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BOUNDEDNESS, PERIODIC SOLUTIONS AND STABILITY IN NEUTRAL FUNCTIONAL DELAY EQUATIONS WITH APPLICATION TO BERNOULLI TYPE DIFFERENTIAL EQUATIONS

YOUSSEF N. RAFFOUL AND MEHMET ÜNAL

Department of Mathematics, University of Dayton
Dayton, OH 45469-2316 USA

Department of Mathematics
Çanakkale Onsekiz Mart University
Çanakkale, Turkey
E-mail: yraffoul1@udayton.edu, munal@comu.edu.tr

ABSTRACT. We use Two fixed point theorems to prove the existence of Bounded solution, periodic solution and stability of solutions of the functional neutral differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -a(t)x(t-r_1) + b(t)f(x(t-r_2(t))).$$

Then we apply our results to the neutral Bernoulli differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -a(t)x(t-r_1) + b(t)x^{\frac{1}{3}}(t-r_2(t)).$$

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1. INTRODUCTION

Motivated by the papers [2], [23], [24], [25] and the references therein, we consider the functional neutral differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -a(t)x(t-r_1) + b(t)f(x(t-r_2(t))). \tag{1.1}$$

where $a : [0,\infty) \to (0,\infty)$, $f : \mathbf{R} \to \mathbf{R}$ and $b : [0,\infty) \to \mathbf{R}$ are continuous, $\tau, r_1 \geq 0$ and, $r_2 : [0,\infty) \to [0,\gamma]$ for $\gamma > 0$ and constant c. In addition, we apply our obtained results to the neutral multiple delays Bernoulli differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -a(t)x(t-r_1) + b(t)x^{\frac{1}{3}}(t-r_2(t)). \tag{1.2}$$

In the case $c = 0$ equation (1.2) reduces to the one's in [2]. In [25], the author proved the existence of positive periodic solutions in neutral nonlinear equation with functional delay of the form

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t-g(t))$$

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which arises in a food-limited population models (see [4], [5]–[8], [10], [11], [12]), [18] and blood cell models, (see [1] and [22]). In addition, the author in [26]) used Krasnosel'skiĭ's fixed point theorem and proved the existence of positive periodic solution of the neutral delay differentia equation

$$\frac{d}{dt}[x(t) - ax(t-\tau)] = r(t)x(t) - f(t, x(t-\tau)).$$

For system (1.1), there may be a stable equilibrium point of the population. In the case the equilibrium point becomes unstable, there may exist a nontrivial periodic solution. Then the oscillation of solutions occurs. The existence of such stable periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of qualitative theory of differential equations. For historical background, basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models we refer the interested reader to [14], [15], [16], [17], [19], [21], [27] and [30].

2. BOUNDEDNESS

In order to transfer our original problem into an integral equation problem, we put (1.1) into the form

$$\begin{aligned} \frac{d}{dt}[x(t) - cx(t-\tau)] &= -a(t+r_1)[x(t) - cx(t-\tau)] - ca(t+r_1)x(t-\tau) \\ &\quad + b(t)f(x(t-r_2(t)) + \frac{d}{dt}\int_{t-r_1}^{t} a(s+r_1)x(s)ds. \end{aligned} \tag{2.1}$$

Let $r \geq \max\{\tau, r_1, \gamma\}$ and assume the existence of an initial continuous function $\psi : [-r, 0] \to \mathbf{R}$ such that $x(t) = x(t, 0, \psi)$ for $t \geq 0$ and $x(t) = \psi(t)$ on $[-r, 0]$. A multiplication of both sides of (2.1) with the integrating factor $e^{\int_0^t a(s+r_1)ds}$ followed by integration from 0 to t yields

$$\begin{aligned} x(t) - cx(t-\tau) &= [x(0) - cx(-\tau)]e^{-\int_0^t a(u+r_1)du} \\ &- c\int_0^t a(s+r_1)x(s-\tau)e^{-\int_s^t a(u+r_1)du}ds + \int_0^t b(s)f(x(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds \\ &+ \int_0^t e^{-\int_s^t a(u+r_1)du}[\frac{d}{ds}\int_{s-r_1}^{s} a(u+r_1)x(u)du]ds. \end{aligned}$$

