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L_p APPROXIMATION WITH RATES BY GENERALIZED DISCRETE SINGULAR OPERATORS

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ABSTRACT. Here we give the approximation properties with rates of generalized discrete versions of Picard, Gauss-Weierstrass, and Poisson-Cauchy singular operators. We treat both the unitary and non-unitary cases of the operators above. We derive quantitatively L_p convergence of these operators to the unit operator by involving the L_p higher modulus of smoothness of an $L_p(\mathbb{R})$ function.

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1. Introduction

This article is motivated mainly by [3, Chapter 15], and [6], where J. Favard in 1944 introduced the discrete version of Gauss-Weierstrass operator

$$\left(F_n f\right)(x)=\frac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu}{n}\right) \exp \left(-n\left(\frac{\nu}{n}-x\right)^2\right), \tag{1.1}$$

$n \in \mathbb{N}$, which has the property that $\left(F_n f\right)(x)$ converges to $f(x)$ pointwise for each $x \in \mathbb{R}$, and uniformly on any compact subinterval of $\mathbb{R}$, for each continuous function f ($f \in C(\mathbb{R})$) that fulfills $|f(t)| \leq A e^{B t^2}$, $t \in \mathbb{R}$, where A, B are positive constants.

The well-known Gauss-Weierstrass singular convolution integral operators are

$$\left(W_n f\right)(x)=\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(u) \exp \left(-n(u-x)^2\right) du. \tag{1.2}$$

We are also motivated by [1], [2], and [3] where the authors studied extensively the approximation properties of particular generalized singular integral operators such as Picard, Gauss-Weierstrass, and Poisson-Cauchy as well as the general cases of singular integral operators. These operators are not necessarily positive linear operators.

In this article, we study quantitatively L_p approximation properties of Picard, Gauss-Weierstrass, and Poisson-Cauchy generalized singular discrete operators regarding convergence to the unit. We examine thoroughly the unitary and non-unitary cases and their interconnections.

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2. **Background**

In [3, p. 289–296], the authors studied the smooth general singular integral operators $\Theta_{r,\xi}(f;x)$ defined as follows. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, they defined

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i}\binom{r}{i} i^{-n}, & j = 0 \end{cases} \tag{2.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $\xi > 0$ and let μ_ξ be Borel probability measure on $\mathbb{R}$. For $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_p(\mathbb{R})$ where $1 \le p < \infty$, and $x \in \mathbb{R}$, they defined the integral

$$\Theta_{r,\xi}(f,x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) d\mu_\xi(t). \tag{2.2}$$

They observed that the operators $\Theta_{r,\xi}(f,x)$ are not positive operators and $\Theta_{r,\xi}(c,x) = c$, c constant. Additionally, they saw that

$$\Theta_{r,\xi}(f,x) - f(x) := \sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x))\, d\mu_\xi(t). \tag{2.3}$$

In [3, p. 290], the rth L_p modulus of smoothness finite given as

$$\omega_r(f^{(n)}, h)_p := \sup_{|t| \le h} \|\Delta_t^r f^{(n)}(x)\|_{p,x} < \infty, \quad h > 0, \tag{2.4}$$

where $\|\cdot\|_{p,x}$ is the L_p norm with respect to x and

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x+jt), \tag{2.5}$$

see also [5, p. 44]. Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

The authors introduced also

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \dots, n \in \mathbb{N}, \tag{2.6}$$

and the integrals

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t),\ k = 1, \dots, n. \tag{2.7}$$

They supposed that $c_{k,\xi} \in \mathbb{R}$, $k = 1, \dots, n$. Then, by using the terminology above, they derived

$$\Delta(x) := \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}. \tag{2.8}$$

In [3, p. 291], they proved

Theorem 2.1. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ and the rest as above. Furthermore suppose that*

$$M_\xi := \int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1}\, d\mu_\xi(t) < \infty. \tag{2.9}$$

Then

$$\|\Delta(x)\|_p \leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \cdot \left(\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}\,d\mu_\xi(t)\right)^{\frac{1}{p}} \xi^{\frac{1}{p}}\omega_r(f^{(n)},\xi)_p. \tag{2.10}$$

If $M_\xi \leq \lambda$, $\forall \xi > 0, \lambda > 0$, and as $\xi \to 0$ we obtain that $\|\Delta(x)\|_p \to 0$.

