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# ON THE MILD SOLUTIONS OF QUANTUM STOCHASTIC EVOLUTION INCLUSIONS

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**ABSTRACT.** Under a Filippov-type assumption, a study of the Quantum stochastic evolution inclusions is done in this paper. Given a quantum stochastic evolution inclusions:

$$\begin{aligned} dx(t) \in Ax(t) + \int_0^t K(t,s)(E(s,x(s))d\Lambda_\pi(s) + F(s,x(s))dA_f(s) \\ + G(s,x(s))dA_g^+(s) + H(s,x(s))ds) \\ x(t_0) = x_0 \end{aligned}$$

where $A$ is the infinitesimal generator of a $C_0$-semigroup of operators, $K$ is a continuous function and $E, F, G, H$ are Lipschitzian multivalued stochastic processes. We established the existence of mild solutions of the quantum stochastic evolution inclusions.

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## 1. Introduction

The problem of existence of solutions of Lipschitzian quantum stochastic differential inclusions was solved in [8]. This gave a multivalued generalization of quantum stochastic calculus of Hudson and Parthasarathy formulation [13]. Some topological properties of the solution sets were established in [3] and [4]. A further analysis of quantum stochastic differential inclusions for the case of hypermaximal monotone type was established in [9] while the existence of solutions of quantum stochastic evolution inclusions was established in [10]. The existence of solutions of quantum stochastic differential inclusions of discontinuous coefficients via fixed point theorem was established in [14]. A detailed account of the theory of differential inclusions involved can be found in [2] and [6].

The existence of mild solutions of evolution inclusions for classical integrodifferential inclusions was established in [7], [5] and the references in them. The continuous selection of solution sets of evolution equations was established in [1] and [16].

In [11] a weaker form of solution of right Hudson-Parthasarathy quantum stochastic differential equations which is mild solution was established. In the same way

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under a Filippov-type assumption, a weaker form of solution, which is mild solution of quantum stochastic evolution inclusions arising from [8] and [10] , was established in this work. Moreover, this in turn gives a multivalued generalization of the result [11].

In the sequel the work shall be as follows: in section 2, preliminaries on notations and basic results are established. Our main result shall be established in section 3.

## 2. Preliminaries

In this section we shall adopt the notations in [8]. Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; $\gamma$ is a fixed Hilbert and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable $\gamma$-valued maps on $\mathbb{R}_+$.

The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $\mathbb{E}$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$. We define the locally convex space $\mathcal{A}$ of noncommutative stochastic processes whose topology $\tau_w$, is generated by the family of seminorms $\{\|x\|_{\eta\xi} = |\langle \eta, x\xi \rangle|, x \in \mathcal{A}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. The completion of $(\mathcal{A}, \tau_w)$ is denoted by $\widetilde{\mathcal{A}}$. The underlying elements of $\widetilde{\mathcal{A}}$ consist of linear maps from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D}\underline{\otimes}\mathbb{E}$. For a fixed Hilbert space $\gamma$, the spaces $L^p_{loc}(\widetilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^p_{loc}(I \times \widetilde{\mathcal{A}})$ are adopted as in [8].

For a topological space $\mathcal{N}$, let $clos(\mathcal{N})$ be the collection of all nonempty closed subsets of $\mathcal{N}$; we shall employ the Hausdorff topology on $clos(\widetilde{\mathcal{A}})$ as defined in [8]. Moreover, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, we define the Hausdorff distance, $\rho(A, B)$ as:

$$\mathbf{d}(x, B) \equiv \inf_{y \in B} |x - y|, \qquad \delta(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B)$$
$$\text{and } \rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then $\rho$ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

By a multivalued stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$, we mean a multifunction on $I$ with values in $clos(\widetilde{\mathcal{A}})$. If $\Phi$ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of $\Phi$ is a stochastic process $X : I \to \widetilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$. A multivalued stochastic process $\Phi$ will be called (i) adapted if $\Phi(t) \subseteq \widetilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ ; (iii) locally absolutely $p$-integrable if $t \mapsto \|\Phi(t)\|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\widetilde{\mathcal{A}})$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

The set of all absolutely $p$-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs}$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \widetilde{\mathcal{A}} \to clos(\widetilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathcal{A}})$.

**Quantum stochastic evolution inclusions.** Let $Y$ be a metric space, an open (resp. closed) ball in $Y$ with centre y and radius $r$ is denoted by $B_Y(y,r)$(resp., $\overline{B}_Y(y,r)$). A multifunction $\Phi : Y \to clos(\widetilde{\mathcal{A}})$ is said to be $\rho_{\eta\xi}$-continuous at $x' \in Y$ if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_{\eta\xi}(\Phi(x), \Phi(x')) \leq \epsilon$ for any $x \in B_Y(x', r)$.