Finally an integration by parts in the last term of the above expression gives

$$\begin{aligned} x(t) = cx(t-\tau) &+ [\psi(0) - c\psi(-\tau)]e^{-\int_0^t a(u+r_1)du} \\ &- c\int_0^t a(s+r_1)x(s-\tau)e^{-\int_s^t a(u+r_1)du}ds \\ &+ \int_0^t b(s)f(x(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds \end{aligned}$$

$$+\int_{t-r_1}^{t} a(u+r_1)x(u)du - e^{-\int_0^t a(u+r_1)du}\int_{-r_1}^{0} a(u+r_1)\psi(u)du$$
$$-\int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du}\int_{s-r_1}^{s} a(u+r_1)x(u)duds. \tag{2.2}$$

Krasnosel'skiĭ fixed point theorem has been extensively used in differential and functional differential equations, by Burton in proving the existence of periodic solutions. Also, Burton was the first to use the theorem to obtain stability results regarding solutions of integral equations and functional differential equations. For a collection of different type of results we refer the reader to [3] and the references therein. The author is unaware of any results regarding the use of Krasnosel'skiĭ to prove the existence of a positive periodic solution.

Theorem 2.1 (Krasnosel'skiĭ). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathbb{S}, \|\cdot\|\big)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{S}$ such that*

(i) *A is compact and continuous,*
(ii) *B is a contraction mapping.*
(iii) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

We are ready to use the above mentioned fixed point theorem to prove that all solutions of (1.1) are bounded. We are emphasizing boundedness over stability because the simple nonlinear Bernoulli equation

$$x' = -2x + x^{1/3},$$

is unstable but all its solutions are bounded. In preparation for our main theorem, we let $h : [-r, \infty) \to [1, \infty)$ be any strictly increasing and continuous function such that $h(-r) = 1$ and $h(s) \to \infty$ as $t \to \infty$. For any continuous function Let $\mathbb{S}$ be the space of all continuous functions $\phi : [-r, \infty) \to \mathbf{R}$. Then $(\mathbb{S}, |\cdot|_h)$ is a Banach space where for $\phi \in \mathbb{S}$,

$$|\phi|_h = \sup_{t\geq -r}\Big|\frac{\phi(t)}{h(t)}\Big| < \infty.$$

The next proposition is useful in proving compactness. For more on compactness we refer the reader to p. 169 of [2].

Proposition 2.2. *For positive constants M and K the set*

$$\mathbb{L} = \{f \in \mathbb{S} : |f(t)| \leq M, \text{ on } [-r,\infty), \text{ and } |f(u) - f(v)| \leq K|u - v|\}$$

is compact.

Let L be a positive constant to be determined later and define

$$\mathbb{M} = \{\phi \in \mathbb{S} : |\phi(t)| \leq L, \text{ for } -r \leq t < \infty, \text{ and } \phi(t) = \psi(t), \text{ on } [-r, 0]\}.$$

In preparation for our main results we define our mappings. Let $\phi \in \mathbb{M}$ and define the mappings $\Gamma, \Upsilon : \mathbb{M} \to \mathbb{M}$ by

$$(\Gamma\phi)(t) = \int_0^t b(s)f(\phi(s - r_2(s))e^{-\int_s^t a(u+r_1)du}ds \tag{2.3}$$

and

$$\begin{aligned}(\Upsilon\phi)(t) =& - c\phi(t-\tau) + [\psi(0) - c\psi(-\tau)]e^{-\int_0^t a(u+r_1)du} \\ &- c\int_0^t a(s+r_1)\phi(s-\tau)e^{-\int_s^t a(u+r_1)du}ds \\ &+ \int_{t-r_1}^t a(u+r_1)\phi(u)du - e^{-\int_0^t a(u+r_1)du}\int_{-r_1}^0 a(u+r_1)\psi(u)du \\ &- \int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du}\int_{s-r_1}^s a(u+r_1)\phi(u)duds. \end{aligned}\tag{2.4}$$

