a bounded self-adjoint operator, and suppose that $B$ satisfies* HB1) *and* HBG). *If $\lambda_0$ is an isolated eigenvalue of $T$ of finite multiplicity, then $S_0$ possesses at least one bifurcation point.*

**Sketch of the proof.** For the proof of this result it is useful to put (in addition to the notations used before)

$$F_\epsilon(u) \equiv Au + \epsilon B(u), \quad \delta_\epsilon(v) \equiv \delta(\epsilon, v), \quad \phi_\epsilon(v) \equiv \phi(\epsilon, v)$$

so that the system (2.13)-(2.14) in the unknowns $\epsilon$ and $v$ can be written

$$PF_\epsilon(v + \phi_\epsilon(v)) = \delta_\epsilon(v)v, \qquad \|v + \phi_\epsilon(v)\|^2 = 1. \tag{2.19}$$

Under the assumptions of Theorem 2.6 we show that for any $\epsilon$ small there exist (at least) *two* distinct solutions $v = v_\epsilon$, $z = z_\epsilon$ of (2.19). To this aim, let $B = \nabla b$ - that is, suppose that (2.18) holds; then $F_\epsilon = \nabla f_\epsilon$ with

$$f_\epsilon(u) = \frac{1}{2}\langle Au, u\rangle + \epsilon b(u).$$

We follow an idea of Stuart [12] to show that for fixed $\epsilon$, the solutions $v$ of (2.19) are precisely the critical points of the functional $\alpha_\epsilon$ defined by

$$\alpha_\epsilon(v) = f_\epsilon(v + \phi_\epsilon(v)) = \frac{1}{2}\langle A\phi_\epsilon(v), \phi_\epsilon(v)\rangle + \epsilon b(v + \phi_\epsilon(v)) \tag{2.20}$$

over the manifold defined by the norm constraint, that is

$$M_\epsilon = \{v \in N : \|v + \phi_\epsilon(v)\|^2 = 1\}. \tag{2.21}$$

Once this is done, the compactness of $M_\epsilon$ implies the existence of $v_\epsilon, z_\epsilon \in M_\epsilon$ such that

$$\alpha_\epsilon(v_\epsilon) = \min_{v \in M_\epsilon} \alpha_\epsilon(v), \qquad \alpha_\epsilon(z_\epsilon) = \max_{v \in M_\epsilon} \alpha_\epsilon(v). \tag{2.22}$$

and therefore implies that (for each $\epsilon$), $v_\epsilon$ and $z_\epsilon$ solve (2.19).

Using for instance $v_\epsilon$ and reasoning as in the proof of Theorem 2.2, we can then construct a sequence $(\delta_n, \epsilon_n, u_n)$ of solutions to (1.1), with $u_n$ converging to some $v_0 \in S_0$ which is therefore a bifurcation point.

**Remark 2.7.** It would be interesting to establish conditions guaranteeing that there are (at least) two different bifurcation points.

## 3. EXAMPLES IN $\mathbb{R}^3$

In this Section we consider (2.1) in the very special case that $X = \mathbb{R}^3$ and that (besides $A$) also the perturbing term $B$ is linear. Moreover we keep fixed a very simple $A$, namely - writing $u = (x, y, z)$ for $u \in \mathbb{R}^3$ - the projection onto the $z$-axis:

$$A(x, y, z) = (0, 0, z).$$

Thus, $A$ can be represented, in the canonical basis, by of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

Consider at first a generic $B$:

$$B = \begin{pmatrix} a & b & m \\ c & d & n \\ p & q & r \end{pmatrix} . \tag{3.2}$$

Then (2.1) is

$$\begin{cases} \epsilon(ax + by + mz) = \delta x, \\ \epsilon(cx + dy + nz) = \delta y, \\ z + \epsilon(px + qy + rz) = \delta z. \end{cases} \tag{3.3}$$

