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THE WENTZELL TELEGRAPH EQUATION: ASYMPTOTICS AND CONTINUOUS DEPENDENCE ON THE BOUNDARY CONDITIONS

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Dedicated to Jeff Webb on his 65th birthday

ABSTRACT. Solutions of the telegraph equation in many unbounded domains are shown to be asymptotically equal to solutions of the corresponding heat equation. This works for many boundary conditions, including general Wentzell boundary conditions. Continuous dependence on the boundary conditions is also shown.

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1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^N$, that is Ω is an open connected set. We assume that the boundary $\partial\Omega$ of Ω consists of a finite number of sufficiently smooth $N-1$ dimensional manifolds. An example is the unbounded shaded region in the figure (with $N = 2$). The figure appears in Section 2. *Sufficiently smooth* means that the divergence theorem can be used in Ω, Stokes' theorem can be used on $\partial\Omega$, and the usual trace theorems for Sobolev classes hold. The assumption that $\partial\Omega$ is of class $C^{2+\epsilon}$ for some $\epsilon > 0$ is more than enough. Let

$$\mathcal{A}(x) = (a_{ij}(x)), \quad i, j = 1, \ldots, N$$

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be an $N \times N$ real Hermitian matrix function on $\overline{\Omega}$ such that $a_{ij} \in C^1(\overline{\Omega})$ for all i and j and there exist $0 < \alpha_0 \le \alpha_1 < \infty$ such that

$$\alpha_0 |\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \alpha_1 |\xi|^2 \tag{1.1}$$

holds for all $x \in \overline{\Omega}$ and all $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N$. Similarly, let

$$\mathcal{B}(x) = (b_{ij}(x)), \quad i, j = 1, \dots, N-1$$

be an $(N-1) \times (N-1)$ real Hermitian matrix on $\partial\Omega$ such that $b_{ij} \in C^1(\partial\Omega)$ for all i and j, and

$$\alpha_0 |\xi|^2 \le \sum_{i,j=1}^{N-1} b_{ij}(x)\xi_i\xi_j \le \alpha_1 |\xi|^2 \tag{1.2}$$

holds for all $x \in \partial\Omega$ and $\xi = (\xi_1, \xi_2, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$; here α_0, α_1 are as in (1.1).

We associate with $\mathcal{A}$ the formal differential operator L,

$$Lu = \nabla \cdot (\mathcal{A}(x)\nabla u), \quad x \in \overline{\Omega}$$

and with $\mathcal{B}$ we associate the operator L_∂,

$$L_\partial u = \nabla_\tau \cdot (\mathcal{B}(x)\nabla_\tau u), \quad x \in \partial\Omega,$$

where ∇_τ is the tangential gradient on $\partial\Omega$. Note that L_∂ becomes the Laplace-Beltrami operator Δ_{LB} when $\mathcal{B} = I$, the identity matrix, for all $x \in \partial\Omega$. With L we associate the General Wentzell Boundary Condition

$$(GWBC) \quad Lu + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad \text{on} \quad \partial\Omega.$$

Here ν is the unit outer normal on $\partial\Omega$,

$$\partial_\nu^{\mathcal{A}} u = (\mathcal{A}\nabla u) \cdot \nu$$

is the conormal derivative with respect to $\mathcal{A}$; $\beta, \gamma \in C^1(\partial\Omega; \mathbb{R})$, $\beta > 0$, $\beta, \frac{1}{\beta}, \gamma$ are bounded, and $q \in [0, \infty)$. The telegraph and heat equations we consider are, with α a positive constant,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} - Lu = 0 & \text{in} \quad \mathbb{R}^+ \times \Omega \\ Lu + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 & \text{on} \quad \mathbb{R}^+ \times \partial\Omega \\ u(0,x) = f_1(x), \quad \frac{\partial u}{\partial t}(0,x) = f_2(x), & x \in \overline{\Omega} \end{cases} \tag{1.3}$$

(where $\mathbb{R}^+ = [0, \infty)$) and

$$\begin{cases} 2\alpha\frac{\partial v}{\partial t} - Lv = 0 & \text{in} \quad \mathbb{R}^+ \times \Omega \\ Lv + \beta\partial_\nu^{\mathcal{A}} v + \gamma v - q\beta L_\partial v = 0 & \text{on} \quad \mathbb{R}^+ \times \partial\Omega \\ v(0,x) = h(x), & x \in \overline{\Omega}. \end{cases} \tag{1.4}$$

Our first main result is that, if $\gamma \geq 0$, if Ω contains arbitrarily large balls, and under certain (mild) restrictions on f_1, f_2, there is an $h = h(\alpha, f_1, f_2)$ such that the solution u of (1.3) satisfies

$$u(t,x) = v(t,x)(1 + o(1))$$

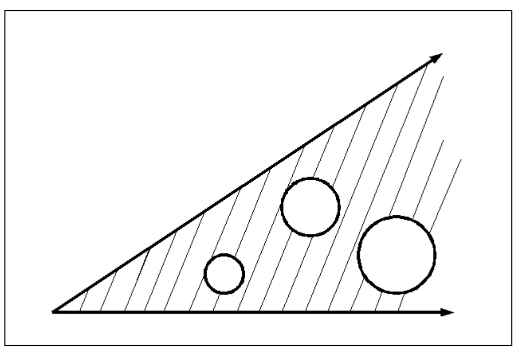
as $t \to \infty$, where v is the solution of (1.4). Moreover the $o(1)$ term decays exponentially for a dense set of initial data.