Since h is strictly increasing and $h : [-r, \infty) \to [1, \infty)$ we have for $0 \le s \le t$, $1 \le h(s) \le h(t) \Rightarrow \frac{1}{h(t)} \le \frac{h(s)}{h(t)} \le 1$. We begin with the following lemma.

Lemma 2.3. *Suppose there positive constants constant β, λ and μ such that*

$$\sup_{t\ge 0}\frac{|b(t)|}{a(t+r_1)} \le \beta \tag{2.5}$$

and for $n = 1, 3, 5, \cdot, \cdot$

$$|f(x)| \le \lambda\|x\|^{\frac{1}{n}}, \textit{ for} |x| \le \mu. \tag{2.6}$$

In addition, if

$$\begin{aligned}&\beta\lambda L^{\frac{1}{n}} + 2|c|L + \|\psi\|e^{-\int_0^t a(u+r_1)du}\left(1 + |c| + \int_{-r_1}^0 a(u+r_1)du\right) \\ &+ 2L\sup_{t\ge 0}\int_{t-r_1}^t a(u+r_1)du < L,\end{aligned}\tag{2.7}$$

for a sufficiently small continuous initial function ψ. Then for $\phi, \eta \in \mathbb{M}$, we have $\Gamma\phi + \Upsilon\eta \in \mathbb{M}$.

Proof. Let $\|.\|$ be the supremum norm on $[-r, \infty)$ of $\phi \in \mathbb{M}$, if ϕ is bounded. Thus, by (2.5) we have that $\phi \in \mathbb{M}$,

$$\begin{aligned}&\Big|\int_0^t b(s)f(\phi(s - r_2(s))e^{-\int_s^t a(u+r_1)du}ds\Big|_h \\ &\le \int_0^t |b(s)||f(\phi(s - r_2(s))|e^{-\int_s^t a(u+r_1)du}/h(t)ds \\ &\le \int_0^t |b(s)||f(\phi(s - r_2(s))|e^{-\int_s^t a(u+r_1)du}h(s)/h(t)ds \\ &\le \int_0^t |b(s)||f(\phi(s - r_2(s))|e^{-\int_s^t a(u+r_1)du}h(t)/h(t)ds\end{aligned}$$

$$
\begin{aligned}
&\le \beta\lambda L^{\frac{1}{n}} \int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du} ds \\
&\le \beta\lambda L^{\frac{1}{n}} \int_0^t \frac{d}{ds}\big(e^{-\int_s^t a(u+r_1)du}\big) ds \\
&= \beta\lambda L^{\frac{1}{n}} \big(1 - e^{-\int_0^t a(u+r_1)du}\big) \\
&\le \beta\lambda L^{\frac{1}{n}}. \qquad (2.8)
\end{aligned}
$$

Similarly, for $\eta \in \mathbb{M}$,

$$
\begin{aligned}
&\Big| \int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du} \int_{s-r_1}^s a(u+r_1)\phi(u) du ds\Big|_h \\
&\le \|\eta\| \int_0^t \frac{d}{ds}\big(e^{-\int_s^t a(u+r_1)du}\big) ds \sup_{t\ge 0} \int_{t-r_1}^t a(u+r_1)du \\
&\le L \sup_{t\ge 0} \int_{t-r_1}^t a(u+r_1)du. \qquad (2.9)
\end{aligned}
$$

