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COMPACT LINEAR OPERATORS VIA NONLINEAR ANALYSIS

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Dedicated to Jeff Webb on his retirement

ABSTRACT. Recent work concerning the representation of compact linear operators acting between Banach spaces is discussed. The abstract results are applied to establish the existence of an infinite sequence of certain types of eigenfunctions and associated eigenvalues of the Dirichlet problem for the $p-$Laplacian; comparison is made with the corresponding quantities obtained by the Lusternik-Schnirelmann procedure.

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1. INTRODUCTION

That linear analysis can help in the study of nonlinear problems will hardly strike anyone as a novel or even interesting observation, but the value of nonlinear techniques in linear questions may seem less obvious. Here we aim to illustrate this usefulness by reporting on recent work on the representation of compact linear operators acting between Banach spaces. In [5] we showed that if $T : X \to Y$ is a compact linear map, where X and Y are real, reflexive Banach spaces with strictly convex duals, there exist a sequence of closed subspaces X_n of X and unit vectors $x_n \in X_n$ which are eigenvectors corresponding to eigenvalues $\lambda_n := \|Tx_n\|_X = \|T_n\|$ of the nonlinear equation

$$T_n^* \tilde{J}_{Y_n} T_n x_n = \lambda_n \tilde{J}_{X_n} x_n,$$

where $\tilde{J}_Z z$ denotes the Gateaux derivative of the norm $\|\cdot\|_Z$ on the Banach space Z at z, T_n is the restriction of T to X_n and the Y_n are closed subspaces of Y containing TX_n. Since the x_n have certain orthogonality properties (in the sense of James), we refer to

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them and the corresponding λ_n as j-eigenvectors and j-eigenvalues respectively.There follows a representation of T in terms of these j-eigenvectors x_n which in the case of X, Y being Hilbert spaces is the celebrated Erhard Schmidt representation of T, namely

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x, x_n)_X y_n, \ y_n = \lambda_n^{-1} T x_n.$$

In this case, $T^* T x_n = \lambda_n^2 x_n$ so that λ_n is a singular value of T, with corresponding eigenvector x_n. For Banach spaces X, Y, Theorem 21 in [5] is of the form

$$Tx = \lim_{n\to\infty} (I - Q_n) \sum_{i=1}^{n-1} \lambda_i \xi_i(x) y_i + Q_\infty T x, \tag{1.1}$$

where Q_n, Q_∞ are (generally nonlinear) projections of Y onto $Y_n, Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n$, respectively, and the $\xi_i(x)$ may be thought of as analogues of Fourier coefficients and are calculable by means of a recursive formula. Furthermore, in Theorem 17 in [5] it is shown that

$$x = \lim_{n\to\infty} (I - P_n) \sum_{i=1}^{n-1} \xi_i x_i + P_\infty x, \tag{1.2}$$

where P_n, P_∞ are the projections of X onto $X_n, X_\infty := \bigcap_{n\in\mathbb{N}} X_n$, respectively. In [5], Corollary 22 and Remark 18, it is shown that if Y is a Hilbert space, then, with $S_n x := \sum_{i=1}^{n-1} \xi_i x_i$, $Q_n T S_n x = 0$, while if X is a Hilbert space $P_n S_n x = 0$. For general Banach spaces X, Y, if the sequence $(S_n x)_{n\in\mathbb{N}}$, can be shown to be bounded in X (as is the case if X is a Hilbert space) then (1.1) can be proved to yield the Schmidt-type representation,

$$Tx = \sum_{n=1}^{\infty} \lambda_i \xi_i(x) y_n, \tag{1.3}$$

and (1.2) becomes

$$x = \sum_{i=1}^{\infty} \xi_i(x) x_i + P_\infty x. \tag{1.4}$$

In this case (x_i) is a Schauder basis of X/X_∞, and of X if T has trivial kernel, since $X_\infty \subseteq$ kerT. To determine whether or not $(S_n x)_{n\in\mathbb{N}}$ is bounded in X has proved to be intractable so far. It is therefore natural to ask if a topology can be imposed on X with respect to which $(S_n x)_{n\in\mathbb{N}}$ is bounded. The *projective limit* fulfills this role, being the coarsest topology on X compatible with the algebraic structure of X under which the maps S_n are continuous; it is a locally convex topology; see [6], Remark 8. In the Banach space case the fact that the λ_n are norms of restrictions of T rather than eigenvalues of linear operators related to T may perhaps be regarded as a disadvantage: we advance the point of view that, in this general context, these norms of restrictions are entirely natural objects, and that a preoccupation with eigenvalues, addictive though it may be, is not always desirable.