Both problems (1.3), (1.4) are well posed on the space $L^2(\Omega) \oplus L^2(\partial\Omega, \frac{dS}{\beta})$. The corresponding result with $\Omega = \mathbb{R}^N$ (and no boundary conditions since $\partial\Omega = \emptyset$) was obtained recently [1].

The wellposedness of (1.4) was shown in bounded domains in [4]; cf. also [2]. In Section 2 we show how to modify the arguments of [4] to show that (1.3), (1.4) are both wellposed in general unbounded domains. In Section 3 we formulate and prove the main asymptotic result. Our second main result deals with the continuous dependence on the boundary conditions for the Wentzell telegraph equation given in (1.3). It is studied in Section 4.

2. THE WENTZELL OPERATOR IN GENERAL DOMAINS

Let $\Omega, \mathcal{A}, \mathcal{B}, \alpha, \alpha_0, \alpha_1, L, L_\partial, \beta, \gamma, q$ be as before. We now take Ω to be an unbounded domain. Thus $\partial\Omega$ may be bounded or have one or more unbounded components. The complement of Ω need not be connected (see the figure). Note the *Swiss cheese* domain pictured here can have infinitely many holes.



Let

$$\mathcal{H} = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta})$$

with inner product

$$\langle U, V \rangle_{\mathcal{H}} = \int_\Omega u_1 \overline{v}_1 \, dx + \int_{\partial\Omega} u_2 \overline{v}_2 \, \frac{dS}{\beta}$$

and norm given by

$$\|U\|_{\mathcal{H}} = \langle U, U \rangle_{\mathcal{H}}^{\frac{1}{2}};$$

here $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $u_1 \in L^2(\Omega, dx) = L^2(\Omega)$ and $u_2 \in L^2(\partial\Omega, \frac{dS}{\beta})$, and similarly for V. If $u \in C(\overline{\Omega}) \cup H^1(\Omega)$, then the trace $u|_{\partial\Omega}$ exists, and we can identify u with

$U = \begin{pmatrix} u|_\Omega \\ u|_{\partial\Omega} \end{pmatrix}$ provided that $u|_\Omega \in L^2(\Omega)$ and $u|_{\partial\Omega} \in L^2(\partial\Omega, \frac{dS}{\beta})$. But, in general, for $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{H}$, u_2 need not be the trace $u_1|_{\partial\Omega}$, even if this trace exists.

Define

$$D(A_0) = \{u \in C^2(\overline{\Omega}) : u|_\Omega \in H^2(\Omega),\ qu|_{\partial\Omega} \in H^2(\partial\Omega, \frac{dS}{\beta})\} \tag{2.1}$$

and

$$A_0 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.$$

That is, $D(A_0)$ is $C^2(\overline{\Omega}) \cap H^2(\Omega)$ if $q = 0$, while $D(A_0)$ is $\{u \in C^2(\overline{\Omega}) \cap H^2(\Omega) : u|_{\partial\Omega} \in H^2(\partial\Omega, \frac{dS}{\beta})\}$ if $q > 0$. More precisely, $u \in D(A_0)$ defines a $U = \begin{pmatrix} u|_\Omega \\ u|_{\partial\Omega} \end{pmatrix} \in \mathcal{H}$, this U is in $D(A_0)$, and $A_0U = W \in \mathcal{H}$ means that W corresponds to some $w \in C(\overline{\Omega})$ such that

$$\begin{aligned} \nabla \cdot (\mathcal{A}\nabla u) = w \quad \text{in} \quad \overline{\Omega}, \\ \nabla \cdot (\mathcal{A}\nabla u) + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.2}$$

Of course, in (2.2), $\nabla \cdot (\mathcal{A}(x)\nabla u)$ can be replaced by w.