Thus, for $\phi, \eta \in \mathbb{M}$, we have by (2.8) and (2.9)

$$
\begin{aligned}
\big|\Gamma\phi + \Upsilon\eta\big|_h &\le \beta\lambda L^{\frac{1}{n}} + |c|L + \|\psi\| e^{-\int_0^t a(u+r_1)du}\big(1+|c|\big) + |c|L \\
&\quad + L \sup_{t\ge 0} \int_{t-r_1}^t a(u+r_1)du + \|\psi\| e^{-\int_0^t a(u+r_1)du} \int_{-r_1}^0 a(u+r_1)du \\
&\quad + L \sup_{t\ge 0} \int_{t-r_1}^t a(u+r_1)du \\
&= \beta\lambda L^{\frac{1}{n}} + 2|c|L + \|\psi\| e^{-\int_0^t a(u+r_1)du}\big(1+|c| + \int_{-r_1}^0 a(u+r_1)du\big) \\
&\quad + 2L \sup_{t\ge 0} \int_{t-r_1}^t a(u+r_1)du \\
&\le L.
\end{aligned}
$$

□

Theorem 2.4. *Suppose there is a positive constant K such that if*

$$
|t_2 - t_1| \le 1, \text{ then } \Big| \int_{t_1}^{t_2} a(u+r_1)du\Big| \le K|t_2 - t_1|. \qquad (2.10)
$$

Also assume (2.5), (2.6) and (2.7) hold. Then $\Gamma\mathbb{M}$ resides is a compact set in the space $(\mathbb{S}, |\cdot|_h)$.

Proof. Let $\phi \in \mathbb{M}$. If $0 \le t_1 < t_2 < t_1 + 1$, then

$$
\begin{aligned}
&|(\Gamma\phi)(t_2) - (\Gamma\phi)(t_1)| \\
&\le \Big| \int_0^{t_2} b(s) f(\phi(s - r_2(s))e^{-\int_s^t a(u+r_1)du} ds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{t_1} b(s)f(\phi(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds\Big| \\
&= \Big| \int_{t_1}^{t_2} b(s)f(\phi(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds \\
&\quad + \Big| \int_0^{t_1} b(s)\big[e^{-\int_s^{t_2} a(u+r_1)du} - e^{-\int_s^{t_1} a(u+r_1)du}\big] f(\phi(s-r_2(s))ds\Big| \\
&\le \beta \int_{t_1}^{t_2} a(s+r_1)ds \\
&\quad + \lambda L^{\frac{1}{n}} \int_0^{t_1} |b(s)|\big|e^{-\int_0^{t_2} a(u+r_1)du} - e^{-\int_0^{t_1} a(u+r_1)du}\big| e^{\int_0^s a(u+r_1)du}ds.
\end{aligned}
$$

This shows that by the Proposition, $\Gamma\mathbb{M}$ resides in a compact set. □

Then there is a solution $x(t,0,\psi)$ of (1.1) on $[0,\infty)$ with $|x(t,0,\psi) < L$.

3. EXISTENCE OF PERIODIC SOLUTIONS

Next we consider a perturbed version of equation (1.1) of the form

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -a(t)x(t-r_1) + b(t)f(x(t-r_2(t)) + p(t), \tag{3.1}$$

where $p : \mathbf{R} \to \mathbf{R}$ is continous and there exist a $T > 0$ such that

$$r_2(t+T) = r_2(t),\ a(t+T) = a(t),\ b(t+T) = b(t), \text{ and } p(t+T) = p(t). \tag{3.2}$$

We note that due to the presence of the term $cx(t-\tau)$, once the equation is inverted then once will face with the term $cx(t-\tau)$ which may not define a compact mapping. Thus, Krasnosel'skiĭ fixed point theorem becomes ideal to prove the existence of periodic solution. In [2] the authors considered a simpler form of (3.1) and used Schaefer fixed point theorem to prove the existence of periodic solution. Again Schaefer fixed point theorem will not work for our equation.