Here we outline the method of proof of these assertions and give some of the properties of the x_n and ξ_n. The final section is devoted to the Dirichlet problem for the

p-Laplacian, special attention being paid to the situation when the underlying space domain is a bounded interval in $\mathbb{R}$, so there is a naturally associated Hardy operator T. For this map the Lusternik-Schnirelmann eigenvectors can be found explicitly, but as numerical evidence suggests that they do not have the j-orthogonality property, it appears that they are distinct from the j-eigenfunctions.

2. **BASIC RESULTS**

Throughout the paper we shall suppose that X and Y are real, reflexive, infinite-dimensional Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and duals X^*, Y^*; the closed unit ball in X is denoted by B_X and the family of all bounded linear maps from X to Y by $B(X,Y)$. The hypotheses that the spaces are real and of infinite-dimension are there just to make the presentation simpler. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle_X$, and given any closed linear subspaces M, N of X, X^* respectively, their polar sets are written as M^0, ${}^0N$; thus,

$$M^0 = \{x^* \in X^* : \langle x, x^* \rangle_X = 0 \text{ for all } x \in M\}$$

and

$${}^0N = \{x \in X : \langle x, x^* \rangle_X = 0 \text{ for all } x^* \in N\}.$$

It is well known that the polar set M^0 of a closed linear subspace M of X is isometrically isomorphic to $(X/M)^*$.

Next, recall that X is called strictly convex if whenever $x, y \in X$ are such that $x \neq y$, $\|x\|_X = \|y\|_X = 1$ and $\lambda \in (0,1)$, then $\|\lambda x + (1-\lambda)y\|_X < 1$; equivalently, no sphere in X contains a line segment. An important result is that X^* is strictly convex if and only if $\|\cdot\|_X$ is Gâteaux differentiable on $X \backslash \{0\}$; the Gâteaux derivative $\widetilde{J}_X(x) := \operatorname{grad} \|x\|_X$ of $\|x\|_X$ at $x \in X\backslash\{0\}$ is the unique element of X^* such that

$$\left\| \widetilde{J}_X(x) \right\|_{X^*} = 1 \text{ and } \left\langle x, \widetilde{J}_X(x) \right\rangle_X = \|x\|_X.$$

In this case the semi-inner product $(x,h)_X$ of x and h is defined by

$$(x,h)_X = \|x\| \langle h, \operatorname{grad} \|x\| \rangle_X \text{ when } x, h \in X, \ x \neq 0;$$

$(0,h)_X$ is defined to be 0 for all $h \in X$. An element x of X is said to be orthogonal to $h \in X$ in the James sense (or j-orthogonal to h), written $x \perp h$, if $(x,h)_X = 0$.

Henceforth we shall assume, in addition to the standing hypotheses mentioned earlier, that X^* and Y^* are strictly convex. A gauge function is a map $\mu : [0,\infty) \to [0,\infty)$ that is continuous, strictly increasing and such that $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$; the map $J_X : X \to X^*$ defined by

$$J_X(x) = \mu(\|x\|_X) \widetilde{J}_X(x) \ (x \in X\backslash\{0\}), \ J_X(0) = 0,$$

is called a duality map on X with gauge function μ. It has the properties that for all $x \in X$,

$$\langle x, J_X(x) \rangle_X = \|J_X\|_{X^*} \|x\|_X, \ \ \|J_X\|_{X^*} = \mu(\|x\|_X).$$

From now on we shall assume that X and Y are equipped with duality maps corresponding to gauge functions μ_X, μ_Y respectively, normalised so that $\mu_X(1) = \mu_Y(1) = 1$. Let M be a closed linear subspace of X. Then if X is strictly convex, so are M and $X \backslash M$; if X^* is strictly convex, so are $(X \backslash M)^*$ and M^0.

