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EXISTENCE OF LOCAL SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH INTEGRAL CONDITIONS

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ABSTRACT. Conditions are given for the existence of local solutions of the nth order ordinary differential equation, $y^{(n)}+f(x,y,y',\dots,y^{(n-1)})=0$, satisfying the respective Dirichlet and nonlocal integral boundary conditions, $y^{(i-1)}(a)=A_i,\ i=1,\dots,n-1$, and $\int_a^b y(x)dx=A_n$.

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1. INTRODUCTION

In this paper, for fixed $n\ge 2$, our primary consideration is with local solutions of the nth order differential equation,

$$y^{(n)}+f(x,y,y',\dots,y^{(n-1)})=0,\quad a\le x\le b, \tag{1.1}$$

satisfying the respective Dirichlet and nonlocal integral boundary conditions,

$$y^{(i-1)}(a)=A_i,\ i=1,\dots,n-1,\ \text{and}\ \int_a^b y(x)dx=A_n, \tag{1.2}$$

where $f(x,r_1,\dots,r_n):[a,b]\times\mathbb{R}^n\to\mathbb{R}$ is continuous and $A_i\in\mathbb{R}$, $i=1,\dots,n$.

A great deal of recent attention has been given to boundary value problems for ordinary differential equations subject to nonlocal boundary conditions in the form of integral boundary conditions. Results in many of these papers have involved a great variety of methods including Krasnosel'skii cone expansion and compression theory, the contraction mapping principle, the Avery and Peterson multiple fixed point theorem, the Leggett and Williams triple fixed point theorem, Leray-Schauder degree theory, Mawhin coincidence degree theory, and so on. For a few papers applying these methods, in the presence of integral boundary conditions, we cite [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 17, 18].

For this work, we first impose a Lipschitz condition on f, along with conditions in terms of the Lipschitz coefficients on the interval length $b-a$, such that an application of the Banach Fixed Point Theorem [7, 16] yields a unique solution of (1.1),

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(1.2). Following that, the Lipschitz condition on f is removed and replaced by other conditions on the interval length $b-a$ and on the values A_i, $i = 1, \ldots, n$, so that an application of the Schauder-Tychonoff Fixed Point Theorem [7, 16] yields a solution of (1.1), (1.2). In the last section, local solutions are obtained via an application of the Leray-Schauder Nonlinear Alternative [7]. Sections 2 and 3 constitute generalizations of the recent paper [8] dealing with (1.1), (1.2), when $n = 2$.

Each of the arguments involves establishing the existence of fixed points for a completely continuous operator whose kernel is a Green's function. In the next section, a few results are presented for an appropriate Green's function.

2. A GREEN'S FUNCTION

Since the only solution of

$$-y^{(n)} = 0, \quad a \le x \le b \tag{2.1}$$

$$y^{(i-1)}(a) = 0,\ i = 1, \ldots, n-1, \quad \int_a^b y(x)dx = 0, \tag{2.2}$$

is $y(x) \equiv 0$, it follows that there exists a Green's function, $G(x,s)$, for (2.1), (2.2). A direct computation gives that

$$G(x,s) = \begin{cases} \frac{(x-a)^{n-1}}{(n-1)!}\frac{(b-s)^n}{(b-a)^n}, & a \le x \le s \le b, \\ \\ \frac{(x-a)^{n-1}}{(n-1)!}\frac{(b-s)^n}{(b-a)^n} - \frac{(x-s)^{n-1}}{(n-1)!}, & a \le s \le x \le b, \end{cases} \tag{2.3}$$

with the properties

(i) For each fixed $a < s < b$, as a function of x on $[a,s]$ and on $[s,b]$, $-\frac{\partial^n G(x,s)}{\partial x^n} = 0$,

(ii) For each fixed $a < s < b$, $\int_a^b G(x,s)dx = 0$, and as a function of x on $[a,s]$, $\frac{\partial^{i-1}G(a,s)}{\partial x^{i-1}} = 0$, $i = 1, \ldots, n-1$,