Define $P_T = \{\phi : C(\mathbf{R},\mathbf{R}), \phi(t+T) = \phi(t)\}$ where $C(\mathbf{R},\mathbf{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x(t)\| = \max_{t\in[0,T]} |x(t)| = \max_{t\in\mathbb{R}} |x(t)|.$$

Let

$$\int_0^T a(s)ds \neq 0. \tag{3.3}$$

Lemma 3.1. *Suppose* (3.2) *and* (3.3) *hold. If* $x(t) \in P_T$*, then* $x(t)$ *is a solution of* (3.1) *if and only if*

$$x(t) = cx(t-\tau) + \Big(1 - e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\Big\{ -c\int_{t-T}^t a(s+r_1)x(s-\tau)e^{-\int_s^t a(u+r_1)du}ds$$

$$+\int_{t-T}^{t} b(s)f(x(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds$$
$$-\int_{t-T}^{t} a(s+r_1)e^{-\int_s^t a(u+r_1)du}\int_{s-r_1}^{s} a(u+r_1)x(u)duds$$
$$+\int_{t-T}^{t} e^{-\int_s^t a(u+r_1)du}p(s)ds.\Big\}$$
$$+\int_{t-r_1}^{t} a(u+r_1)x(u)du \tag{3.4}$$

Proof. Let $x(t) \in P_T$ be a solution of A multiplication of both sides of (2.1) with the integrating factor $e^{\int_0^t a(s+r_1)ds}$ followed by integration from $t-T$ to t yields

$$\begin{aligned} x(t)\Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)e^{\int_0^t a(u+r_1)du} &= c\Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)e^{\int_0^t a(u+r_1)du}x(t-\tau) \\ &- c\int_{t-T}^{t} a(s+r_1)x(s-\tau)e^{\int_0^s a(u+r_1)du}ds \\ &+ \int_{t-T}^{t} b(s)f(x(s-r_2(s))e^{\int_0^s a(u+r_1)du}ds \\ &+ \int_{t-T}^{t} e^{\int_0^s a(u+r_1)du}[\frac{d}{ds}\int_{s-r_1}^{s} a(u+r_1)x(u)du]ds \\ &+ \int_{t-T}^{t} e^{\int_0^s a(u+r_1)du}p(s)ds. \end{aligned} \tag{3.5}$$

Let

$$U = e^{\int_0^s a(u+r_1)du}$$

and

$$dV = \frac{d}{ds}\int_{s-r_1}^{s} a(u+r_1)x(u)du$$

we obtain

$$\begin{aligned} &\int_{t-T}^{t} e^{\int_0^s a(u+r_1)du}[\frac{d}{ds}\int_{s-r_1}^{s} a(u+r_1)x(u)du]ds \\ &= e^{\int_0^t a(u+r_1)du}\int_{t-r_1}^{t} a(u+r_1)x(u)du \\ &\quad - e^{\int_0^{t-T} a(u+r_1)du}\int_{t-T-r_1}^{t-T} a(u+r_1)x(u)du \\ &\quad - \int_{t-T}^{t} a(s+r_1)e^{\int_0^s a(u+r_1)du}\int_{s-r_1}^{s} a(u+r_1)x(u)duds \\ &= e^{\int_0^t a(u+r_1)du}\Big(1-e^{\int_{t-T}^t a(s+r_1)ds}\Big)\int_{t-r_1}^{t} a(u+r_1)x(u)du \\ &\quad - \int_{t-T}^{t} a(s+r_1)e^{\int_0^s a(u+r_1)du}\int_{s-r_1}^{s} a(u+r_1)x(u)duds \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) and then dividing by $e^{\int_0^t a(u+r_1)du}\Big(1-e^{\int_{t-T}^t a(s+r_1)ds}\Big)$ gives (3.4). This completes the proof. □

Let L be a positive constant and define the set

$$\mathbb{M} = \{\phi \in P_T : \|\phi\| \le L\}.$$

Then the set $\mathbb{M}$ is a closed convex and bounded subset of the Banach space P_T. Let $\gamma = \sup_{t\ge 0}\Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\Big|\int_{t-T}^t e^{-\int_s^t a(u+r_1)du}p(s)ds\Big|$.