To quantify the degree of strict convexity of X its modulus of convexity $\delta_X : [0,2] \to [0,1]$ is introduced: it is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \|x+y\|_X/2 : x, y \in X; \|x\|_X, \|y\|_X \leq 1, \|x-y\|_X \geq \varepsilon\right\}.$$

The space X is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Every uniformly convex space is strictly convex; the converse is false. If M is a closed linear subspace of a uniformly convex space X, then both M and $X \backslash M$ are uniformly convex. We say that X is uniformly smooth if its modulus of smoothness $\rho_X : (0,\infty) \to [0,\infty)$, defined by

$$\rho_X(\varepsilon) = \sup\left\{\frac{\|x+y\|_X + \|x-y\|_X}{2} - 1 : \|x\|_X = 1, \|y\|_X = \varepsilon\right\},$$

has the property that

$$\lim_{\varepsilon \to 0} \rho_X(\varepsilon)/\varepsilon = 0.$$

It turns out that X is uniformly convex (respectively, uniformly smooth) if and only if X^* is uniformly smooth (respectively, uniformly convex). If X is uniformly convex, it is reflexive and has the following useful property: if (x_n) is a sequence in X that converges weakly to $x \in X$ (written $x_n \rightharpoonup x$) and $\|x_n\|_X \to \|x\|_X$, then $\|x_n - x\|_X \to 0$. We refer to [9] II, Chapter 1 for further details and proofs of these claims.

Now we sketch the arguments given in [5] that lead to the representation of the action of a compact map. In addition to the standing hypotheses already made we shall assume that X and Y have strictly convex duals; $T : X \to Y$ is supposed to be linear and compact. The starting point is the elementary assertion that there exists $x_1 \in X$, with $\|x_1\|_X = 1$, such that $\|T\| = \|Tx_1\|_Y$, and that x_1 satisfies the equation

$$T^* \widetilde{J}_Y T x_1 = \nu \widetilde{J}_X x_1, \nu = \|T\|,$$

or equivalently,

$$T^* J_Y T x_1 = \nu_1 J_X x_1, \nu_1 = \|T\| \mu_Y(\|T\|). \tag{2.1}$$

Motivated by the Hilbert space procedure, we set $X_1 = X, M_1 = \text{sp}\,\{J_X x_1\}$ (where sp denotes the linear span), $X_2 = {}^0M_1, N_1 = \text{sp}\,\{J_Y T x_1\}$, $Y_2 = {}^0N_1$ and $\lambda_1 = \|T\|$. Since X_2 and Y_2 are closed subspaces of reflexive spaces they are reflexive. Also, $X_2^* = ({}^0M_1)^*$ is isometrically isomorphic to X_1^*/M_1, from which it follows that X_2^* is strictly convex; the same argument applies to Y_2^*. Because

$$\langle Tx, J_Y T x_1\rangle_Y = \nu_1 \langle x, J_X x_1\rangle_X \quad \text{for all } x \in X,$$

we see that T maps X_2 to Y_2. The restriction T_2 of T to X_2 is thus a compact linear map from X_2 to Y_2, and if it is not the zero operator we can repeat the above argument: there exists $x_2 \in X_2 \backslash \{0\}$ such that

$$\langle T_2 x, J_{Y_2} T_2 x_2 \rangle_{Y_2} = \nu_2 \langle x, J_{X_2} x_2 \rangle_{X_2} \quad \text{for all } x \in X_2,$$

where $\nu_2 = \lambda_2 \mu_Y(\lambda_2)$, $\lambda_2 = \|Tx_2\|_Y = \|T_2\|$. Evidently $\lambda_2 \leq \lambda_1$ and $\nu_2 \leq \nu_1$. Continuing in this way we obtain elements $x_1, ..., x_n$ of X, all with unit norm, subspaces $M_1, ..., M_n$ of X^* and $N_1, ..., N_n$ of Y^*, where

$$M_k = \text{sp } \{J_X x_1, ..., J_X x_k\} \text{ and } N_k = \text{sp } \{J_Y T x_1, ..., J_Y T x_k\}, \ k = 1, ..., n, \tag{2.2}$$

and decreasing families $X_1, ..., X_n$ and $Y_1, ..., Y_n$ of subspaces of X and Y respectively given by