(iii) $\frac{\partial^{i-1}G(x,s)}{\partial x^{i-1}}$ is continuous on $[a,b] \times [a,b]$, $i = 1, \ldots, n-1$,

and

(iv) For each fixed $a < s < b$, as a function of x, $\frac{\partial^{n-1}G(x,s)}{\partial x^{n-1}}$ is continuous on $[a,s]$ and on $[s,b]$, and

$$\frac{\partial^{n-1}G(s+,s)}{\partial x^{n-1}} - \frac{\partial^{n-1}G(s-,s)}{\partial x^{n-1}} = -1.$$

Next, let $w(x)$ be the solution of (2.1), (1.2). Then,

$$w(x) = \frac{n}{(b-a)^n}\left\{A_n - \sum_{i=1}^{n-1}\frac{A_i}{i!}(b-a)^i\right\}(x-a)^{n-1} + \sum_{j=0}^{n-2}\frac{A_{j+1}}{j!}(x-a)^j. \tag{2.4}$$

It is immediate that $y \in C^{(n)}[a,b]$ is a solution of (1.1), (1.2) iff $y \in C^{(n-1)}[a,b]$ is a solution of

$$y(x) = w(x) + \int_a^b G(x,s)f(s, y(s), y'(s), \ldots, y^{(n-1)})ds. \tag{2.5}$$

In subsequent sections, we also will have need of the values $\gamma_i > 0$, $i = 1, \ldots, n$, defined by

$$\gamma_i := (b-a)^{i-1-n} \max_{a \le x \le b} \int_a^b \left| \frac{\partial^{i-1} G(x,s)}{\partial x^{i-1}} \right| ds. \tag{2.6}$$

3. UNIQUE LOCAL SOLUTIONS OF (1.1), (1.2)

In this section, we put restrictions on f and the length of the interval $[a, b]$ which are sufficient for the existence of unique solutions of (1.1), (1.2). Use will be made of the constants in (2.6) and the Banach Fixed Point Theorem.

Theorem 3.1. *Let $f(t, r_1, \ldots, r_n) : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ be continuous, and for some $K_i \ge 0$, $i = 1, \ldots, n$, satisfy a Lipschitz condition,*

$$|f(x, r_1, \ldots, r_n) - f(x, s_1, \ldots, s_n)| \le \sum_{i=1}^{n} K_i |r_i - s_i|, \tag{3.1}$$

on $[a, b] \times \mathbb{R}^n$. If

$$\sum_{i=1}^{n} K_i \gamma_i (b-a)^{n-i+1} < 1, \tag{3.2}$$

then, for each $A_i \in \mathbb{R}$, $i = 1, \ldots, n$, the boundary value problem (1.1), (1.2) *has a unique solution on $[a, b]$.*

Proof. Our proof involves an application of the Banach Fixed Point Theorem. We let $\mathcal{M} := C^{(n-1)}[a, b]$, with norm, $\|h\| := \sum_{i=1}^{n} K_i |h^{(i-1)}|_\infty$, for $h \in \mathcal{M}$ and where $|\cdot|_\infty = \max_{a \le x \le b} |\cdot|$.

Next, let $A_i \in \mathbb{R}$, $i = 1, \ldots, n$, be given, and define a mapping $T : \mathcal{M} \to \mathcal{M}$ by

$$(Th)(x) := w(x) + \int_a^b G(x,s) f(s, h(s), h'(s), \ldots, h^{(n-1)}(s)) ds,$$

where $a \le x \le b$, $h \in \mathcal{M}$, $w(x)$ is defined by (2.4), and $G(x, s)$ is the Green's function given in (2.3). In view of (2.5), fixed points of T are solutions of (1.1), (1.2).