$\Phi$ will be said to be $\rho_{\eta\xi}$-continuous if it is so at each $x' \in Y$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Let $\mathcal{L}$ be the $\sigma$-algebra of the Lebesgue measurable subsets of $\mathbb{R}$ and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of $A$, with $\mu(A) < \infty$. A multifunction $\Phi : Y \to clos(\widetilde{\mathcal{A}})$ is said to be *Lusin measurable* if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\epsilon > 0$, there exists a compact set $K_\epsilon^{\eta\xi} \subset A$ with $\mu(A \setminus K_\epsilon^{\eta\xi}) < \epsilon$ such that $\Phi$ restricted to $K_\epsilon^{\eta\xi}$ is $\rho_{\eta\xi}$-continuous.

A map $\Phi : I \times \widetilde{\mathcal{A}} \to \mathrm{clos}(\widetilde{\mathcal{A}})$ is said to be *Lipschitzian* if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there exists $l_{\eta\xi}^{\Phi} : I \to (0,\infty)$ in $L_{loc}^1(I)$ such that

$$\rho_{\eta\xi}(\Phi(t,x), \Phi(t,y)) \leq l_{\eta\xi}^{\Phi}(t)\|x-y\|_{\eta\xi}$$

for $x, y \in \widetilde{\mathcal{A}}$ and almost all $t \in I$. The functions $\{l_{\eta\xi}^{\Phi}(\cdot) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ are called *Lipschitz functions* for $\Phi$. Let $E, F, G, H \in L_{loc}^2(I \times \widetilde{\mathcal{A}})_{mvs}$, in this paper we are concerned with the quantum stochastic evolution inclusions

$$\begin{aligned} dx(t) \in Ax(t) + \int_0^t & K(t,s)(E(s,x(s))d\Lambda_\pi(s) + F(s,x(s))dA_f(s) \\ & + G(s,x(s))dA_g^+(s) + H(s,x(s))ds) \\ x(t_0) = x_0 & \end{aligned} \tag{2.1}$$

As established in [8], using the relations:

$$\begin{aligned} (\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\} \\ (\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\} \\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\ H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(\cdot, X(\cdot)) \end{aligned}$$

is a selection of $H(\cdot, X(\cdot)) \forall\ X \in L_{loc}^2(\widetilde{\mathcal{A}})\}$

$$\begin{aligned} \mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) \\ &\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \end{aligned}$$

problem (2.1) can be rewritten in a non-classical form

$$\begin{aligned} \frac{d}{dt}\langle \eta, x(t)\xi\rangle &\in \langle \eta, Ax(t)\xi\rangle + \int_0^t K(t,s)\mathbb{P}(s,x(s))(\eta,\xi)ds \\ x(t_0) &= x_0 \end{aligned} \tag{2.2}$$

where $\mathbb{P}: I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is a sesquilinear form-valued multifunction; $A$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $\{G(t); t \geq 0\}$ from $\widetilde{\mathcal{A}}$ into $\widetilde{\mathcal{A}}$. Also, $D = \{(t,s) \in I \times I; t \geq s\}$ and $K : D \to \mathbb{R}$ is continuous.

Let $L^1(I, \widetilde{\mathcal{A}})$ be the space of all Bochner integrable maps from $I$ to $\widetilde{\mathcal{A}}$ and $C(I, \widetilde{\mathcal{A}})$ the space of continuous maps from $I$ to $\widetilde{\mathcal{A}}$. The spaces $L^1(I, \widetilde{\mathcal{A}})$ and $C(I, \widetilde{\mathcal{A}})$ are locally convex spaces with topologies $\tau_1$ and $\tau_{con}$ respectively, generated by the family of seminorms:

$$\tau_1 : \{\|\cdot\|_{1,\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\} \text{ with } \|z\|_{1,\eta\xi} = \int_I dt |\langle \eta, z(t)\xi\rangle|$$

and

$$\tau_{con} : \{\|\cdot\|_{con,\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\} \text{ with } \|z\|_{con,\eta\xi} = \sup_{t\in I} |\langle \eta, z(t)\xi\rangle|$$

An adapted stochastic process $x : I \to \widetilde{\mathcal{A}}$ is said to be a *mild solution* of (2.2) or equivalently (2.1) if $x(\cdot) \in C(I, \widetilde{\mathcal{A}})$ and there exists a Bochner integrable function $f(\cdot) \in L^1(I, \widetilde{\mathcal{A}})$ such that

$$\begin{aligned} &\langle \eta, f(t)\xi\rangle \in \mathbb{P}(t, x(t))(\eta, \xi) \text{ a.e. } t \in I \\ &\langle \eta, x(t)\xi\rangle = \langle \eta, G(t)x_0\xi\rangle + \int_0^t G(t) \int_0^\tau K(\tau, s)\langle \eta, f(s)\xi\rangle ds d\tau, \quad t \in I \end{aligned} \tag{2.3}$$

$(x(\cdot), f(\cdot))$ shall be called a *trajectory selection pair* of problem (2.2).