For $U, V \in D(A_0)$, by the divergence theorem,

$$\begin{aligned} \langle A_0U, V\rangle_{\mathcal{H}} &= \int_\Omega \nabla \cdot (\mathcal{A}\nabla u)\overline{v}\, dx + \int_{\partial\Omega} \nabla \cdot (\mathcal{A}\nabla u)\overline{v}\, \frac{dS}{\beta} \\ &= -\int_\Omega (\mathcal{A}\nabla u) \cdot \nabla\overline{v}\, dx - \int_{\partial\Omega} \gamma u\overline{v}\, \frac{dS}{\beta} + q\int_{\partial\Omega} (L_\partial u)\overline{v} dS \\ &\quad \text{by the divergence theorem and the boundary condition (2.2)} \\ &= -\int_\Omega (\mathcal{A}\nabla u) \cdot \nabla\overline{v}\, dx - \int_{\partial\Omega} \gamma u\overline{v}\, \frac{dS}{\beta} - q\int_{\partial\Omega} (\mathcal{B}\nabla_\tau u) \cdot \nabla_\tau\overline{v} dS \qquad (2.3) \\ &\quad \text{by Stokes' theorem on the boundary.} \end{aligned}$$

Thus $\langle A_0U, V\rangle_{\mathcal{H}} = \langle U, A_0V\rangle_{\mathcal{H}}$, establishing the symmetry of A_0 since $D(A_0)$ is dense in $\mathcal{H}$.

To show that A_0 is essentially selfadjoint, we must solve $\lambda U - A_0U = F$ for some fixed large enough $\lambda > 0$ and all $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in a dense subspace of $\mathcal{H}$. Taking the inner product of $\lambda U - A_0U = F$ with $V \in D(A_0)$ leads to, as above,

$$\begin{aligned} \lambda\langle U, V\rangle_{\mathcal{H}} + &\int_\Omega (\mathcal{A}\nabla u) \cdot \nabla\overline{v}\, dx + \int_{\partial\Omega} \gamma u\overline{v}\, \frac{dS}{\beta} \\ &+ q\int_{\partial\Omega} (\mathcal{B}\nabla_\tau u) \cdot \nabla_\tau\overline{v} dS \\ &= \int_\Omega f_1\overline{v}\, dx + \int_{\partial\Omega} f_2\overline{v}\, \frac{dS}{\beta}. \end{aligned} \tag{2.4}$$

Let $B(U,V)$ be the left hand side of (2.4) and let $C(V)$ be the right hand side. For $q \geq 0$, let us introduce $\mathcal{V}_q$ as follows.

$$\mathcal{V}_0 := H^1(\Omega),$$
$$\mathcal{V}_q := \{u \in \mathcal{V}_0 : \, u|_{\partial\Omega} \in H^1(\partial\Omega, \frac{dS}{\beta})\} \quad if \quad q > 0.$$

The norm defined by

$$\|V\|^2_{\mathcal{V}_q} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\partial\Omega, \frac{dS}{\beta})} + q\|\nabla_\tau v\|^2_{L^2(\partial\Omega, dS)}$$

makes $\mathcal{V}_q$ into a Hilbert space such that $\mathcal{V}_q$ embeds continuously into $\mathcal{H}$. Then, for $q \geq 0$, $B(\cdot,\cdot)$ is a sesquilinear form on $\mathcal{V}_q$ and $C(\cdot)$ is a bounded conjugate linear functional on $\mathcal{V}_q$. From now on, $q \geq 0$ is fixed. B is hermitian since $B(U,V) = \overline{B(V,U)}$ for all $U, V \in \mathcal{V}_q$. For $\lambda > 0$ and all $U, V \in \mathcal{V}_q$,

$$\begin{aligned}|B(U,V)| &\leq \alpha_1 \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \lambda \|U\|_{\mathcal{H}} \|V\|_{\mathcal{H}} \\ &\quad + \|\gamma\|_\infty \|u\|_{L^2(\partial\Omega, \frac{dS}{\beta})} \|v\|_{L^2(\partial\Omega, \frac{dS}{\beta})} + q\alpha_1 \|\nabla_\tau u\|_{L^2(\partial\Omega, dS)} \|\nabla_\tau v\|_{L^2(\partial\Omega, dS)} \\ &\leq c_1(\lambda) \|U\|_{\mathcal{V}_q} \|V\|_{\mathcal{V}_q}\end{aligned}$$

for some positive constant $c_1(\lambda) = c_1(\lambda; q, \alpha_1, \beta, \gamma)$. Next,

$$\begin{aligned}-Re\, B(U,U) &\geq \alpha_0 \|\nabla u\|^2_{L^2(\Omega)} + \lambda \|U\|^2_{\mathcal{H}} \\ &\quad - \|\gamma_-\|_\infty \|u\|^2_{L^2(\partial\Omega, \frac{dS}{\beta})} + q\alpha_0 \|\nabla_\tau u\|^2_{L^2(\partial\Omega, dS)} \\ &\geq c_0(\lambda) \|U\|^2_{\mathcal{V}_q}\end{aligned}$$

for some constant $c_0(\lambda) = c_0(\lambda; q, \alpha_0, \beta, \gamma) > 0$, all $U \in \mathcal{V}_q$ and all $\lambda > \|\gamma_-\|_\infty$, the supremum of the negative part of γ.