The last equation can be solved for $z$ to yield

$$z = z(\delta, \epsilon, x, y) = \frac{\epsilon}{\delta - (1 + \epsilon r)}(px + qy) \tag{3.4}$$

and we are thus reduced to solve the system (in the unknowns $\delta, \epsilon, x, y$)

$$\begin{cases} ax + by + mz(\delta, \epsilon, x, y) = (\delta/\epsilon)x, \\ cx + dy + nz(\delta, \epsilon, x, y)) = (\delta/\epsilon)y. \end{cases} \tag{3.5}$$

**Example 3.1.** Consider

$$B = \begin{pmatrix} a & b & m \\ c & d & n \\ 0 & 0 & r \end{pmatrix} , \tag{3.6}$$

that is,

$$B(x, y, z) = (ax + by + mz, cx + dy + nz, rz).$$

We see from (3.4) that in this case $z(\delta, \epsilon, x, y) \equiv 0$, so that the bifurcation system reduces to

$$\begin{cases} ax + by = (\delta/\epsilon)x, \\ cx + dy = (\delta/\epsilon)y. \end{cases} \tag{3.7}$$

The solutions $(x, y) \neq (0, 0)$ of this system - if any - are the eigenvectors of the reduced $2 \times 2$ matrix

$$\hat{B} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.8}$$

corresponding to real eigenvalues. Suppose first that $\hat{B}$ has two real eigenvalues $\mu_1, \mu_2$ with $\mu_1 \neq \mu_2$. If $v_1, v_2$ are corresponding normalized eigenvectors, then the bifurcation branches defined putting

$$\delta_i(\epsilon) = \epsilon\mu_i, \qquad u_i(\epsilon) = v_i \qquad (i = 1, 2) \tag{3.9}$$

provide a (trivial) continuation of $v_i$ as solution of (1.1) for $\epsilon \neq 0$; the same clearly holds for $-v_i$. Thus each eigenvector of $\hat{B}$ is continuable as a unit eigenvector of $A + \epsilon B$.

The same conclusion holds true when $\mu_1 = \mu_2 \equiv \mu_0$, save that either the geometric multiplicity of $\mu_0$ is two - in which case all vectors of $\mathbb{R}^2$ are eigenvectors of $\hat{B}$ - or it is one, and there is (modulo reflections) just one normalized eigenvector $v_0$ of $\hat{B}$.

**Remark 3.2.** If $\hat{B}$ has no real eigenvalue there cannot be bifurcation points. On the grounds of Proposition 2.1, this holds for any $B$ (and not only for $B$ as in (3.6)).

**Example 3.3.** Consider

$$B = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ p & q & r \end{pmatrix}, \tag{3.10}$$

that is,

$$B(x, y, z) = (ax + by, cx + dy, px + qy + rz).$$

This time $z(\delta, \epsilon, x, y)$ is given by its general expression (3.4), however since $m = n = 0$ this does not affect the bifurcation system - which maintains its reduced form (3.7) - nor the conclusion that each eigenvector of $\hat{B}$ is a bifurcation point. The difference with Ex.1.1 is that here the solutions of the full system (3.3) have a nonzero $z-$component, and consequently the bifurcation branch continuing a given eigenvector $v_0 = (x_0, y_0)$ of $\hat{B}$ corresponding to the eigenvalue $\mu_0$ is less trivial and is given by the equations

$$\delta(\epsilon) = \epsilon\mu_0, \qquad u(\epsilon) = \big(x_0, y_0, z(\epsilon\mu_0, \epsilon, x_0, y_0)\big). \tag{3.11}$$

**Remark 3.4.** The above examples can be clearly seen in the context of Equation (2.1). We keep the notations used in Section 2 for $N = \operatorname{Ker} A, W = \operatorname{Im} A$ as well as for the projections $P, Q$ onto these subspaces. Pick a $v_0 \in S_0$ and consider the complementary equation (2.4) with $v = v_0$:

$$Aw + \epsilon QB(v_0 + w) = \delta w. \tag{3.12}$$

If we suppose that

$$QB(v_0) = 0, \tag{3.13}$$

then $w = 0$ solves (3.12); by uniqueness, it follows that $w(\delta, \epsilon, v_0) = 0$ for any $\delta$ and $\epsilon$. The bifurcation equation (2.8) thus reduces (for $v = v_0$) to