$$X_k = \ {}^0M_{k-1}, Y_k = \ {}^0N_{k-1}, \ k = 2, ..., n. \tag{2.3}$$

For each $k \in \{1, ..., n\}$, T maps X_k into Y_k, $x_k \in X_k$ and, with T_k the restriction of T to X_k, $\lambda_k(T) = \lambda_k = \|T_k\|, \nu_k = \lambda_k \mu(\lambda_k)$, we have

$$\langle T_k x, J_{Y_k} T_k x_k \rangle_{Y_k} = \nu_k \langle x, J_{X_k} x_k \rangle_{X_k} \text{ for all } x \in X_k, \tag{2.4}$$

and so

$$T_k^* J_{Y_k} T_k x_k = \nu_k J_{X_k} x_k. \tag{2.5}$$

In fact, (2.4) is equivalent to

$$\langle Tx, J_Y T x_k \rangle_Y = \nu_k \langle x, J_X x_k \rangle_X \text{ for all } x \in X_k. \tag{2.6}$$

Since $Tx_k \in Y_k = \ {}^0N_{k-1}$, we have

$$\langle Tx_k, J_Y T x_l \rangle_Y = 0 \text{ if } l < k. \tag{2.7}$$

The process stops with λ_n, x_n and X_{n+1} if and only if the restriction of T to X_{n+1} is the zero operator while $T_n \neq 0$. It turns out that $x_i \perp x_k$ in the James sense whenever $i < k$; and for this reason the sequences (λ_i) and (x_i) are called j-eigenvalues and j-eigenvectors respectively of $T^* J_Y T x = \nu J_X x, \ \nu_i = \lambda_i \mu_Y(\lambda_i)$; and (x_i) is said to be j-orthogonal.

If the rank of T is infinite, then the sequence (λ_n) is infinite and converges to 0. For then, since $Tx_n \in {}^0N_{n-1}$,

$$\left\langle Tx_n, \widetilde{J}_Y T x_m \right\rangle_Y = 0 \text{ if } m < n. \tag{2.8}$$

Thus if $m < n$,

$$\begin{aligned} \lim_{k \to \infty} \lambda_k \leq \|Tx_m\|_Y &= \left\langle Tx_m, \widetilde{J}_Y T x_m \right\rangle_Y = \left\langle Tx_m - Tx_n, \widetilde{J}_Y T x_m \right\rangle_Y \\ &\leq \|Tx_m - Tx_n\|_Y \left\| \widetilde{J}_Y T x_m \right\|_{Y^*} = \|Tx_m - Tx_n\|_Y . \end{aligned}$$

Since (x_n) is bounded and T is compact, some subsequence of (Tx_n) must converge and hence the assertion follows. Moreover, if $x \in \cap_{n \in \mathbb{N}} X_n$, then for all $n \in \mathbb{N}$, $\|Tx\|_Y \leq \lambda_n \|x\|_X \to 0$ as $n \to \infty$: hence

$$X_\infty := \cap_{n \in \mathbb{N}} X_n \subset \ \ker\,(T). \tag{2.9}$$

Next, we introduce the family of maps

$$S_k : X \to \mathcal{M}'_{k-1} := \text{ sp } \{x_1, ..., x_{k-1}\} \ (k \geq 2) \tag{2.10}$$

determined by the condition that $x - S_k x \in X_k$ for all $x \in X$. It turns out that S_k is uniquely given by

$$S_k x := \sum_{j=1}^{k-1} \xi_j(x) x_j, \tag{2.11}$$

where

$$\xi_j(x) = \left\langle x - \sum_{i=1}^{j-1} \xi_i(x) x_i, J_X x_j \right\rangle_X \text{ for } j \geq 2, \text{ and } \xi_1(x) = \langle x, J_X x_1 \rangle_X . \tag{2.12}$$

From the uniqueness it follows that $S_k^2 = S_k : S_k$ is a linear projection of X onto $\mathcal{M}'_{k-1}$ and, for each $k \geq 2$, X and X^* have the direct sum decompositions

$$X = X_k \oplus \mathcal{M}'_{k-1}, \quad X^* = M_{k-1} \oplus \left(\mathcal{M}'_{k-1}\right)^0.$$

The following results are established in [5].