The Lax-Milgram Lemma (cf. e.g. [7, Theorem 6, p. 57]) shows that $\lambda U - A_0 U = F$ has a weak solution U for each $\lambda > \|\gamma_-\|_\infty$ and all $F \in \mathcal{V}_q$. Let $F \in C^{2+\epsilon}(\overline{\Omega}) \cap \mathcal{V}_q$ for some $\epsilon > 0$. Then a standard elliptic regularity argument (as in [4]) shows that $U \in D(A_0)$ and $\lambda U - A_0 U = F$ holds. Thus A, which we define to be the closure of A_0, is selfadjoint and bounded above by $\|\gamma_-\|_\infty I$. In particular, $A = A^* \leq 0$ if $\gamma \geq 0$ on $\partial\Omega$.

Note that the previous arguments are analogous to those used in [4], but we prefer to insert them explicitly in order to show that the case of Ω unbounded and of much more general expressions of (GWBC) are allowed.

3. THE TELEGRAPH EQUATION AND ITS ASYMPTOTICS

Let $A = \overline{A}_0$ be as above, with $\gamma \geq 0$. Observe that A is injective if $\partial\Omega$ has infinite $N-1$ dimensional surface measure. If $\int_{\partial\Omega} dS < \infty$, then A will be injective if, in addition, we assume $\gamma(x) > 0$ for some $x \in \partial\Omega$. Then the initial value problem for the telegraph equation (where ' denotes $\frac{d}{dt}$)

$$\text{(3.1)} \quad \begin{cases} u''(t) + 2\alpha u'(t) - Au = 0 \quad (t \in \mathbb{R}^+) \\ u(0) = f_1, \quad u'(0) = f_2 \end{cases}$$

is wellposed for $\alpha > 0$ by the spectral theorem in the space $L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta})$. The corresponding heat equation problem

$$\text{(3.2)} \quad \begin{cases} 2\alpha v'(t) - Av(t) = 0 \quad (t \in \mathbb{R}^+) \\ v(0) = h \end{cases}$$

is also wellposed for $\alpha > 0$, again by the spectral theorem in the same space. We want to show that, under suitable hypotheses, given f_1, f_2 (in some suitable dense set of initial data) there is an $h = h(\alpha, f_1, f_2)$ such that the solution u of (3.1) and the solution v of (3.2) satisfy

$$u(t) = v(t)(1 + o(1))$$

as $t \to \infty$, i.e.

$$\|u(t) - v(t)\|_{\mathcal{H}} = \|v(t)\|_{\mathcal{H}}(o(1))$$

as $t \to \infty$. This condition requires that $h \neq 0$.

Hypothesis 3.1. *Let Ω be a sufficiently smooth unbounded domain in $\mathbb{R}^N$ containing arbitrarily large balls, i.e. given $R > 0$ there is an $x_R \in \Omega$ such that the ball $B(x_R, R) := \{y \in \mathbb{R}^N : |y - x_R| < R\}$ is in Ω.*

"Sufficiently smooth" is explained in Section 1. Exterior domains satisfy Hypothesis 3.1, as do halfspaces, the "inside" and "outside" of the paraboloid $\partial\Omega = \{x \in \mathbb{R}^N : x_N = \sum_{j=1}^{N-1} x_j^2\}$, and the domain pictured in the figure in Section 2. In all these cases it is clear that $x = x_R$ cannot be chosen to be independent of R.

Hypothesis 3.2. *$\mathcal{A}(x) = (a_{i,j}(x))$ is a real hermitian matrix function in $C^{1+\delta}(\overline{\Omega})$ for some $\delta > 0$, and (1.1) holds, and similarly for $\mathcal{B}(x) = (b_{i,j}(x))$ with (1.2) holding; $\beta, \gamma \in C^1(\partial\Omega, \mathbb{R})$ with $\beta > 0, \gamma \geq 0$ and $\frac{1}{\beta}$ all bounded; $q \in [0, \infty)$. $Lu = \nabla\cdot(\mathcal{A}(x)\nabla u)$ with boundary condition (2.2), $A_0 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$, $D(A_0)$ is defined by (2.1).*

Hypotheses 3.1 and 3.2 imply that $A = \overline{A}_0$ is selfadjoint, injective and nonpositive on $\mathcal{H}$. By the spectral theorem, there is a unitary operator U_0 from $\mathcal{H}$ onto some concrete L^2 space, $L^2(\Lambda, \Sigma, \lambda)$, such that $U_0 A U_0^{-1} = M_m$, the operator of multiplication by the $\Sigma-$ measurable function $m : \Lambda \to (-\infty, 0]$; here $M_m g = mg$ and $g \in D(M_m)$ if and only if $g, mg \in L^2(\Lambda, \Sigma, \lambda)$. The spectrum of A is

$$\sigma(A) = ess\,Range(m) \subset (-\infty, 0].$$

If $F : \sigma(A) \to \mathbb{C}$ is Borel measurable, then $F(A) = U_0^{-1} M_{F(m)} U_0$, and $\chi_\Gamma(x) = 1$ or 0, according as $x \in \Gamma$ or $x \notin \Gamma$.