The second relation in equation (2.3) may be rewritten as

$$\langle \eta, x(t)\xi\rangle = \langle \eta, G(t)x_0\xi\rangle + \int_0^t U(t,s)\langle \eta, f(s)\xi\rangle ds d\tau, \quad t \in I$$

where $U(t,s) = \int_s^t G(t)K(\tau,s)d\tau$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$; $B_{\eta\xi}$ and $B$ are defined as :

$$B_{\eta\xi} = \{x \in \widetilde{\mathcal{A}} : \|x\|_{\eta\xi} \leq 1\} \text{ and } B = \{x \in \mathbb{C} : |x| \leq 1\}$$

A map $\Psi : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is said to be *Lipschitzian* if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there exists $l_{\eta\xi} : I \to (0, \infty)$ in $L^1_{loc}(I)$ such that

$$\rho(\Psi(t,x)(\eta,\xi), \Psi(t,y)(\eta,\xi)) \leq l_{\eta\xi}(t)\|x-y\|_{\eta\xi}$$

for $x, y \in \widetilde{\mathcal{A}}$ and almost all $t \in I$.

Let $Y$ be a metric space, a multifunction $\Psi : Y \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is said to be $\rho$-continuous at $x' \in Y$ if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(\Psi(x)(\eta,\xi), \Psi(x')(\eta,\xi)) \leq \epsilon$ for any $x \in B_Y(x', r)$. A sesquilinear form valued multifunction, $\Psi : I \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is said to be *Lusin measurable* if for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\epsilon > 0$, there exists a compact set $K_\epsilon^{\eta\xi} \subset A$, $A \subset I$ with $\mu(A \setminus K_\epsilon^{\eta\xi}) < \epsilon$ such that $\Psi$ restricted to $K_\epsilon^{\eta\xi}$ is $\rho$-continuous.

We shall assume the following hypotheses in what follows.

**Hypothesis 1** (i) $A$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $\{G(t); 0 \leq t \leq T\}$.
(ii) Let $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$ and $\Phi \in \{E, F, G, H\}$, $\Phi(\cdot,\cdot) : I \times \widetilde{\mathcal{A}} \to clos(\widetilde{\mathcal{A}})$ is nonempty such that for any $x \in \widetilde{\mathcal{A}}$, $\Phi(\cdot, x)$ is Lusin measurable on $I$.
(iii) There exists $l^{\Phi}_{\eta\xi} : I \to (0, \infty)$ in $L^1_{loc}(I)$ such that

$$\rho_{\eta\xi}(\Phi(t, x), \Phi(t, y)) \leq l^{\Phi}_{\eta\xi}(t)\|x - y\|_{\eta\xi}$$

for $x, y \in \widetilde{\mathcal{A}}$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
(iv) There exists $q^{\Phi}_{\eta\xi}(\cdot) \in L^1_{loc}(I, (0, \infty))$ such that for each $t \in I$;

$$\Phi(t, 0) \subset q^{\Phi}_{\eta\xi}(t)B_{\eta\xi}.$$

(v) $D = \{(t, s) \in I \times I; t \geq s\}$ and $K : D \to \mathbb{R}$ is continuous.

By proposition (6.1) in [8], $\mathbb{P}$ is Lipschitzian whenever, $E$, $F$, $G$, $H$ are Lipschitzian. We remark that in the same manner, if $E(\cdot, x)$, $F(\cdot, x)$, $G(\cdot, x)$, $H(\cdot, x)$ are Lusin measurable then $\mathbb{P}(\cdot, x)(\eta, \xi)$ is Lusin measurable. Moreover, if there exists $q^{\Phi}_{\eta\xi}(\cdot) \in L^1_{loc}(I, (0, \infty))$ such that for each $t \in I$;

$$\Phi(t, 0) \subset q^{\Phi}_{\eta\xi}(t)B_{\eta\xi}.$$

Then there exists $q_{\eta\xi}(\cdot) \in L^1_{loc}(I, (0, \infty))$ such that for each $t \in I$;

$$\mathbb{P}(t, 0)(\eta, \xi) \subset q_{\eta\xi}(t)B.$$

where $q_{\eta\xi}(t) = \max\{q^{\Phi}_{\eta\xi}(t);\ \text{for each } t \in I\}$. Therefore Hypothesis 1 can be restated as:

**Hypothesis 2** (i) $A$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $\{G(t); 0 \leq t \leq T\}$.
(ii) For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\mathbb{P}(\cdot,\cdot) : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ has nonempty closed and bounded values in $\mathbb{C}$, and for any $x \in \widetilde{\mathcal{A}}$, $\mathbb{P}(\cdot, x)(\eta, \xi)$ is Lusin measurable on $I$.
(iii) There exists $l_{\eta\xi} : I \to (0, \infty)$ in $L^1_{loc}(I)$ such that

$$\rho(\mathbb{P}(t, x)(\eta, \xi), \mathbb{P}(t, y)(\eta, \xi)) \leq l_{\eta\xi}(t)\|x - y\|_{\eta\xi}$$

for $x, y \in \widetilde{\mathcal{A}}$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
(iv) There exists $q_{\eta\xi}(\cdot) \in L^1_{loc}(I, (0, \infty))$ such that for each $t \in I$;

$$\mathbb{P}(t, 0)(\eta, \xi) \subset q_{\eta\xi}(t)B.$$

(v) $D = \{(t, s) \in I \times I; t \geq s\}$ and $K : D \to \mathbb{R}$ is continuous.
Set $n_{\eta\xi}(t) = \int_0^t l_{\eta\xi}(u)du$, $t \in I$, $M = \sup_{t\in I} \|G(t)\|_{\eta\xi}$ and $M_0 = \sup_{(t,s)\in D} |K(t, s)|$, then $|U(t, s)| \leq MM_0(t - s) \leq MM_0T$. The following results are analogues of Lemmas 3.1 and 3.2 in [7].

**Lemma 2.1.** *Let $\Psi_1, \Psi_2 : I \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ be two Lusin measurable multifunctions and let $\epsilon_1, \epsilon_2 > 0$ be such that*

$$H(t)(\eta,\xi) = \big(\Psi_1(t)(\eta,\xi) + \epsilon_1 B\big) \cap \big(\Psi_2(t)(\eta,\xi) + \epsilon_2 B\big) \neq \emptyset, \quad \forall t \in I$$

*Then the multifunction $H : I \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ has a Lusin measurable selection $h : I \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$*

*Proof.* Since $\Psi_1$ and $\Psi_2$ are Lusin measurable, we can construct a sequence $\{J_n\}$ of pairwise disjoint compact sets $J_n \subset I$ satisfying, for each $n \in \mathbb{N}$, the following properties:
(I) $\Psi_1$ and $\Psi_2$ restricted to $J_n$ are $\rho$-continuous.
(II) $J_n \subset I \setminus \cup_{i=1}^{n} J_i$;
(III) $\mu\big(I \setminus \cup_{i=1}^{n} J_i\big) < \frac{1}{2^n}$
Set $J_0 = I \setminus \cup_n J_n$ and observe that, by (iii), $\mu(J_0) = 0$. $\{J_n\}_{n\geq 0}$ is partition of $I$.

We claim that for each $n = 0, 1, \ldots$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there is a Lusin measurable function $h_n : J_n \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ which is a selector of the multifunction $H$ restricted to $J_n$. To show this, fix an arbitrary $n \in \mathbb{N}$. For each $t \in J_n$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, pick out a point $u_{t,\eta\xi} \in H(t)(\eta,\xi)$. Since $H(t)(\eta,\xi)$ is open and $\Psi_1$ and $\Psi_2$ restricted to $J_n$ are $\rho$-continuous, there is a $\delta_t > 0$ such that

$$u_{t_k,\eta\xi} \in \big(\Psi_1(s)(\eta,\xi) + \epsilon_1 B\big) \cap \big(\Psi_2(s)(\eta,\xi) + \epsilon_2 B\big) \tag{2.4}$$

for every $s \in B^{J_n}(t, \delta_t)$.

The family $\{B^{J_n}(t,\delta_t)\}_{t\in J_n}$ is an open covering of $J_n$. As $J_n$ is compact, it admits a finite subcovering, $\{B^{J_n}(t_k, \delta_{t_k})\}_{k=1}^{q}$, say. Now consider the partition $\{I_k\}_{k=1}^{q}$ of $J_n$ given by

$$I_1 = B^{J_n}(t_1,\delta_{t_1}) \ I_k = B^{J_n}(t_k,\delta_{t_k}) \setminus \cup_{i=1}^{k-1} I_i, \quad 2 \leq k \leq q$$

and define $h_n : J_n \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ by

$$h_n(t)(\eta,\xi) = \sum_{k=1}^{q} u_{t_k} \chi I_k(t)(\eta,\xi).$$

Then $h_n$ is Lusin measurable and $h_n$ is a selector of $H$ restricted to $J_n$.