Theorem 2.1. *Let X be uniformly convex and X^* strictly convex, and let P_k, P_∞ denote the projections of X onto $X_k, X_\infty := \bigcap_{k\in\mathbb{N}} X_k$, respectively. Then for all $x \in X$,*

$$x = \lim_{k\to\infty} (I - P_k) \sum_{i=1}^{k-1} \xi_i(x) x_i + P_\infty x.$$

Theorem 2.2. *Let Y be uniformly convex and Y^* strictly convex, and let Q_k, Q_∞ denote the projections of Y onto $Y_k, Y_\infty := \bigcap_{k\in\mathbb{N}} Y_k$, respectively. Then*

$$Tx = \lim_{k\to\infty} (I - Q_k) \sum_{i=1}^{k-1} \lambda_i \xi_i(x) y_i + Q_\infty x, \quad y_i = Tx_i / \|Tx_i\|_Y.$$

3. EIGENVALUES OF THE p-LAPLACIAN

Let ν_1 be a j-eigenvalue with corresponding j-eigenfunction x: thus $\|x\|_X = 1, \|T\| = \|Tx\|_Y$ and

$$T^* J_Y T x = \nu_1 J_X x, \quad \nu_1 = \|T\| \mu_Y(\|T\|). \tag{3.1}$$

On setting $x^* := J_X^{-1} x$, (3.1) can be written in the form

$$T^* J_Y T J_X^{-1} x^* = \nu_1 x^*, \quad \nu_1 = \|T\| \mu_Y(\|T\|) \tag{3.2}$$

so that ν_1 is an eigenvalue of the nonlinear operator $T^* J_Y T J_X^{-1} : X^* \to X^*$, with corresponding eigenvector x^*. In the case when X and Y are Hilbert spaces, the natural gauge functions are $\mu_X(t) = \mu_X(t) = t$, so that $\nu_1 = \|T\|^2$ is an eigenvalue of T^*T and hence $\|T\|$ is a singular value of T. We now examine the natural (Carathéodory) and variational ways of defining eigenvalues in examples involving the p-Laplacian, and then compare them with the j-eigenvalues.

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and let $W_p^1(\Omega)$ be the Sobolev space of all real-valued functions $u \in L_p(\Omega)$ all of whose first-order distributional derivatives $D_j u$ also belong to $L_p(\Omega)$. The norm on $W_p^1(\Omega)$ is defined to be

$$\left(\int_\Omega \left\{ |u|^p + \sum_{j=1}^n |D_j u|^p \right\} dx \right)^{1/p}.$$

We take X to be $\overset{0}{W}{}_p^1(\Omega)$, the closure in $W_p^1(\Omega)$ of the set $C_0^\infty(\Omega)$ of all infinitely differentiable functions with compact support in Ω, and define the norm on X by

$$\|u\|_X = \left(\int_\Omega \sum_{j=1}^n |D_j u|^p \, dx \right)^{1/p}. \tag{3.3}$$

Because of the Friedrichs inequality (see [4], Theorem V.3.22), this norm is equivalent to the norm on X inherited from $W_p^1(\Omega)$. Let $Y = L_p(\Omega)$, $T = \mathrm{id}\colon X \to Y$; id is compact. It is plain that both X and Y are reflexive and strictly convex. Obviously Y^* is strictly convex; that the same holds for X^* follows from the observation that $\|\cdot\|_X$ is Gâteaux-differentiable on $X \backslash \{0\}$. Direct verification shows that

$$\widetilde{J}_Y u = \|u\|_p^{-(p-1)} |u|^{p-2} u, \tag{3.4}$$

where $\|\cdot\|_p$ is the usual norm on $L_p(\Omega)$. As for $\widetilde{J}_X$, we claim that

$$\widetilde{J}_X u = - \|u\|_X^{-(p-1)} \Delta_p u \text{ in the sense of distributions}, \tag{3.5}$$

where

$$\Delta_p u = \sum_{j=1}^n D_j \left(|D_j u|^{p-2} D_j u \right), \tag{3.6}$$

corresponding to a version of the p-Laplacian. To verify this, note that for all $u \in X$,

$$\begin{aligned} \left\langle u, - \|u\|_X^{-(p-1)} \Delta_p u \right\rangle_X &= - \|u\|_X^{-(p-1)} \langle u, \Delta_p u \rangle_X \\ &= \|u\|_X^{-(p-1)} \int_\Omega \sum_{j=1}^n D_j u . |D_j u|^{p-2} D_j u dx \\ &= \|u\|_X . \end{aligned}$$