Hypothesis 3.3 *Suppose $\alpha^2 I + A$ is injective, i.e. $-\alpha^2$ is not an eigenvalue of A. Let*

$$K_\delta = \chi_{[\delta,\alpha^2-\delta]}(-A) + \chi_{[\alpha^2,\infty)}(-A)$$

for $\delta > 0$ and let

$$\mathcal{K} = \bigcup_{\delta>0} Range(K_\delta).$$

Assume

$$f_2 + \alpha f_1 \in Range((\alpha^2 I + A)^{\frac{1}{2}}) \cap \mathcal{K}$$

and suppose

$$h := \chi_{(0,\alpha^2)}(-A)(\frac{f_2}{2} + (\alpha^2 I + A)^{-\frac{1}{2}}(\frac{f_2+\alpha f_1}{2})) \neq 0. \tag{3.3}$$

Note that $\mathcal{K}$ is dense in $\mathcal{H}$, as is the set of h_1 defined by the version of (3.3) obtained by deleting $\chi_{(0,\alpha^2)}(-A)$.

Theorem 3.1. *Let Hypotheses* 3.1, 3.2 *and* 3.3 *hold. Let u be the unique solution to* (3.1). *Then*

$$u(t) = v(t)(1 + o(1))$$

where v is the unique solution to (3.2) *with h given by* (3.3).

Proof. This will follow from Theorem 2.1 in [1], once we show that A is injective and $sup\, \sigma(A) = 0$.

Assume $AU = 0$. Then

$$\nabla \cdot (\mathcal{A}\nabla u) = 0 \quad in \quad \overline{\Omega},$$

$$\nabla \cdot (\mathcal{A}\nabla u) + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad on \quad \partial\Omega.$$

Taking the inner product $\langle AU, U\rangle_{\mathcal{H}} = 0$ yields

$$-\int_\Omega (\mathcal{A}\nabla u)\cdot\nabla\overline{u}\,dx - \int_{\partial\Omega} \gamma|u|^2\frac{dS}{\beta} - q\int_{\partial\Omega} |\mathcal{B}^{\frac{1}{2}}\nabla_\tau u|^2\,dS = 0.$$

Since $\gamma \geq 0$ we conclude that u coincides with a constant on Ω. Since $u|_{\partial\Omega} = trace(u|_\Omega)$, u is a constant on $\overline{\Omega}$. In addition, since $u \in L^2(\Omega)$ and $\int_\Omega dx = \infty$ by Hypothesis 3.1, it follows that $u \equiv 0$. Thus A is injective.

Let $R > 0$ be given. Choose $x_R \in \Omega$ so that the ball $B(x_R, R) \subset \Omega$. Assume further, without loss of generality, that $B(x_R, R)$ is compactly contained in Ω.

Any function supported in $B(x_R, R)$ will satisfy the boundary condition (2.2), since the function vanishes on and near $\partial\Omega$. Let

$$\psi_1(x) = e^{-\frac{1}{x}} \quad for \quad x > 0, \quad \psi_1(x) = 0 \quad for \quad x \leq 0.$$

Then $\psi_1 \in C^\infty(\mathbb{R})$. In $\mathbb{R}^N$, let $r = |x|$ and let $\widetilde{\psi_2}(x) = \psi_2(r) = \psi_1(r)\psi_1(1-r)$. Then $\psi_2 \in C_c^\infty(\mathbb{R})$, $\psi_2 > 0$ inside $B(0,1)$ and $\psi_2(r) = 0$ for $r \geq 1$. Given $R > 1$, let $r = |x|$ and

$$\widetilde{\psi_R}(x) = \begin{cases} \psi_2(r) & for \quad 0 < r < \frac{1}{2} \\ \psi_2(\frac{1}{2}) & for \quad \frac{1}{2} \leq r < R - \frac{1}{2} \\ \psi_2(R-r) & for \quad R - \frac{1}{2} \leq r < R \\ 0 & for \quad r \geq R. \end{cases}$$

Finally, let

$$\phi(x) = \widetilde{\psi_R}(x - x_R),$$

which is defined on $\mathbb{R}^N$, be viewed as a function on Ω. Let $\omega_N = \int_{\partial B(0,1)} dS$ be the surface area of the unit sphere in $\mathbb{R}^N$. Then

$$\begin{aligned} \langle \phi, \phi \rangle_{\mathcal{H}} = \omega_N \int_0^{\frac{1}{2}} [\psi_2(r)]^2 r^{N-1}\, dr + \omega_N \int_{\frac{1}{2}}^{R-\frac{1}{2}} [\psi_2(\frac{1}{2})]^2 r^{N-1}\, dr \\ + \omega_N \int_{R-\frac{1}{2}}^{R} [\psi_2(R-r)]^2 r^{N-1}\, dr. \end{aligned}$$