Moreover, they showed [3, p. 293].

Theorem 2.2. *Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}$. Suppose that*

$$\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,d\mu_\xi(t) < \infty. \tag{2.11}$$

Then

$$\|\Delta(x)\|_1 \leq \frac{1}{(r+1)(n-1)!}\left(\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}d\mu_\xi(t)\right)\xi\omega_r(f^{(n)},\xi)_1. \tag{2.12}$$

Additionally assume that

$$\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,d\mu_\xi(t) \leq \lambda, \lambda > 0, \tag{2.13}$$

$\forall \xi > 0$. Hence as $\xi \to 0$ we get $\|\Delta(x)\|_1 \to 0$.

They also demonstrated the case of $n = 0$ [3, p. 295].

Proposition 2.3. *Let $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$ and the rest as above. Suppose that*

$$\rho_\xi := \int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) < \infty. \tag{2.14}$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_p \leq \omega_r(f,\xi)_p\left(\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t)\right)^{\frac{1}{p}}. \tag{2.15}$$

Additionally assume that $\rho_\xi \leq \lambda, \lambda > 0$, $\forall \xi > 0$, then as $\xi \to 0$ we get $\Theta_{r,\xi} \to$ unit operator I in the L_p norm, $p > 1$.

Finally, they gave also [3, p. 296].

Proposition 2.4. *Suppose*

$$\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r} d\mu_\xi(t) < \infty. \tag{2.16}$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_1 \leq \omega_r(f,\xi)_1\left(\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r} d\mu_\xi(t)\right). \tag{2.17}$$

Additionally assuming that

$$\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r}d\mu_{\xi}(t)\leq\lambda,\lambda>0,\tag{2.18}$$

$\forall\xi>0$, *we obtain as* $\xi\to0$ *that* $\Theta_{r,\xi}\to I$ *in the* L_1 *norm.*

On the other hand, in [4], the authors defined important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ as follows:

Let $f\in C^n(\mathbb{R})$, $n\in\mathbb{Z}^+$, $0<\xi\leq1$, $x\in\mathbb{R}$.

i) When

$$\mu_{\xi}(\nu)=\frac{e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}},\tag{2.19}$$

they defined the generalized discrete Picard operators as

$$P_{r,\xi}^{*}\left(f;x\right):=\frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}.\tag{2.20}$$

ii) When

$$\mu_{\xi}(\nu)=\frac{e^{\frac{-\nu^{2}}{\xi}}}{\sum_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}},\tag{2.21}$$

they defined the generalized discrete Gauss-Weierstrass operators as

$$W_{r,\xi}^{*}\left(f;x\right):=\frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)e^{\frac{-\nu^{2}}{\xi}}}{\sum_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}.\tag{2.22}$$

iii) Let $\alpha\in\mathbb{N}$, and $\beta>\frac{1}{\alpha}$. When

$$\mu_{\xi}(\nu)=\frac{\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},\tag{2.23}$$

they defined the generalized discrete Poisson-Cauchy operators as

$$\Theta_{r,\xi}^{*}\left(f;x\right):=\frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}.\tag{2.24}$$

They observed that for c constant they have

$$P_{r,\xi}^{*}\left(c;x\right)=W_{r,\xi}^{*}\left(c;x\right)=\Theta_{r,\xi}^{*}\left(c;x\right)=c.\tag{2.25}$$

They assumed that the operators $P_{r,\xi}^{*}\left(f;x\right)$, $W_{r,\xi}^{*}\left(f;x\right)$, and $\Theta_{r,\xi}^{*}\left(f;x\right)\in\mathbb{R}$, for $x\in\mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}}<\infty$.

iv) Let $f \in C_u(\mathbb{R})$ (uniformly continuous functions) or $f \in C_b(\mathbb{R})$ (continuous and bounded functions). When

$$\mu_\xi(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}, \tag{2.26}$$

they defined the generalized discrete non-unitary Picard operators as

$$P_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{2.27}$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$m_{\xi,P} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{2.28}$$