And we shall show that T is a contraction with respect to the norm $\|\cdot\|$. So, let $A_i \in \mathbb{R}$ be as above, and let $h, g \in \mathcal{M}$. Then, by the Lipschitz condition (3.1) and the constants in (2.6), we have, for $a \le x \le b$ and $i = 1, \ldots, n$,

$$\begin{aligned} |(Th)^{(i-1)}(x) - ((Tg)^{(i-1)}(x)| &\le \int_a^b \left|\frac{\partial^{i-1}G(x,s)}{\partial x^{i-1}}\right| \big|f(s, h(s), \ldots, h^{(n-1)}(s)) \\ &\quad -f(s, g(s), \ldots, g^{(n-1)}(s))\big|\, ds \\ &\le \int_a^b \left|\frac{\partial^{i-1}G(x,s)}{\partial x^{i-1}}\right| \sum_{j=1}^{n} K_j |h^{(j-1)}(s) - g^{(j-1)}(s)| ds \\ &\le \int_a^b \left|\frac{\partial^{i-1}G(x,s)}{\partial x^{i-1}}\right| \sum_{j=1}^{n} K_j |h^{(j-1)} - g^{(j-1)}|_\infty ds \\ &= \int_a^b \left|\frac{\partial^{i-1}G(x,s)}{\partial x^{i-1}}\right| \|h - g\| ds \\ &\le \gamma_i (b-a)^{n-i+1} \|h - g\|, \end{aligned}$$

so that $|(Th)^{(i-1)} - (Tg)^{(i-1)}|_\infty \le \gamma_i (b-a)^{n-i+1}\|h-g\|$. Consequently,

$$\begin{aligned} \|Th - Tg\| &= \sum_{i=1}^{n} K_i |(Th)^{(i-1)} - (Tg)^{(i-1)}|_\infty \\ &\le \left(\sum_{i=1}^{n} K_i \gamma_i (b-a)^{n-i+1} \right) \|h - g\|. \end{aligned}$$

So by (3.2), $T : \mathcal{M} \to \mathcal{M}$ is a contraction, and by the Banach Fixed Point Theorem, there exists a unique $y \in \mathcal{M}$ such that $Ty = y$, and as such from (2.5), y is the unique solution of (1.1), (1.2). □

4. LOCAL SOLUTIONS OF (1.1), (1.2)

In this section, we remove the Lipschitz condition on f, but impose restrictions on both interval length and boundary conditions so that local solvability can still be established. In particular, we will make use of the Schauder-Tychonoff Fixed Point Theorem to show the existence of local solutions of (1.1), (1.2) in the absence of condition (3.1).

In what follows, let $\mathcal{M} := C^{(n-1)}[x_1, x_2]$, let $N_i > 0$, $i = 1, \ldots, n$, be given and define

$$\mathcal{K} := \{h \in \mathcal{M} \mid |h^{(i-1)}|_\infty \le 2N_i,\ i = 1, \ldots, n\},$$

and where for $h \in \mathcal{M}$, we define $\|h\| := \max_{1 \le i \le n}\{|h^{(i-1)}|_\infty\}$. The conditions of the Schauder-Tychonoff Fixed Point Theorem are satisfied relative to $(\mathcal{M}, \|\cdot\|)$ and $\mathcal{K}$, in that $(\mathcal{M}, \|\cdot\|)$ is a Banach space and $\mathcal{K}$ is closed, bounded and convex.

Theorem 4.1. *Assume $f(x, r_1, \ldots, r_n) : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and let $N_i > 0$, $i = 1, \ldots, n$, be given. Let $Q := \max\{|f(x, r_1, \ldots, r_n)| \mid a \le x \le b, |r_i| \le 2N_i,\ i = 1, \ldots, n\}$. Then, for any $[x_1, x_2] \subseteq [a, b]$, the boundary value problem*

$$y^{(n)} + f(x, y, \ldots, y^{(n-1)}) = 0, \quad x_1 \le x \le x_2, \tag{4.1}$$

$$y^{(i-1)}(x_1) = y_i,\ i = 1, \ldots, n-1, \quad \int_{x_1}^{x_2} y(x)dx = y_n, \tag{4.2}$$