It is easily seen that there are positive constants k_1, k_2, such that

$$k_1 (R - \frac{1}{2})^N \leq \langle \phi, \phi \rangle_{\mathcal{H}} \leq k_2\, R^N. \tag{3.4}$$

Next,

$$\begin{aligned} 0 > \quad \langle A\phi, \phi \rangle_{\mathcal{H}} &= - \int_\Omega (\mathcal{A} \nabla \phi) \cdot \nabla \phi\, dx \\ &\geq -\alpha_1 \int_\Omega |\nabla \phi|^2\, dx \\ &= -\alpha_1 \omega_N \int_0^R |\frac{\partial}{\partial r} \widetilde{\psi_R}(x)|^2 r^{N-1}\, dr \\ &= -\alpha_1 \omega_N \left[\int_0^{\frac{1}{2}} |\psi_2'(r)|^2 r^{N-1}\, dr + \int_{R-\frac{1}{2}}^{R} |\psi_2'(R-r)|^2 r^{N-1}\, dr \right] \\ &\geq -\alpha_1 \omega_N \|\psi_2'\|_\infty \left[\left(\frac{R^N - (R-\frac{1}{2})^N}{N} \right) + \frac{2^{-N}}{N} \right]. \end{aligned}$$

But by Taylor's theorem

$$R^N - (R - \frac{1}{2})^N = \frac{N}{2} \xi^{N-1} \leq \frac{N}{2} R^{N-1}$$

for some $\xi \in (R - \frac{1}{2}, R)$.

Thus

$$0 > \quad \langle A\phi, \phi \rangle_{\mathcal{H}} \geq -\alpha_1 \omega_N \|\psi_2'\|_\infty \left(\frac{R^{N-1} + 2^{-N}}{2} \right). \tag{3.5}$$

Combining (3.4), (3.5) yields

$$0 > \frac{\langle A\phi, \phi \rangle_{\mathcal{H}}}{\langle \phi, \phi \rangle_{\mathcal{H}}} \geq \frac{-\alpha_1 \omega_N \|\psi_2'\|_\infty (\frac{R^{N-1} + 2^{-N}}{2})}{k_1 (R - \frac{1}{2})^N} \to 0$$

as $R \to \infty$.

In the multiplicative representation of A, $A = U_0^{-1} M_m U_0$ for a Σ-measurable function $m : \Lambda \to (-\infty, 0]$, where U_0 is unitary from $\mathcal{H}$ to $L^2(\Lambda, \Sigma, \lambda)$. Rewriting ϕ as ϕ_R, we have, for $\widehat{\phi}_R = U_0 \phi_R$,

$$0 > \quad \frac{\langle A\phi_R, \phi_R \rangle_{\mathcal{H}}}{\langle \phi_R, \phi_R \rangle_{\mathcal{H}}} = \frac{\int_\Lambda m |\widehat{\phi}_R|^2 \, d\lambda}{\int_\Lambda |\widehat{\phi}_R|^2 \, d\lambda} \to 0$$

as $R \to \infty$. Thus $-m$ must take arbitrarily small positive values on a set of positive $\lambda-$ measure, since $\lambda(\{\omega \in \Lambda : m(\omega) = 0\}) = 0$ since A is injective. But taking into account that $ess\, Range(m) = \sigma(A)$, it follows that $sup\, \sigma(A) = 0$. The assertion now follows. □

Remark 3.2 Suppose A as before satisfies $A = A^* \leq 0$ and Ω is a bounded domain. Then A has an orthonormal basis $\{\phi_n\}$ of eigenvectors with eigenvalues $\{\lambda_n\}$ satisfying

$$0 \geq \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ \cdots \geq \lambda_n \to -\infty$$

as $n \to \infty$ and λ_1 is a simple eigenvalue whose eigenspace is spanned by a positive function ϕ_1 on Ω. Problems (1.3) and (1.4) can be solved by separation of variables,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x),$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x),$$

where $A\phi_n = \lambda_n \phi_n$. The solutions have the form

$$u(x,t) = a_1(t)\phi_1(x)(1 + o(1)),$$

$$v(x,t) = b_1(t)\phi_1(x)(1 + o(1)),$$

provided $a_1(0), b_1(0)$ (which depend on ϕ_1, λ_1 and α) are both nonzero. One then readily shows that, if one chooses h as before, namely

$$h(x) = \frac{1}{2}(< f, \phi_1 > + (\alpha^2 + \lambda_1)^{-\frac{1}{2}}(< g, \phi_1 > + \alpha < f, \phi_1 >)\phi_1(x)$$

provided $-\lambda_1 < \alpha^2$, we have

$$\lim_{\alpha \to 0^+} \frac{|u(x,t) - v(x,t)|}{|v(x,t)|} = 0$$

in various senses (e.g., $|\cdot|$ can denote absolute value or the $\mathcal{H}$ norm). We omit the elementary but slightly tedious details. The main point is that, in the compact resolvent case, the asymptotic behavior of the telegraph equation is generically one dimensional. This contrasts strongly with the nontrivial infinite dimensional asymptotics described by Theorem 3.1.