Let $s \in J_n$ be arbitrary, thus $s \in I_k$ for some $1 \leq k \leq q$. Since $s \in I_k \subset B^{J_n}(t_k, \delta_{t_k})$. In view of (2.4) (with $t = t_k$) we have

$$u_{t_k,\eta\xi} \in \big(\Psi_1(s)(\eta,\xi) + \epsilon_1 B\big) \cap \big(\Psi_2(s)(\eta,\xi) + \epsilon_2 B\big)$$

thus $h_n(s)(\eta,\xi) \in H(s)(\eta,\xi)$, for $h_n(s) = u_{t_k}$. Hence $h_n$ is a Lusin measurable selector of $H$ restricted to $J_n$. Then for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$; $h : I \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ given by

$$h(t)(\eta,\xi) = \sum_{n\geq 0} h_n(t) \chi J_n(t)(\eta,\xi).$$

is a Lusin measurable selector of $H$. $\square$

**Lemma 2.2.** *Let $\mathbb{P} : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ satisfy Hypothesis 2. Then for arbitrary adapted stochastic process $x : I \to \widetilde{\mathcal{A}}$ continuous; $t \mapsto \langle \eta, u(t)\xi\rangle$ Lusin measurable and $\epsilon > 0$, for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ we have:*
*(i) the multifunction $t \mapsto \mathbb{P}(t, x(t))(\eta, \xi)$ is Lusin measurable on $I$;*
*(ii) the multifunction $t \mapsto \langle \eta, G(t)\xi\rangle$ defined by*

$$\begin{aligned}\langle \eta, G(t)\xi\rangle = \big(&\mathbb{P}(t, x(t))(\eta, \xi) + \epsilon B\big)\\ &\cap B\big(u(t)(\eta, \xi), d(u(t)(\eta, \xi), \mathbb{P}(t, x(t))(\eta, \xi)) + \epsilon\big)\end{aligned}$$

*has a Lusin measurable selection $g : I \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$.*

*Proof.* Let $x_n$ be a sequence of piecewise continuous functions $x_n : I \to \widetilde{\mathcal{A}}$ converging to $x$ uniformly on $I$. Given $\epsilon > 0$, let $K_\epsilon \subset I$ be a compact set, with $\mu(I \setminus K_\epsilon) < \epsilon$, such that $l_{\eta\xi}$ restricted to $K_\epsilon$ is continuous and for each $n \in \mathbb{N}$, the multifunction $t \mapsto \mathbb{P}(t, x_n(t))(\eta, \xi)$ restricted to $K_\epsilon$ is $\rho$-continuous.

Set $M_\epsilon = \sup_{t\in K_\epsilon} l_{\eta\xi}(t)$. Let $t_0, t \in K_\epsilon$ be arbitrary. We have:

$$\begin{aligned}\rho\big(\mathbb{P}(t, x(t))(\eta, \xi), \mathbb{P}(t_0, x(t_0))(\eta, \xi)\big) &\le \rho\big(\mathbb{P}(t, x(t))(\eta, \xi), \mathbb{P}(t, x_n(t))(\eta, \xi)\big)\\ &\quad + \rho\big(\mathbb{P}(t, x_n(t))(\eta, \xi), \mathbb{P}(t_0, x_n(t_0))(\eta, \xi)\big)\\ &\quad + \rho\big(\mathbb{P}(t_0, x_n(t_0))(\eta, \xi), \mathbb{P}(t_0, x(t_0))(\eta, \xi)\big)\\ &\le M_\epsilon \|x_n(t) - x(t)\|_{\eta\xi} + \rho\big(\mathbb{P}(t, x_n(t))(\eta, \xi), \mathbb{P}(t_0, x_n(t_0))(\eta, \xi)\big)\\ &\quad + M_\epsilon \|x_n(t_0) - x(t_0)\|_{\eta\xi}\\ &\le M_\epsilon \sigma_n + \rho\big(\mathbb{P}(t, x_n(t))(\eta, \xi), \mathbb{P}(t_0, x_n(t_0))(\eta, \xi)\big)\end{aligned}$$

where $\sigma_n = \sup_{t\in I} \|x_n(t) - x(t)\|_{\eta\xi}$. Since $\sigma_n \to 0$ as $n \to \infty$ and $t \mapsto \mathbb{P}(t, x_n(t))(\eta, \xi)$ restricted to $K_\epsilon$ is $\rho$-continuous. The multifunction $t \mapsto \mathbb{P}(t, x(t))(\eta, \xi)$ restricted to $K_\epsilon$ is $\rho$-continuous and (i) is proved.