They observed that

$$\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{2.29}$$

which is the probability measure (2.19) defining the operators $P^*_{r,\xi}$.

v) Let $f \in C_u(\mathbb{R})$ or $f \in C_b(\mathbb{R})$. When

$$\mu_\xi(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}, \tag{2.30}$$

with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\operatorname{erf}(\infty) = 1$, they defined the generalized discrete non-unitary Gauss-Weierstrass operators as

$$W_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{2.31}$$

Here $\mu_{\xi,W}(\nu)$ has mass

$$m_{\xi,W} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{2.32}$$

They observed that

$$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{2.33}$$

which is the probability measure (2.21) defining the operators $W^*_{r,\xi}$.

The authors observed that $P_{r,\xi}(f;x)$, $W_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$.

In [4], for $k = 1, \ldots, n$, the authors defined the sums

$$c_{k,\xi}^{*} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{2.34}$$

$$p_{k,\xi}^{*} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-\nu^{2}}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^{2}}{\xi}}}, \tag{2.35}$$

and for $\alpha \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, they introduced

$$q_{k,\xi}^{*} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^{k} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{2.36}$$

Furthermore, they proved that these sums $c_{k,\xi}^{*}$, $p_{k,\xi}^{*}$, and $q_{k,\xi}^{*}$ are finite.

In [4], the authors also proved

$$m_{\xi,P} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \to 1 \text{ as } \xi \to 0^{+} \tag{2.37}$$

and

$$m_{\xi,W} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^{2}}{\xi}}}{1 + \sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)} \to 1 \text{ as } \xi \to 0^{+}. \tag{2.38}$$

Additionally, in [4], the authors defined the following error quantities:

$$\begin{aligned} E_{0,P}(f,x) &:= P_{r,\xi}(f;x) - f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_{j} f(x+j\nu)\right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} - f(x), \end{aligned} \tag{2.39}$$

$$\begin{aligned} E_{0,W}(f,x) &:= W_{r,\xi}(f;x) - f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_{j} f(x+j\nu)\right) e^{\frac{-\nu^{2}}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} - f(x). \end{aligned} \tag{2.40}$$

Furthermore, they introduced the errors ($n \in \mathbb{N}$):

$$E_{n,P}(f,x) := P_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_{k} \frac{\sum_{\nu=-\infty}^{\infty} \nu^{k} e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \tag{2.41}$$

and

$$E_{n,W}(f,x) := W_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_{k} \frac{\sum_{\nu=-\infty}^{\infty} \nu^{k} e^{-\frac{\nu^{2}}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{2.42}$$

Next, they obtained the inequalities

$$\left|E_{0,P}(f,x)\right| \leq m_{\xi,P} \left|P_{r,\xi}^{*}(f;x) - f(x)\right| + |f(x)| \left|m_{\xi,P} - 1\right|, \tag{2.43}$$

$$|E_{0,W}(f,x)| \leq m_{\xi,W} \left|W^*_{r,\xi}(f;x) - f(x)\right| + |f(x)| \, |m_{\xi,W} - 1| \,, \tag{2.44}$$

and

$$\begin{aligned} &|E_{n,P}(f,x)| \\ &\leq m_{\xi,P} \left| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} \right| + |f(x)| \, |m_{\xi,P} - 1| \,, \end{aligned} \tag{2.45}$$

with

$$\begin{aligned} &|E_{n,W}(f,x)| \\ &\leq m_{\xi,W} \left| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right| + |f(x)| \, |m_{\xi,W} - 1| \,. \end{aligned} \tag{2.46}$$

3. **Main Results**

Let here $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_p(\mathbb{R})$ where $1 \leq p < \infty$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

First, we present our results for generalized discrete Picard operators.