4. CONTINUOUS DEPENDENCE ON THE BOUNDARY CONDITIONS OF THE WENTZELL TELEGRAPH EQUATION

In [3], we studied the continuous dependence on the boundary conditions of the solutions of the Wentzell heat equation in a bounded domain. Using the framework of this paper, the results of [3] extend to the case of arbitrary (smooth enough) domains. In [3] we treated the special case of $\mathcal{B} = I$ for each $x \in \partial\Omega$, but the extension of [3] to the more general $\mathcal{B}(x)$ used here is trivial. Now we prove the analogous continuous dependence result in the context of the Wentzell wave and telegraph equations in arbitrary domains. Thus we consider (1.3) for $\alpha \geq 0$.

Here is our continuous dependence result.

Theorem 4.1. *Let $\mathcal{A}_k, \mathcal{B}_k$ for $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ satisfy hypotheses (3.2) with positive ellipticity constants α_0, α_1 in (1.1), (1.2) being independent of k. Let $\beta_k, \gamma_k \in C^1(\partial\Omega)$ be real with $\beta_k > 0$,*

$$\inf\{\gamma_k(x) : \; k \in \mathbb{N}_0, x \in \partial\Omega\} = -\omega > -\infty$$

$$\sup\{\beta_k(x) + \frac{1}{\beta_k(x)} + \gamma_k(x) : k \in \mathbb{N}_0, \; x \in \partial\Omega\} = M < \infty.$$

Let $q_k \in (0, \infty)$ for all $k \in \mathbb{N}_0$. Suppose

$$q_k \to q_0, \quad \beta_k \to \beta_0, \quad \gamma_k \to \gamma_0, \quad \mathcal{A}_k \to \mathcal{A}_0, \quad \mathcal{B}_k \to \mathcal{B}_0$$

as $k \to \infty$, uniformly on their respective domain.

Let

$$\mathcal{H}_k = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta_k}), \quad k \in \mathbb{N}_0,$$

and let A_k be the corresponding selfadjoint operator on $\mathcal{H}_k$ corresponding to $(1.3)_k$, by which we mean (1.3) with $u, \beta, \gamma, \dots$ replaced by $u_k, \beta_k, \gamma_k, \dots$, except that we require α, f_1, f_2 to be independent of k. Finally, assume $f_1, (-A_k)^{\frac{1}{2}} f_1, f_2 \in \mathcal{H}_0$. Then for the unique solution u_k of $(1.3)_k$ we have

$$U_k(t) \to U_0(t), \quad (-A_k)^{\frac{1}{2}} u_k(t) \to (-A_0)^{\frac{1}{2}} u_0(t), \quad u_k'(t) \to u_0'(t) \tag{4.1}$$

as $k \to \infty$, uniformly in $\mathcal{H}_0$ for t in bounded subsets of $\mathbb{R}$.

Proof. First note that $\mathcal{H}_k$ and $\mathcal{H}_0$ are equal as sets and have uniformly equivalent Hilbert space norms. The uniform boundedness of β_k and $\frac{1}{\beta_k}$ implies there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \|f\|_k \leq \|f\|_0 \leq c_2 \|f\|_k$$

holds for all $f \in \mathcal{H}_0$ and all $k \in \mathbb{N}_0$, where $\|\cdot\|_k$ denotes the $\mathcal{H}_k$ norm; we will also let $\|\cdot\|_k$ denote the $\mathbb{B}(\mathcal{H}_k)$ (operator) norm. Let $\omega = \sup\{(\gamma_k)_-(x) : \quad x \in \partial\Omega, k \in \mathbb{N}_0\}$ where $(\gamma_k)_-$ is the negative part of γ_k. Then $0 \leq \omega < \infty$ by our assumption, and

$$\|e^{tA_k}\|_k \leq e^{\omega t}$$

for all $t \geq 0, k \in \mathbb{N}_0$. Note also that A_k generates a strongly continuous cosine function (see [5, 6]) on $\mathcal{H}_k$, given by

$$C_k(t) = \frac{e^{it(-A_k)^{\frac{1}{2}}} + e^{-it(-A_k)^{\frac{1}{2}}}}{2}, \quad t \in \mathbb{R}, \tag{4.2}$$

and

$$\|C_k(t)\|_k \leq e^{\omega|t|}$$

holds for all $t \in \mathbb{R}$ and all $k \in \mathbb{N}_0$. Combining this estimate with (4.2) we deduce

$$\|C_k(t)\|_0 \leq M_1 e^{\omega|t|}$$

holds for some constant M_1 and all t, k.