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $t \in I$ set $\langle \eta, G^1(t)\xi\rangle = \mathbb{P}(t, x(t))(\eta, \xi)$, $\langle \eta, G^2(t)\xi\rangle = B\big(u(t)(\eta, \xi), d\big(u(t)(\eta, \xi), \langle \eta, G^1(t)\xi\rangle\big)\big)$ and observe that $t \mapsto \langle \eta, G^1(t)\xi\rangle$ and $\langle \eta, G^2(t)\xi\rangle$ are Lusin measurable on $I$. Furthermore, for each $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ we have

$$\langle \eta, G(t)\xi\rangle = \big(\langle \eta, G^1(t)\xi\rangle + \epsilon B\big) \cap \big(\langle \eta, G^2(t)\xi\rangle + \epsilon B\big) \text{ and } \langle \eta, G(t)\xi\rangle \neq \emptyset.$$

Hence by Lemma (2.1), $\langle \eta, G(t)\xi\rangle$ has a Lusin measurable selection $g : I \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$, thus (ii) holds. □

## Main Result

**Theorem 3.1.** *If Hypothesis 2 is satisfied, then for every $x_0 \in \widetilde{\mathcal{A}}$, the Cauchy problem (2.2) has a mild solution $x(\cdot) \in C(I, \widetilde{\mathcal{A}})$.*

*Proof.* We note that if an adapted stochastic process $z(\cdot) : I \to \widetilde{\mathcal{A}}$ is continuous, then every Lusin measurable selection $t \mapsto \langle \eta, u(t)\xi\rangle$ of the multifunction $t \mapsto \mathbb{P}(t, z(t))(\eta,\xi) + B$ is Bochner integrable on $I$. Therefore, for any $t \in I$, we have

$$\begin{aligned}|\langle \eta, u(t)\xi\rangle| &\le \rho\big(\mathbb{P}(t, z(t))(\eta,\xi) + B, \{0\}\big)\\ &\le \rho\big(\mathbb{P}(t, z(t))(\eta,\xi), \mathbb{P}(t,0)(\eta,\xi)\big) + \rho\big(\mathbb{P}(t,0)(\eta,\xi), \{0\}\big) + 1\\ &\le l_{\eta\xi}(t)\|z(t)\|_{\eta\xi} + q_{\eta\xi}(t) + 1.\end{aligned}$$

Let $0 < \epsilon < 1$, $\epsilon_n = \frac{\epsilon}{2^{n+2}}$.

Consider $f_0 : I \to \widetilde{\mathcal{A}}$ an arbitrary Lusin measurable, Bochner integrable function and define

$$\langle \eta, x_0(t)\xi\rangle = \langle \eta, G(t)x_0\xi\rangle + \int_0^t U(t,s)\langle \eta, f_0(s)\xi\rangle ds, \quad t \in I$$

Since $x_0(\cdot)$ is continuous, by Lemma 2.2 there exists a Lusin measurable function $f_1 : I \to \widetilde{\mathcal{A}}$ which, for each $t \in I$, the map $t \mapsto \langle \eta, f_1(t)\xi\rangle$ satisfies

$$\begin{aligned}\langle \eta, f_1(t)\xi\rangle \in &\Big(\mathbb{P}(t, x_0(t))(\eta,\xi) + \epsilon_1 B\Big)\\ &\cap B\Big(\langle \eta, f_0(t)\xi\rangle, d\big(\langle \eta, f_0(t)\xi\rangle, \mathbb{P}(t, x_0(t))(\eta,\xi)\big) + \epsilon_1\Big)\end{aligned}$$

Obviously, $\langle \eta, f_1(\cdot)\xi\rangle$ is Bochner integrable on $I$. Let $x_1(\cdot) : I \to \widetilde{\mathcal{A}}$ such that for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we define the map $t \mapsto \langle \eta, x_1(t)\xi\rangle$ as:

$$\langle \eta, x_1(t)\xi\rangle = \langle \eta, G(t)x_0\xi\rangle + \int_0^t U(t,s)\langle \eta, f_1(s)\xi\rangle ds, \quad t \in I.$$

By induction, we construct a sequence $t \mapsto \langle \eta, x_n(t)\xi\rangle$, $n \ge 2$ given by

$$\langle \eta, x_n(t)\xi\rangle = \langle \eta, G(t)x_0\xi\rangle + \int_0^t U(t,s)\langle \eta, f_n(s)\xi\rangle ds, \quad t \in I \tag{3.1}$$

where $t \mapsto \langle \eta, f_n(t)\xi\rangle$ is a Lusin measurable function which for $t \in I$ satisfies:

$$\begin{aligned}\langle \eta, f_n(t)\xi\rangle \in &\big(\mathbb{P}(t, x_{n-1}(t))(\eta,\xi) + \epsilon_n B\big)\\ &\cap B\big(\langle \eta, f_{n-1}(t)\xi\rangle, d(\langle \eta, f_{n-1}(t)\xi\rangle, \mathbb{P}(t, x_{n-1}(t))(\eta,\xi)) + \epsilon_n\big).\end{aligned} \tag{3.2}$$