Proposition 3.1. *Let $0 < \xi \leq 1$, $1 \leq p < \infty$, $n \in \mathbb{N}$ such that $np \neq 1$. Then, there exists $K_1 > 0$ such that*

$$\begin{aligned} M^*_{p,\xi} &:= \frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} \, e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \\ &\leq K_1 < \infty \end{aligned} \tag{3.1}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} > 1,$$

then

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} < 1.$$

Therefore, we obtain

$$\begin{aligned} M^*_{p,\xi} &< \sum_{\nu=-\infty}^{\infty} |\nu|^{np-1} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) e^{\frac{-|\nu|}{\xi}} \\ &< \sum_{\nu=-\infty}^{\infty} |\nu|^{np-1} \left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} e^{\frac{-|\nu|}{\xi}} \\ &:= R_1. \end{aligned} \tag{3.2}$$

We notice that

$$\begin{aligned} R_1 &= 2\sum_{\nu=1}^{\infty} \nu^{np-1}\left(1+\frac{\nu}{\xi}\right)^{rp+1} e^{\frac{-\nu}{\xi}} \\ &= 2\sum_{\nu=1}^{\infty}\left(\nu^{np-1}e^{\frac{-\nu}{2\xi}}\right)\left(\left(1+\frac{\nu}{\xi}\right)^{rp+1} e^{\frac{-\nu}{2\xi}}\right). \end{aligned} \tag{3.3}$$

Since we have $\frac{\nu}{\xi} \geq 1$ for $\nu \geq 1$, we get

$$\left(1+\frac{\nu}{\xi}\right)^{rp+1} e^{\frac{-\nu}{2\xi}} \leq \frac{2^{rp+1}\nu^{rp+1}}{\xi^{rp+1}e^{\frac{\nu}{2\xi}}} = \frac{2^{rp+1}z^{rp+1}}{e^{\frac{z}{2}}} \tag{3.4}$$

where $z := \frac{\nu}{\xi}$. Additionally, since

$$e^{\frac{z}{2}} = \sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^k}{k!} \geq \frac{z^{\lceil rp\rceil+1}}{2^{\lceil rp\rceil+1}\left(\lceil rp\rceil+1\right)!}, \tag{3.5}$$

where $\lceil\cdot\rceil$ is the ceiling of the number, we obtain

$$\frac{z^{\lceil rp\rceil+1}}{e^{\frac{z}{2}}} \leq 2^{\lceil rp\rceil+1}\left(\lceil rp\rceil+1\right)!. \tag{3.6}$$

Hence, by (3.3), (3.4), and (3.6), we have

$$\begin{aligned} R_1 &\leq 2^{2\lceil rp\rceil+3}\left(\lceil rp\rceil+1\right)!\sum_{\nu=1}^{\infty}\nu^{np-1}e^{\frac{-\nu}{2\xi}} \\ &\leq 2^{2\lceil rp\rceil+3}\left(\lceil rp\rceil+1\right)!\sum_{\nu=1}^{\infty}\nu^{np-1}e^{\frac{-\nu}{2}}. \end{aligned} \tag{3.7}$$

Now, we define the function $f(\nu) = \nu^{np-1}e^{\frac{-\nu}{2}}$ for $\nu \geq 1$. Then, we have $f'(\nu) = \nu^{np-2}e^{\frac{-\nu}{2}}\left(np-1-\frac{\nu}{2}\right)$. Thus, $f(\nu)$ is positive, continuous, and decreasing for $\nu > 2(np-1)$. Let $A := \lceil 2(np-1)\rceil$. Hence, by shifted triple inequality similar to [7], we get

$$\begin{aligned} &\sum_{\nu=1}^{\infty}\nu^{np-1}e^{\frac{-\nu}{2}} \\ &= \sum_{\nu=1}^{A}\nu^{np-1}e^{\frac{-\nu}{2}} + \sum_{\nu=A+1}^{\infty}\nu^{np-1}e^{\frac{-\nu}{2}} \\ &\leq \sum_{\nu=1}^{A}\nu^{np-1}e^{\frac{-\nu}{2}} + \int_{A+1}^{\infty}\nu^{np-1}e^{\frac{-\nu}{2}}d\nu + f(A+1) \\ &\leq \sum_{\nu=1}^{A}\nu^{np-1}e^{\frac{-\nu}{2}} + \int_{0}^{\infty}\nu^{\lceil np-1\rceil}e^{\frac{-\nu}{2}}d\nu + (A+1)^{np-1}e^{-\frac{A+1}{2}} \\ &= \lambda_n + (A+1)^{np-1}e^{-\frac{A+1}{2}} + \int_{0}^{\infty}\nu^{\lceil np-1\rceil}e^{\frac{-\nu}{2}}d\nu, \end{aligned} \tag{3.8}$$