The problem $(1.3)_k$ can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} A_k^{\frac{1}{2}} u_k(t) \\ u_k'(t) \end{pmatrix} = \left[\begin{pmatrix} 0 & A_k^{\frac{1}{2}} \\ (-A)_k^{\frac{1}{2}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha \end{pmatrix}\right] \begin{pmatrix} A_k^{\frac{1}{2}} u_k(t) \\ u_k'(t) \end{pmatrix},$$

$$\begin{pmatrix} A_k^{\frac{1}{2}} u_k(0) \\ u_k'(0) \end{pmatrix} = \begin{pmatrix} A_k^{\frac{1}{2}} f_1 \\ f_2 \end{pmatrix},$$

or, in simpler notation,

$$\frac{d}{dt} W_k = (G_k + P) W_k, \quad W_k(0) = F_k.$$

Next, G_k and $G_k + P$ generate (C_0) groups on $\mathcal{H}_0^2 = \mathcal{H}_0 \oplus \mathcal{H}_0$, and we shall use exponential notation for them, even though these generators are unbounded operators.

Write

$$A_k = \int_{(-\infty,\omega]} \lambda E_k(d\lambda)$$

as a spectral measure representation of the selfadjoint operator A_k whose spectrum is in $(-\infty, \omega]$. Define

$$A_k^{\frac{1}{2}} = \int_{[0,\omega]} \lambda^{\frac{1}{2}} E_k(d\lambda) + i \int_{(-\infty,0)} (-\lambda)^{\frac{1}{2}} E_k(d\lambda) = R_k + S_k,$$

where $R_k = R_k^*$ is nonnegative and bounded by $\sqrt{\omega}$, and $S_k = -S_k^*$ is unbounded. Thus A_k is a normal operator on $\mathcal{H}_k$, as is $G_k = \begin{pmatrix} 0 & A_k^{\frac{1}{2}} \\ (-A_k)^{\frac{1}{2}} & 0 \end{pmatrix}$ on $\mathcal{H}_k^2$, and $\|e^{tG_k}\|_k \leq e^{|t|\sqrt{\omega}}$ holds for all $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$.

Moreover, by the version of the Neveu-Trotter-Kato approximation theorem used in [3], the second and third convergence assertions in (4.1) are equivalent to

$$e^{t(G_k+P)} H \to e^{t(G_0+P)} H$$

in $\mathcal{H}_0^2$ as $k \to \infty$ for all $H \in \mathcal{H}_0^2$, and this is equivalent to

$$e^{tG_k} H \to e^{tG_0} H$$

in $\mathcal{H}_0^2$ as $k \to \infty$ for all $H \in \mathcal{H}_0^2$, since P is a fixed bounded operator.

But the unique solution of $(1.3)_k$ is also given by

$$u_k(t) = C_k(t)f_1 + \int_0^t C_k(s)f_2\,ds$$

for all $t \in \mathbb{R}$. Since also

$$A_k^{\frac{1}{2}} u_k(t) = C_k(t)(A_k^{\frac{1}{2}} f_1) + A_k^{\frac{1}{2}} \int_0^t C_k(s)f_2\,ds$$

and

$$f_2 \to A_k^{\frac{1}{2}} \int_0^t C_k(s)f_2\,ds$$

is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}_0$ for all $t \in \mathbb{R}$, it follows that (4.2) is equivalent to each of

$$(\lambda - G_k)^{-1}H \to (\lambda - G_0)^{-1}H$$

for all $H \in \mathcal{H}_0^2$ and all λ with $|Re\,\lambda| \geq M_2$ for some $M_2 > 0$ (cf. [6,5]), and

$$(\lambda^2 - A_k)^{-1}h \to (\lambda^2 - A_0)^{-1}h$$

in $\mathcal{H}_0$ for all λ with $|Re\,\lambda| \geq M_2$ for some $M_2 > 0$. But this last convergence assertion follows from [3]. □

Remark 4.2 With a little extra work which we omit, we can in Theorem 4.1 allow α to vary, so that $\alpha_k \geq 0$ can converge to $\alpha_0 \geq 0$. Two points are worth noting. This limit α_0 should perhaps be called α_0^*, as it has nothing to do with the α_0 representing the lower modulus of ellipticity of the matrices $\mathcal{A}_k(x), \mathcal{B}_k(x)$. We could also let f_1, f_2 vary with k. This is standard and requires no new ideas.

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