has a solution, provided $x_2 - x_1 \leq \delta(N_1, \ldots, N_n) := \min_{1\leq i\leq n} \left\{ \sqrt[n-i+1]{\frac{N_i}{\gamma_i Q}} \right\}$, *and for* $i = 1, \ldots, n$, $\max_{x_1\leq x\leq x_2} \left| w^{(i-1)}(x) \right| \leq N_i$, *where* $w(x)$ *is the solution of* $-w^{(n)} = 0$ *satisfying* $w^{(i-1)}(x_1) = y_i,\ i = 1, \ldots, n-1$, *and* $\int_{x_1}^{x_2} w(x)dx = y_n$.

Proof. Let $N_i > 0$, $i = 1, \ldots, n$, be given, let $a \leq x_1 \leq x_2 \leq b$, with $x_2 - x_1 \leq \delta(N_1, \ldots, N_n)$, let $y_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $w(x)$ satisfy the conditions of the statement of the theorem, and let $(\mathcal{M}, \|\cdot\|)$ and $\mathcal{K}$ be as above.

Define the mapping $T : \mathcal{M} \to \mathcal{M}$ by

$$(Th)(x) := w(x) + \int_{x_1}^{x_2} G(x,s) f(s, h(s), \ldots, h^{(n-1)}(s))ds,$$

where $G(x,s)$ is the Green's function in (2.3) relative to the interval endpoints x_1 and x_2. Moreover, by (2.5) to obtain a solution of (4.1), (4.2), it suffices to establish a fixed point of T in $\mathcal{K}$. We proceed via a few claims.

Claim 1. T maps $\mathcal{K}$ into $\mathcal{K}$.
Choose $h \in \mathcal{K}$. Then, for $x_1 \leq x \leq x_2$ and $i = 1, \ldots, n$,

$$\begin{aligned}
|(Th)^{(i-1)}(x)| &\leq |w^{(i-1)}(x)| + \int_{x_1}^{x_2} \left| \frac{\partial^{i-1} G(x,s)}{\partial x^{i-1}} \right| |f(s, h(s), \ldots, h^{(i-1)}(s))| ds \\
&\leq N_i + \int_{x_1}^{x_2} \left| \frac{\partial^{i-1} G(x,s)}{\partial x^{i-1}} \right| Q ds \\
&\leq N_i + Q\gamma_i (x_2 - x_1)^{n-i+1} \\
&\leq N_i + Q\gamma_i \left(\sqrt[n-i+1]{\frac{N_i}{\gamma_i Q}} \right)^{n-i+1} \\
&= 2N_i.
\end{aligned}$$

So, $|(Th)^{(i-1)}|_\infty \leq 2N_i$, and $Th \in \mathcal{K}$.

Claim 2. T is continuous on $\mathcal{K}$.
Let the compact subset $H \subseteq [a,b] \times \mathbb{R}^n$ be defined by $H := [a,b] \times \prod_{i=1}^n [-2N_i, 2N_i]$. Then, given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for $(\eta_1, r_1, \ldots, r_n), (\eta_2, s_1, \ldots, s_n) \in H$, with $|\eta_1 - \eta_2| < \delta$ and $|r_i - s_i| < \delta$, $i = 1, \ldots, n$, it follows that $|f(\eta_1, r_1, \ldots, r_n) - f(\eta_2, s_1, \ldots, s_n)| < \epsilon$.

Now, choose $h, g \in \mathcal{K}$ with $\|h - g\| < \delta$, so that $|h^{(i-1)} - g^{(i-1)}|_\infty < \delta$, $i = 1, \ldots, n$. Similar arguments to those in Claim 1 show that, for $x_1 \leq x \leq x_2$ and $i = 1, \ldots, n$,

$$|(Th)^{(i-1)}(x) - (Tg)^{(i-1)}(x)| \leq \epsilon\gamma_i (x_2 - x_1)^{n-i+1},$$

and so

$$\|Th - Tg\| \leq \epsilon \cdot \left[\sum_{i=1}^n \gamma_i (x_2 - x_1)^{n-i+1} \right].$$

Therefore, T is continuous $\mathcal{K}$.