With $\mu_X(t) = \mu_Y(t) = t^{p-1}$, the corresponding duality maps J_X, J_Y are given by

$$J_X(u) = -\Delta_p u, \quad J_Y(u) = |u|^{p-2} u$$

and in (3.2), $\nu_1 = \lambda_1^p$ where $\lambda_1 = \|T\|$. The Euler equation $T^* J_Y T u_1 = \nu_1 J_X u_1$, is equivalent to

$$\int_\Omega \phi \, |u_1|^{p-2} u_1 dx = \lambda_1^p \int_\Omega \sum_{j=1}^n (D_j \phi) \, |D_j u_1|^{p-2} D_j u_1 dx, \quad (\forall \phi \in \overset{0}{W}{}_p^1(\Omega)) \tag{3.7}$$

so that u_1 is a weak solution of the Dirichlet eigenvalue problem

$$-\Delta_p u_1 = \lambda_1^{-p} |u_1|^{p-2} u_1, \; u_1 = 0 \text{ on } \partial\Omega. \tag{3.8}$$

In one dimension, with $\Omega = (a,b), \lambda := \lambda_1^{-p}$, and using the notation $[x]^\alpha := x|x|^{\alpha-1}$, this is of the form

$$-\left([u_1']^{p-1}\right)' = \lambda[u_1]^{p-1}. \tag{3.9}$$

with Dirichlet boundary conditions

$$u_1(a) = u_1(b) = 0. \tag{3.10}$$

If (3.9) and (3.10) are satisfied for some non-zero u_1 which is such that u_1 and $[u_1']^{p-1}$ are absolutely continuous on (a,b), then λ is said to be an eigenvalue in the Carathéodory sense with eigenvector u_1.

The kth Lusternik-Schnirelmann eigenvalue μ_k of (3.9) and (3.10) is defined in the following variational sense. Let

$$M := \{u \in \overset{0}{W}{}_p^1(\Omega) : \|u\|_{L_p(a,b)} = 1.\}$$

Then

$$\mu_k = \inf_{A \in \mathcal{F}_{k+1}} \sup_{u \in A} \left\{ \|u\|_{\overset{0}{W}{}_p^1(\Omega)} \right\}, \tag{3.11}$$

where

$$\mathcal{F}_m := \{A \in \mathcal{A} : \gamma(A) \geq m\},$$

$\mathcal{A}$ is the set of all non-empty, compact and symmetric (i.e. $A = -A$) subsets of M and $\gamma(A)$ is the Krasnoselskij genus of A defined by

$$\gamma(A) := \inf\{j \in \mathbb{N} : \exists \text{ a continuous, odd function } f : A \to \mathbb{R}^j \setminus \{0\}\}.$$

In [2] it is shown that if λ is a Lusternik-Schnirelmann eigenvalue, there exists a non-zero $u \in \overset{0}{W}{}_p^1(\Omega)$ satisfying the weak form of (3.9) and (3.10), namely,

$$\int_a^b \{[u']^{p-1}\phi' - \lambda[u]^{p-1}\phi\}dt = 0, \quad \forall \phi \in \overset{0}{W}{}_p^1(a,b).$$

Since $\overset{0}{W}{}_p^1(a,b)$ is continuously embedded in $C(\overline{\Omega})$, it follows that $u(a) = u(b) = 0$. Furthermore, u is absolutely continuous (see [4], Corollary V.3.12). On setting

$$v(t) = -\lambda \int_0^t [u]^{p-1} dx.$$

we see that v is absolutely continuous, $[u']^{p-1} = v$ and (3.9) is satisfied. It follows that λ is a Carathéodory eigenvalue and u is an associated eigenvector. In fact it is shown in [2], Theorem 5.1, that the converse is also true, so that the Lusternik-Schnirelmann and Carathéodory eigenvalues coincide.