$\langle \eta, f_n(\cdot)\xi\rangle$ is also Bochner integrable. From (3.2), for $n \ge 2$ and $t \in I$, we obtain:

$$\begin{aligned}|\langle \eta, (f_n(t) - f_{n-1}(t))\xi\rangle| &\le d(\langle \eta, f_{n-1}(t)\xi\rangle, \mathbb{P}(t, x_{n-1}(t))(\eta,\xi)) + \epsilon_n\\ &\le d(\langle \eta, f_{n-1}(t)\xi\rangle, \mathbb{P}(t, x_{n-2}(t))(\eta,\xi))\\ &\quad + \rho(\mathbb{P}(t, x_{n-2}(t))(\eta,\xi), \mathbb{P}(t, x_{n-1}(t))(\eta,\xi)) + \epsilon_n\\ &\le \epsilon_{n-1} + l_{\eta\xi}(t)\|x_{n-1}(t) - x_{n-2}(t)\|_{\eta\xi} + \epsilon_n.\end{aligned}$$

Since $\epsilon_{n-1} + \epsilon_n < \epsilon_{n-2}$, for $n \ge 2$, we deduce that

$$|\langle \eta, (f_n(t) - f_{n-1}(t))\xi\rangle| \le \epsilon_{n-2} + l_{\eta\xi}(t)\|x_{n-1}(t) - x_{n-2}(t)\|_{\eta\xi}. \tag{3.3}$$

For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, denote $p_{0,\eta\xi} = d\big(\langle\eta, f_0(t)\xi\rangle, \mathbb{P}(t,x_0(t))(\eta,\xi)\big)$, $t \in I$. We then prove by recurrence, that for $n \geq 2$ and $t \in I$:

$$\begin{aligned}\|x_n(t)-x_{n-1}(t)\|_{\eta\xi} \leq \sum_{k=0}^{n-2}\int_0^t \epsilon_{n-2-k}\frac{(MM_0T)^{k+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^k}{k!}du \\ + \epsilon_0\int_0^t \frac{(MM_0T)^n(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n-1}}{(n-1)!}du \\ + \int_0^t \frac{(MM_0T)^n(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n-1}}{(n-1)!}p_{0,\eta\xi}(u)du.\end{aligned} \tag{3.4}$$

We start with $n=2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ there is

$$\begin{aligned}\|x_2(t)-x_1(t)\|_{\eta\xi} &= |\langle\eta,(x_2(t)-x_1(t))\xi\rangle| \\ &\leq \int_0^t |U(t,s)|\cdot|\langle\eta,(f_2(s)-f_1(s))\xi\rangle|ds \\ &\leq \int_0^t MM_0T\big[\epsilon_0 + l_{\eta\xi}(s)\|x_1(s)-x_0(s)\|_{\eta\xi}\big]ds \\ &\leq \epsilon_0 MM_0Tt + \int_0^t \Bigg[MM_0Tl_{\eta\xi}(s) \\ &\quad \int_0^s |U(s,r)|\cdot|\langle\eta,(f_1(r)-f_0(r))\xi\rangle dr\Bigg]ds \\ &\leq \epsilon_0 MM_0Tt \\ &\quad + \int_0^t \big[(MM_0T)^2 l_{\eta\xi}(s)\int_0^s (p_{0,\eta\xi}(u)+\epsilon_1)du\big]ds \\ &\leq \epsilon_0 MM_0Tt \\ &\quad + \int_0^t \big[(MM_0T)^2(p_{0,\eta\xi}(u)+\epsilon_1)\int_u^t l_{\eta\xi}(s)ds\big]du \\ &= \epsilon_0 MM_0Tt \\ &\quad + \int_0^t (MM_0T)^2(n_{\eta\xi}(t)-n_{\eta\xi}(s))[p_{0,\eta\xi}(s)+\epsilon_0]ds,\end{aligned}$$

that is, (3.4) is verified for $n=2$.

Using again (3.3) and (3.4), we conclude:

$$\begin{aligned}\|x_{n+1}(t)-x_n(t)\|_{\eta\xi} &= |\langle\eta,(x_{n+1}(t)-x_n(t))\xi\rangle| \\ &\leq \int_0^t |U(t,s)|\cdot|\langle\eta,(f_{n+1}(s)-f_n(s))\xi\rangle|ds \\ &\leq \int_0^t MM_0T\big[\epsilon_{n-1} + l_{\eta\xi}(s)\|x_n(s)-x_{n-1}(s)\|_{\eta\xi}\big]ds \\ &\leq \epsilon_{n-1}MM_0Tt + \int_0^t l_{\eta\xi}(s)\end{aligned}$$