Theorem 3.2. *Suppose* (2.5), (2.6), (2.10), *and* (3.2) *hold. In addition, if*

$$2|c|L + \beta\lambda L^{\frac{1}{n}} + 2L\sup_{t\ge 0}\int_{t-r_1}^t a(u+r_1)du + \gamma < L, \tag{3.7}$$

then $x(t)$ is a solution of (3.1) *in* $\mathbb{M}$.

Proof. It is clear from (3.7) that

$$2|c| + 2\sup_{t\ge 0}\int_{t-r_1}^t a(u+r_1)du < 1.$$

Let $\phi \in \mathbb{M}$ and define the mappings $\Gamma, \Upsilon : \mathbb{M} \to \mathbb{M}$ by

$$(\Gamma\phi)(t) = \Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\int_{t-T}^t b(s)f(x(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds$$

and

$$\begin{aligned}(\Upsilon\phi)(t) = {} & cx(t-\tau) + \Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\Big\{-c\int_{t-T}^t a(s+r_1)x(s-\tau)e^{-\int_s^t a(u+r_1)du}ds \\ & - \int_{t-T}^t a(s+r_1)e^{-\int_s^t a(u+r_1)du}\int_{s-r_1}^s a(u+r_1)x(u)duds \\ & + \int_{t-T}^t e^{-\int_s^t a(u+r_1)du}p(s)ds.\Big\} \\ & + \int_{t-r_1}^t a(u+r_1)x(u)du.\end{aligned}$$

Then, we note that

$$\begin{aligned}&\Big|\Big(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\int_{t-T}^t b(s)f(x(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds\Big| \\ &\le \beta\lambda L^{\frac{1}{n}}(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}\int_{t-T}^t [\frac{d}{ds}e^{-\int_s^t a(u+r_1)du}]ds \\ &= \beta\lambda L^{\frac{1}{n}}(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big)^{-1}(1-e^{-\int_{t-T}^t a(s+r_1)ds}\Big) \\ &= \beta\lambda L^{\frac{1}{n}}.\end{aligned}$$

The rest of the proof of the theorem follows along the lines of the proofs of Lemma 2.1 and Theorem 2.2. □

4. POSITIVE PERIODIC SOLUTIONS

In this section we obtain necessary conditions for the the existence of a positive periodic solution of (3.1). As a consequence of conditions (3.2) and (4.2) there are positive constants m_1^*, m_2^*, p_1, and p_2 so that

$$m_1^* = \inf_{t\geq 0} \int_{t-r_1}^{t} a(u+r_1)du \text{ and } m_2^* = \sup_{t\geq 0} \int_{t-r_1}^{t} a(u+r_1)du,$$

and

$$p_1^* \leq \Big(1 - e^{-\int_{t-T}^{t} a(s+r_1)ds}\Big)^{-1} \int_{t-T}^{t} e^{-\int_s^t a(u+r_1)du} p(s)ds \leq p_2^*. \tag{4.1}$$

We modify condition (2.5) by assuming the existence of positive constants α and β so that

$$0 \leq \alpha\; a(t+r_1) \leq b(t) \leq \beta\; a(t+r_1). \tag{4.2}$$

In order to show one of the mapping is a contraction, we must require that

$$|c| < 1. \tag{4.3}$$

As a consequence, we are led to consider to consider the two case; $0 \leq c < 1$, and $-1 < c \leq 0$. For some nonnegative constant L and a positive constant K we define the set

$$\mathbb{M} = \{\phi \in P_T : L \leq \|\phi\| \leq K\},$$

which is a closed convex and bounded subset of the Banach space P_T. Assume $0 \leq c < 1$. Then there are nonnegative constants m_1, m_2 such that