Let $a = 0, b = 1$. Then, the eigenvalues and corresponding eigenvectors of (3.9)-(3.10) are (see [3])

$$\lambda_n = (p-1)(n\pi_p)^p, \quad u_n(t) = \sin_p(n\pi_p t), \quad (n \in \mathbb{N}), \tag{3.12}$$

where

$$\pi_p = \frac{2\pi}{p \sin_p(\pi/p)}$$

and $\sin_p$ is the function defined on $[0, \pi_p/2]$ to be the inverse of the function $F_p : [0,1] \to \mathbb{R}$ given by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt,$$

extended to $[0, \pi_p]$ so it is symmetric about $\pi_p/2$ and then extended to the whole of the real line in a natural way to be an odd, $2\pi_p$-periodic function.

It is natural to ask if the eigenvectors u_n in (3.12) (normalised in $X = \overset{0}{W}{}^1_p(0,1)$) coincide with the j-eigenfunctions x_n arising in the general theory outlined in Section 2 above. This would mean that for all $m \le n-1$,

$$\begin{aligned} 0 &= \langle u_n, J_X u_m \rangle_X \\ &= \langle u_n, -\Delta_p u_m \rangle_X = \lambda_m \langle u_n, [u_m]^{p-1} \rangle_X \\ &= (p-1)(m\pi_p^p) \int_0^1 \sin_p(n\pi_p t)[\sin_p m\pi_p t]^{p-1} dt \\ &= (p-1)(m\pi_p^p) \int_0^1 \sin_p(n\pi_p t) |\sin_p(m\pi_p t)|^{p-2} \sin_p(m\pi_p t) dt. \end{aligned} \tag{3.13}$$

Another example in which Lusternik-Schnirelmann and Carathéodory eigenfunctions coincide is provided by the Hardy operator $T : L_p(0,1) \to L_p(0,1),\ 1 < p < \infty$, given by

$$Tf(x) := \int_0^x f(t) dt, \quad x \in (0,1). \tag{3.14}$$

This operator is compact and T^* is given by

$$T^* g(x) = \int_x^1 g(t) dt, \quad x \in (0,1).$$

Consider now the equation

$$T^* \tilde{J}_Y T x = \nu \tilde{J}_X, \quad \nu = \|Tx\|_Y, \quad \|x\|_X = 1, \tag{3.15}$$

which is the Euler equation for maximising $\|Tx\|_Y$ under the restriction $\|x\|_X = 1$. Then, with $X = Y = L_p(0,1)$ we have that for $f \in L_p(0,1)$, $\tilde{J}_X(f) = \|f\|^{p-1} |f|^{p-2} f$, $\tilde{J}_Y(Tf) = \|Tf\|^{-(p-1)} |Tf|^{p-2} Tf$, and, on setting $g(t) = Tf(t)$ and $\|\cdot\|_p := \|\cdot\|_X$, (3.15) becomes

$$\int_t^1 |g|^{p-2} g ds = \|g\|_p^p |g'(t)|^{p-2} g'(t),$$

whence

$$\begin{aligned} -([g']^{p-1})' &= \lambda [g]^{p-1}, \quad \lambda = \|g\|_Y^{-p} \\ g(0) &= g'(1) = 0; \end{aligned} \tag{3.16}$$

see [1], (1.3), where a more general form of (3.14) is considered.

From [3] it follows that the eigenvalues and eigenfunctions of (3.16) are

$$\lambda_n = [(n-1/2)\pi_p]^p(p-1), \quad u_n(t) = \frac{1}{(n-1/2)\pi_p}\sin_p[(n-1/2)\pi_p t], \quad n \in \mathbb{N}. \quad (3.17)$$

It is also proved in [1] and [7] that $\lambda_n^{-1/p}$ is equal to the nth approximation number of T. In fact, this result is established in [1] for the generalised Hardy operator

$$T_{u,v}f(x) := v(x)\int_0^x u(t)f(t)dt, \quad f \in L_p(0,1),$$

under the assumptions $u \in L_{p'}(0,1), v \in L_p(0,1)$, which ensure that T is compact. An asymptotic formula for the eigenvalues of $T_{u,v}$ is also obtained in [1].

From [7], Theorem 5.2, it follows that the Lusternik-Schnirelmann and Carathéodory eigenvalues of (3.16) coincide. Numerical computations given in [7] moreover suggest that the Lusternik-Schnirelmann eigenfunctions of T do not have the j-orthogonality property and so are distinct from the j-eigenfunctions.

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