$$\Big[\sum_{k=0}^{n-2}\int_0^s \epsilon_{n-2-k}\frac{(MM_0T)^{k+2}(n_{\eta\xi}(s)-n_{\eta\xi}(u))^k}{k!}du$$
$$+\int_0^s \frac{(MM_0T)^{n+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n-1}}{(n-1)!}$$
$$(p_{0,\eta\xi}(u)+\epsilon_0)du\Big]ds$$
$$=\epsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\epsilon_{n-2-k}$$
$$\int_0^t\big[\frac{(MM_0T)^{k+2}(n_{\eta\xi}(s)-n_{\eta\xi}(u))^k}{k!}l_{\eta\xi}(s)du\big]ds$$
$$+\int_0^t l_{\eta\xi}(s)$$
$$\Big(\int_0^s \frac{(MM_0T)^{n+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n-1}}{(n-1)!}l_{\eta\xi}(s)$$
$$(p_{0,\eta\xi}(u)+\epsilon_0)du\Big)ds$$
$$=\epsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\epsilon_{n-2-k}$$
$$\int_0^t\Big(\int_u^t \frac{(MM_0T)^{k+2}(n_{\eta\xi}(s)-n_{\eta\xi}(u))^k}{k!}l_{\eta\xi}(s)ds\Big)du$$
$$+\int_0^t\big(\int_u^t \frac{(MM_0T)^{n+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n-1}}{(n-1)!}l_{\eta\xi}(s)ds\big)$$
$$(p_{0,\eta\xi}(u)+\epsilon_0)du$$
$$=\epsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\epsilon_{n-2-k}$$
$$\int_0^t \frac{(MM_0T)^{k+2}(n_{\eta\xi}(s)-n_{\eta\xi}(u))^{k+1}}{(k+1)!}du$$
$$+\int_0^t \frac{(MM_0T)^{n+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n}}{n!}(p_{0,\eta\xi}(u)+\epsilon_0)du$$
$$=\sum_{k=0}^{n-1}\epsilon_{n-1-k}\int_0^t \frac{(MM_0T)^{k+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{k}}{k!}du$$
$$+\int_0^t \frac{(MM_0T)^{n+1}(n_{\eta\xi}(t)-n_{\eta\xi}(u))^{n}}{n!}(p_{0,\eta\xi}(u)+\epsilon_0)du,$$

therefore the relation (3.4) is true for $n+1$.

From (3.4), it follows that for $n\geq 2$ and $t\in I$, $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$:

$$\|x_n(t)-x_{n-1}(t)\|_{\eta\xi}\leq a_{n,\eta\xi}, \tag{3.5}$$

where

$$a_{n,\eta\xi} = \sum_{k=0}^{n-2} \epsilon_{n-2-k} \frac{(MM_0T)^{k+1} n_{\eta\xi}(T)^k}{k!} + \frac{(MM_0T)^n n_{\eta\xi}(T)^{n-1}}{(n-1)!} \left[ \int_0^t p_{0,\eta\xi}(u)du + \epsilon_0 \right],$$

The series $\{a_{n,\eta\xi}\}$ converges. We infer from (3.5) that $x_n(\cdot)$ converges to a continuous function, $x(\cdot) : I \to \widetilde{\mathcal{A}}$. Moreover, from the definition of $x_n(\cdot)$ in (3.1)and the completeness of $\widetilde{\mathcal{A}}$ we conclude that $x(\cdot)$ is an adapted stochastic process belonging to $C(I, \widetilde{\mathcal{A}})$.

On the other hand, in view of (3.3), there is

$$|\langle \eta, (f_n(t) - f_{n-1}(t))\xi \rangle| \leq \epsilon_{n-2} + l_{\eta\xi}(t) a_{n-1,\eta\xi}, \quad t \in I, n \geq 3$$

which implies that the sequence $\langle \eta, f_n(\cdot)\xi \rangle$ converges to $t \mapsto \langle \eta, f(\cdot)\xi \rangle$, where $f(\cdot) : I \to \widetilde{\mathcal{A}}$ is a Lusin measurable function. Since $x_n(\cdot)$ is bounded and

$$\|f_n(t)\|_{\eta\xi} = |\langle \eta, f_n(t)\xi \rangle| \leq l_{\eta\xi}(t) \|x_{n-1}(t)\|_{\eta\xi} + q_{\eta\xi}(t) + 1,$$

hence $f(\cdot)$ is Bochner integrable.

By passing with $n \to \infty$ in (3.1) and using Lebesgue dominated convergence theorem, we obtain

$$\langle \eta, x(t)\xi \rangle = \langle \eta, G(t)x_0\xi \rangle + \int_0^t U(t,s) \langle \eta, f(s)\xi \rangle ds, \quad t \in I.$$

On the other hand, from (3.2) we get

$$\langle \eta, f_n(t)\xi \rangle \in \big( \mathbb{P}(t, x_n(t))(\eta, \xi) + \epsilon_n B \big), \quad t \in I, n \geq 1$$

and letting $n \to \infty$ we obtain

$$\langle \eta, f(t)\xi \rangle \in \big( \mathbb{P}(t, x(t))(\eta, \xi) \quad t \in I.$$

Hence $x(\cdot)$ is a mild solution of the Cauchy problem (2.2) and the trajectory selection pair is $(x(\cdot), f(\cdot))$. □

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