Next, we choose a sequence $\{h_\ell\}_{\ell=1}^\infty \subset \mathcal{K}$, and we consider the sequence $\{(Th_\ell)\}_{\ell=1}^\infty \subset \mathcal{K}$. First, for each $\ell \in \mathbb{N}$, $(Th_\ell)^{(n)}(x) = -f(x, h_\ell(x), \ldots, h_\ell^{(n-1)}(x))$, which implies $|(Th_\ell)^{(n)}(x)| \leq Q, x_1 \leq x \leq x_2$. Consequently, for $\rho, \sigma \in [x_1, x_2]$, $|(Th_\ell)^{(n-1)}(\rho) - (Th_\ell)^{(n-1)}(\sigma)| \leq Q|\rho - \sigma|$, for all $\ell \in \mathbb{N}$, and so $\{(Th_\ell)^{(n-1)}\}_{\ell=1}^\infty$ is a uniformly equicontinuous family of functions. Since $|(Th_\ell)^{(n-1)}|_\infty \leq 2N_n$, for all $\ell \in \mathbb{N}$, it follows from the Arzelà-Ascoli Theorem that there exists a subsequence $\{(Th_{\ell_p})^{(n-1)}\}$ which converges uniformly on $[x_1, x_2]$. Similarly, $|(Th_{\ell_p})^{(n-1)}|_\infty \leq 2N_n$, for all $p \in \mathbb{N}$, implies that $\{(Th_{\ell_p})^{(n-2)}\}$ is a uniformly equicontinuous family of functions, and this, coupled with $|(Th_{\ell_p})^{(n-2)}|_\infty \leq 2N_{n-1}$ and the Arzelà-Ascoli Theorem, implies there exists a further subsequence $\{(Th_{\ell_{p_q}})^{(n-2)}\}$ which is uniformly convergent on $[x_1, x_2]$. Repeating this argument leads eventually to a subsequence, labeled for convenience $\{\ell_\nu\} \subseteq \{\ell\}$, such that $\{(Th_{\ell_\nu})^{(i-1)}\}_{\nu=1}^\infty$ converges uniformly on $[x_1, x_2]$, for each $i = 1, \ldots, n$.

Consequently, $\{(Th_{\ell_\nu})\}$ converges in the the norm of $\mathcal{M}$. It follows from the Schauder-Tychonoff Fixed Point Theorem that T has a fixed point $y \in \mathcal{K}$, and as such y is a desired solution of the boundary value problem (4.1), (4.2). □

Corollary 4.1. *Let $f(x, r_1, \ldots, r_n) : [a, b] \times \mathbb{R}^n$ be continuous and bounded. Then, for any $A_i \in \mathbb{R}$, $i = 1, \ldots, n$, the boundary value problem* (1.1), (1.2) *has a solution.*

Proof. Let $Q := \sup\{|f(x, r_1, \ldots, r_n)| \mid a \leq x \leq b, |r_i| < \infty,\ i = 1, \ldots, n\}$. Let $A_i \in \mathbb{R}$, $i = 1, \ldots, n$ be given, and let $w(x)$ be defined by (2.4). Choose $N_i > 0$, $i = 1, \ldots, n$, such that $\max_{a \leq x \leq b} \left|w^{(i-1)}(x)\right| \leq N_i$, $i = 1, \ldots, n$, and $\min_{1 \leq i \leq n} \left\{ \sqrt[n-i+1]{\frac{N_i}{\gamma_i Q}} \right\} \geq b - a$. The conclusion follows since $b - a \leq \delta(N_1, \ldots, N_n)$ of Theorem 4.1. □

5. LOCAL SOLUTIONS BY LERAY-SCHAUDER NONLINEAR ALTERNATIVE

In this section, we make application of the Leray-Schauder Nonlinear Alternative in obtaining solutions of the differential equation,

$$y^{(n)} + f(x, y) = 0, \quad 0 \leq x \leq L, \tag{5.1}$$

satisfying the boundary conditions,

$$y^{(i-1)}(0) = 0,\ i = 1, \ldots, n-1, \quad \int_0^L y(x)dx = 0, \tag{5.2}$$

where $f(x, r) : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. In making application of the nonlinear alternative, we again impose growth conditions on f as well as on the interval length L.