$$0 \leq m_1 \leq c \leq m_2 < 1. \tag{4.4}$$

In addition, we assume that for all $u \in \mathbb{R}$ and $0 \leq L \leq \rho \leq K$

$$\begin{aligned} 0 &\leq \frac{L(1-m_1-m_1^*) + K(m_2+m_2^*) - p_1^*}{\alpha} \\ &\leq f(u,\rho) \leq \frac{K(1-m_2-m_2^*) + L(m_1+m_1^*) - p_2^*}{\beta}. \end{aligned} \tag{4.5}$$

To apply Theorem 2.1, we will need to construct two mappings; one is contraction and the other is compact. Thus, we set the map $\mathbf{A} : \mathbb{M} \to P_T$

$$\begin{aligned} (\mathbf{A}\phi)(t) = \Big(1 - e^{-\int_{t-T}^{t} a(s+r_1)ds}\Big)^{-1} \Big\{ &- c \int_{t-T}^{t} a(s+r_1)\phi(s-\tau) e^{-\int_s^t a(u+r_1)du} ds \\ &+ \int_{t-T}^{t} b(s) f(\phi(s-r_2(s)) e^{-\int_s^t a(u+r_1)du} ds \\ &- \int_{t-T}^{t} a(s+r_1) e^{-\int_s^t a(u+r_1)du} \int_{s-r_1}^{s} a(u+r_1)\phi(u)duds \\ &+ \int_{t-T}^{t} e^{-\int_s^t a(u+r_1)du} p(s)ds. \Big\} \end{aligned}$$

$$+\int_{t-r_1}^{t} a(u+r_1)\phi(u)du$$

In a similar way we set the map $\mathbf{B} : \mathbb{M} \to P_T$

$$(\mathbf{B}\varphi)(t) = c\varphi(t-\tau), t \in \mathbb{R}.$$

Theorem 4.1. *Assume* (2.10), (3.2), (4.2), (4.3) *and* (4.5) *with* $0 \le c < 1$ *hold. If*

$$\int_{t-T}^{t} a(s+r_1)ds > 0 \tag{4.6}$$

then (3.1) *has a positive periodic solution* z *satisfying* $0 \le L \le z \le K$.

Proof. Let $\varphi \in \mathbb{M}$. Then by (4.3), $\mathbf{B}$ defines a contraction mapping under the supremum norm. Moreover, for $\varphi, \phi \in P_T$, the mappings $\mathbf{B}$ and $\mathbf{A}$ are periodic. Showing $\mathbf{A}$ is compact is similar to the previous proof and hence we omit it. Left to show that for $\varphi, \phi \in \mathbb{M}$ implies $\mathbf{B}\varphi + \mathbf{B}\phi \in \mathbb{M}$. Let $\varphi, \phi \in \mathbb{M}$, then

$$\begin{aligned}
(\mathbf{B}\varphi)(t) + (\mathbf{A}\psi)(t) &= c\varphi(t-\tau) \\
&+ \left(1 - e^{-\int_{t-T}^{t} a(s+r_1)ds}\right)^{-1} \Big\{ -c\int_{t-T}^{t} a(s+r_1)\phi(s-\tau)e^{-\int_s^t a(u+r_1)du}ds \\
&+ \int_{t-T}^{t} b(s)f(\phi(s-r_2(s))e^{-\int_s^t a(u+r_1)du}ds \\
&+ \int_{t-r_1}^{t} a(u+r_1)\phi(u)du \\
&\le m_2K - m_1L + \beta\frac{K(1-m_2-m_2^*) + L(m_1+m_1^*) - p_2^*}{\beta} \\
&- m_1^*L + p_2^* + m_2^*K \\
&\le K.
\end{aligned}$$