$$\epsilon PB(v_0) = \delta v_0, \tag{3.14}$$

which is precisely - taking $\mu_0 = \delta/\epsilon$ - the necessary condition (2.5). This remark is not new, for (3.13) and (2.5) are equivalent to saying that $B(v_0) = \mu_0 v_0$ and in this case, as already noted in the Introduction, we can immediately solve (2.1) for all $\epsilon$. Perhaps more interesting is to observe that requiring the condition (3.13) for *all* $v_0 \in S_0$ amounts to requiring that $B$ map $S_0$ into $N$ and therefore - when $B$ is linear, of course - that $B$ map $N$ into itself, i.e., that $N$ be an *invariant subspace* for $B$. Indeed, this is what happens in Example 1.1.

Consider instead the dual situation in which $W$, rather than $N$, is an invariant subspace for $B$; the $B$ in Example 1.2 is chosen to enjoy this property. This is expressed by the condition that $PB(w) = 0$ for any $w \in W$; so that if we pick a $v_0 \in S_0$ satisfying the necessary condition (2.5), then we have in particular

$$\epsilon PB(v_0 + w(\epsilon\mu_0, \epsilon, v_0)) = \epsilon PB(v_0) = \epsilon\mu_0 v_0 \tag{3.15}$$

for any $\epsilon$. This shows that $(\epsilon\mu_0, \epsilon, v_0)$ is a solution of (2.8) for any $\epsilon$, implying that $v_0$ is continuable via the equations

$$\delta(\epsilon) = \epsilon\mu_0, \qquad u(\epsilon) = v_0 + w(\epsilon\mu_0, \epsilon, v_0). \tag{3.16}$$

Of course, in order to have unit eigenvectors we shall take $U(\epsilon) \equiv u(\epsilon)/\|u(\epsilon)\|$ rather than $u(\epsilon)$ itself and use the linearity of the equation.

**Example 3.5.** Here we consider the case

$$B = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

For such a $B$, (3.4) becomes

$$z = z(\delta, \epsilon, x, y) = \frac{\epsilon}{\delta - 1} x \tag{3.18}$$

and the bifurcation system (3.5) is

$$\begin{cases} by = (\delta/\epsilon)x, \\ cx + \frac{\epsilon}{\delta-1}x = (\delta/\epsilon)y. \end{cases} \tag{3.19}$$

We have to distiguish the following cases:

- $bc > 0$
- $b > 0$ (or $b < 0$) and $c = 0$
- $b = c = 0$
- $bc < 0$.

The last case is not of interest, for the reduced matrix

$$\hat{B} \equiv \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \tag{3.20}$$

has no real eigenvalue.

- **Case** $bc > 0$**:**

The first equation in (3.19) yields

$$y = \frac{\delta}{\epsilon b} x, \tag{3.21}$$

so that the solutions $u = (x, y, z)$ of the full system (3.3) have the form

$$u = x(1, \frac{\delta}{\epsilon b}, \frac{\epsilon}{\delta - 1}). \tag{3.22}$$

Moreover replacing ( 3.21) in the second equation of the system (3.19) gives the condition

$$c + \frac{\epsilon}{\delta - 1} = \frac{\delta^2}{\epsilon^2 b} \tag{3.23}$$

provided that $x \neq 0$; however, (3.22) implies that $u = 0$ if $x = 0$, and we look for solutions $u \neq 0$. (Note that the above equations make sense whenever $b \neq 0$, however since the l.h.s. of (3.23) has - for $\epsilon$ small and $c \neq 0$ - the sign of $c$, it follows that $b$

and $c$ must have the same sign in order that (real) solutions to (3.23) exist). Now the latter equation - expressing the eigenvalues as a function of the parameter $\epsilon$ - can be written equivalently as

$$(\delta - 1)(\delta^2 - \epsilon^2 bc) = \epsilon^3 b, \tag{3.24}$$

and by direct inspection we then find that it has for each $\epsilon$ three real solutions $\delta_i(\epsilon), 1 \le i \le 3$, with the property

$$\delta_i(\epsilon) \to 0,\ i = 1,2; \qquad \delta_3(\epsilon) \to 1 \qquad (\epsilon \to 0). \tag{3.25}$$