Theorem 5.1. *Assume*

(A) *There exist* $\sigma \in C([0, L], \mathbb{R}^+)$ *and a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|f(x, r)| \leq \sigma(x)\psi(|r|), \quad (x, r) \in [0, L] \times \mathbb{R},$$

and

(B) *There exists* $M > 0$ *such that*

$$\frac{M}{\gamma_1 \psi(M) L^n \|\sigma\|} > 1.$$

Then (5.1), (5.2) *has a solution on* $[0, L]$.

Proof. Let $E = C[0, L]$ with norm, $\|\cdot\| = \max_{0 \leq x \leq L} |\cdot|$. We seek fixed points of the mapping $T : E \to E$ defined by

$$(Th)(x) = \int_0^L G(x, s) f(s, h(s)) ds, \quad h \in E,$$

were G is the Green's function of (2.3) relative to the endpoints 0 and L.

We first show that T maps bounded sets into bounded sets. For $r > 0$, let $B_r := \{h \in E \mid \|h\| \leq r\}$ be a bounded subset of E. Then, for $0 \leq x \leq L$ and $h \in B_r$,

$$\begin{aligned}
|(Th)(x)| &\leq \int_0^L |G(x, s)||f(x, h(s))| ds \\
&\leq \int_0^L |G(x, s)| \sigma(s) \psi(|h(s)|) ds \\
&\leq \int_0^L |G(x, s)| \sigma(s) \psi(\|h\|) ds \\
&\leq \int_0^L |G(x, s)| \|\sigma\| \psi(r) ds \\
&\leq \gamma_1 L^n \psi(r) \|\sigma\|.
\end{aligned}$$

Hence,

$$\|Th\| \leq \gamma_1 L^n \psi(r) \|\sigma\|,$$

and so T maps B_r into a bounded set.

Next, we show T maps bounded sets into equicontinuous sets. In that direction, let $0 \leq p < q \leq L$ and let $h \in B_r$, with B_r above. Then,

$$\begin{aligned}
|(Th)(p) - (Th)(q)| &\leq \int_0^L |G(p, s) - G(q, s)||f(s, h(s))| ds \\
&\leq \|\sigma\| \psi(r) \int_0^L |G(p, s) - G(q, s)| ds.
\end{aligned}$$

The right hand side of the inequality tends to zero, as $|p - q| \to 0$, independent of $h \in B_r$. So T maps B_r into an equicontinuous set. It follows by the Arzelà-Ascoli Theorem that T is completely continuous.

Now, suppose for some $h \in E$ and some $0 < \mu < 1$, we have $h = \mu Th$. Then, for $0 \le x \le L$,

$$\begin{aligned} |h(x)| &= |\mu(Th)(x)| \\ &\le \int_0^L |G(x,s)||f(x,h(s))|ds \\ &\le \gamma_1 L^n \psi(\|h\|)\|\sigma\|, \end{aligned}$$

which yields

$$\frac{\|h\|}{\gamma_1 \psi(\|h\|) L^n \|\sigma\|} \le 1.$$

By (B), $\|h\| \ne M$. If we set

$$V := \{h \in E \mid \|h\| < M\},$$

then the operator $T : \overline{V} \to E$ is completely continuous (i.e., continuous and compact). From the choice of V, there is no $h \in \partial V$ such that $h = \mu Th$, for some $0 < \mu < 1$. By the Leray-Schauder Nonlinear Alternative, it follows that T has a fixed point $y \in \overline{V}$ which is a solution of (5.1), (5.2). □

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