This implies that

$$\|\mathbf{B}\varphi) + (\mathbf{A}\psi)\| \le K.$$

On the other hand,

$$\begin{aligned}
(\mathbf{B}\varphi)(t) + (\mathbf{A}\psi)(t) &\ge m_1L - m_2^*K + \alpha\frac{L(1-m_1-m_1^*) + K(m_2+m_2^*) - p_1^*}{\alpha} \\
&\quad - m_2K + p_1^* + m_1^*L \\
&\ge L.
\end{aligned}$$

This implies that

$$\|\mathbf{B}\varphi) + (\mathbf{A}\psi)\| \ge L.$$

This shows that $\mathbf{B}\varphi + \mathbf{A}\psi \in \mathbb{M}$. All the hypothesis of Theorem 2.1 are satisfied and therefore equation (3.1) has a periodic solution, say z residing in $\mathbb{M}$. $\square$

For the next theorem we assume $-1 < c \le 0$. Then there are non positive constants m_1, m_2 such that

$$-1 < m_1 \le c \le m_2 \le 0.$$

In addition, we assume that for all $u \in \mathbb{R}$ and $0 \le L \le \rho \le K$

$$\begin{aligned} 0 &\le \frac{L(1 - m_1 - m_1^* + m_2) + m_2^* K - p_2^*}{\beta} \\ &\le f(u, \rho) \le \frac{K(1 - m_2 - m_2^* + m_1) + m_1^* L - p_1^*}{\alpha}. \end{aligned} \tag{4.7}$$

Theorem 4.2. *Assume* (2.10), (3.2), (4.2), (4.3) *and* (4.7) *with* $-1 < c \le 0$.

Then (3.1) *has a positive periodic solution* z *satisfying* $0 \le L \le z \le K$.

Proof. The proof follows along the lines of the proof of Theorem 4.1 and hence we omit. □

Next we display an example in which we show the existence of positive periodic solution.

5. EXAMPLE

The neutral differential equation

$$x'(t) = -\frac{1}{2}\sin^2(t)x(t-\pi) + \frac{1}{2}x'(t-\pi) + \frac{\sin^2(t)/2}{x^2(t-\pi)+1} + \frac{1}{25} \tag{5.1}$$

has a positive π-periodic solution x satisfying

$$0.64 \le x \le 1.$$

Here we have $c = \frac{1}{2}$ and hence we use Theorem 4.1. By looking at the equation we can see that

$$a(t) = b(t) = \frac{1}{2}\sin^2(t),\ f(t, x(t-\pi)) = \frac{1}{x^2(t-\pi)+1}, \quad T = \pi,\ \text{and}\ p(t) = \frac{1}{25}.$$

Then,

$$m_1 = m_2 = \frac{1}{2},\ \text{and}\ \alpha = \beta = 1.$$

Since,

$$e^{\int_{t-\pi}^{t} a(s+r_1)ds} = e^{\int_{t-\pi}^{t} \frac{1}{2}\sin^2(s-\pi)ds} = \pi/4,$$

we have

$$m_1^* = m_2^* = \pi/4.$$

Moreover, after some calculation we arrive at

$$\left(1 - e^{-\int_{t-T}^{t} a(s+r_1)ds}\right)^{-1} \int_{t-T}^{t} e^{-\int_s^t a(u+r_1)du} p(s)ds = \frac{1}{25(1-e^{-\pi/4})} e^{\left(-\pi^2/8 + (\pi/8)\sin(2t)\right)}.$$

As a consequence, we have

$$p_1^* = \frac{1}{25(1-e^{-\pi/4})}e^{-\pi^2/8} = .02141,$$

and

$$p_2^* = \frac{1}{25(1-e^{-\pi/4})}e^{-\pi^2/8+\pi/8} = .031708.$$

By letting $K = 1$, and $L = 0.64$ we see that all the conditions of Theorem 4.1 are satisfied and hence Equation (5.1) has a positive π-periodic solution x satisfying $0.64 \leq x \leq 1$.

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