Therefore, using (3.23) in (3.22), we see that the eigenvectors of interest are given by the formula

$$u(\epsilon) = x(1, \frac{\delta}{\epsilon b}, \frac{\delta^2}{\epsilon^2 b} - c) \qquad (x \neq 0) \tag{3.26}$$

where $\delta = \delta_i(\epsilon),\ i = 1,2$. Equation (3.26) shows that the ratio $\delta/\epsilon$ is the significant parameter here. Now since $\epsilon/(\delta - 1)$ approaches zero as $\epsilon \to 0$, it follows from (3.23) that $\delta^2/(\epsilon^2 b) \to c$ as $\epsilon \to 0$, and therefore

$$\frac{\delta}{\epsilon} \to \pm\sqrt{bc} \equiv \pm\mu_0 \qquad (\epsilon \to 0). \tag{3.27}$$

Thus if we let in (3.26) $x = 1$ and $\delta = \delta_i(\epsilon)\,(i = 1,2)$, and denote with $u_i(\epsilon)$ the corresponding vector, then as $\epsilon \to 0$

$$u_i(\epsilon) = (1, \frac{\delta_i(\epsilon)}{\epsilon b}, \frac{\delta_i^2(\epsilon)}{\epsilon^2 b} - c) \to (1, \pm\frac{\mu_0}{b}, 0) \equiv u_\pm \tag{3.28}$$

where the signs $+$ and $-$ refer to $i = 1$ and $i = 2$ respectively. It follows that if $U_i(\epsilon)$, $U_\pm$ denote the normalized vectors corresponding respectively to $u_i(\epsilon)$ and $u_\pm$, then

$$U_1(\epsilon) \to U_+, \qquad U_2(\epsilon) \to U_- \qquad (\epsilon \to 0), \tag{3.29}$$

and this finally shows that $U_\pm$ (together of course with their opposites $-U_\pm$) are the bifurcation points in this case. Since $\mu_0 = \sqrt{bc} > 0$, we conclude that there are precisely four bifurcation points. Note that $\pm\mu_0$ are the eigenvalues of the matrix in (3.20) and $U_\pm$ (together with their opposites $-U_\pm$) the corresponding unit eigenvectors. Thus also in this case (as in the Examples 1.1 and 1.2), every $v_0 \in S_0$ satisfying the necessary condition (2.5) is in fact a bifurcation point.

- **Case $b > 0$ (or $b < 0$) and $c = 0$:**

  The previous analysis remains true save that in this case $\mu_0 = 0$. Therefore,

$$U_\pm = (1, 0, 0) \equiv e_1$$

is the only bifurcation point (modulo reflections). However, it is important to note that there exist two distinct *bifurcation branches* bifurcating from $e_1$: indeed it is easily seen from (3.24) that $\delta_1(\epsilon) \neq \delta_2(\epsilon)$ for each $\epsilon \neq 0$, and this shows - via the formula (3.28) - that $u_1(\epsilon) \neq u_2(\epsilon)$ for $\epsilon \neq 0$.

- **Case $b = c = 0$:**

The situation is quite different in this case, for the bifurcation system (3.19) reduces to

$$\begin{cases} 0 = \delta x \\ \frac{\epsilon^2}{\delta - 1} x = \delta y. \end{cases} \tag{3.30}$$

Solutions $(x, y) \neq (0, 0)$ of (3.30) exist only for $\delta = 0$, in which case they are (for $\epsilon \neq 0$)

$$(0, y), \qquad y \neq 0.$$

Thus, the only nontrivial normalized solution of the full system (3.3) are (for any $\epsilon \neq 0$)

$$(0, \pm 1, 0) \equiv \pm e_2.$$

This shows that $\pm e_2$ are also the only bifurcation points of the system in this case (and we have the trivial bifurcation branch $\delta(\epsilon) = 0, U_\epsilon = \pm e_2$). On the other hand, as $\hat{B} = 0$, any $v = (x, y) \in N = \mathbb{R}^2$ satisfies the necessary condition with $\mu_0 = 0$.

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