where

$$\lambda_n := \sum_{\nu=1}^{A} \nu^{np-1} e^{\frac{-\nu}{2}} < \infty \tag{3.9}$$

for all $\xi \in (0,1]$. Furthermore, by the integral calculation in [3, p. 86], we obtain

$$\int_0^\infty \nu^{\lceil np-1 \rceil} e^{\frac{-\nu}{2}} d\nu = (\lceil np-1 \rceil)! 2^{\lceil np-1 \rceil + 1}. \tag{3.10}$$

Thus, by (3.7), (3.8), and (3.10), we get

$$\begin{aligned} R_1 &\leq 2^{2\lceil rp \rceil + 3} (\lceil rp \rceil + 1)! \\ &\quad \times \left[\lambda_n + (A+1)^{np-1} e^{-\frac{A+1}{2}} + (\lceil np-1 \rceil)! 2^{\lceil np-1 \rceil + 1} \right] \\ &:= K_1 < \infty \end{aligned} \tag{3.11}$$

for all $\xi \in (0,1]$. Then, by (3.2) and (3.11), the proof is done. □

We have the following quantitative result.

Theorem 3.2. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$, and the rest as above in this section. Then*

$$\begin{aligned} &\left\| P_{r,\xi}^{*} (f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^{*} \right\|_p \\ &\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(M_{p,\xi}^{*} \right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p. \end{aligned} \tag{3.12}$$

Additionally, as $\xi \to 0^+$ we obtain that R.H.S. of (3.12) *goes to zero.*

Proof. By *Theorem* 2.1 and *Proposition* 3.1. □

We present the related result for the case of $p = 1$.

Theorem 3.3. *Let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\begin{aligned} &\left\| P_{r,\xi}^{*} (f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^{*} \right\|_1 \\ &\leq \frac{1}{(n-1)! \, (r+1)} M_{1,\xi}^{*} \xi \omega_r(f^{(n)}, \xi)_1 \end{aligned} \tag{3.13}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.13) *goes to zero.*

Proof. By *Theorem* 2.2 and *Proposition* 3.1. □

Next, we demonstrate the following result.

Proposition 3.4. *Let $0 < \xi \leq 1$, and $1 \leq p < \infty$. Then there exists $K_2 > 0$such that*

$$\bar{M}^*_{p,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \leq K_2 < \infty \tag{3.14}$$

for all $\xi \in (0,1]$.

Proof. For $n \geq 2$, we observe that

$$\begin{aligned} \bar{M}^*_{p,\xi} &\leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \quad (3.15) \\ &\leq \sum_{\nu=-\infty}^{\infty} |\nu|^{2p-1} \left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} e^{\frac{-|\nu|}{\xi}} \\ &\leq R_1. \end{aligned}$$

Therefore, by *Proposition* 3.1, we get the desired result. □

We give the special case of $n = 0$.

Proposition 3.5. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above in this section. Then*

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_p \leq \left(\bar{M}^*_{p,\xi}\right)^{1/p} \omega_r(f,\xi)_p \tag{3.16}$$

holds. Hence, as $\xi \to 0^+$, we obtain that $P^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

Proof. By *Proposition* 2.3 and *Proposition* 3.4. □

Next result is for the special case of $n = 0$ and $p = 1$.

Proposition 3.6. *The inequality*

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_1 \leq \bar{M}^*_{1,\xi} \omega_r(f,\xi)_1 \tag{3.17}$$

holds. Furthermore, we get $P^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

Proof. By *Proposition* 2.4 and *Proposition* 3.4. □

Next, we present our results for generalized discrete Gauss-Weierstrass operators.

Proposition 3.7. *Let $0 < \xi \leq 1$, $1 \leq p < \infty$, $n \in \mathbb{N}$ such that $np \neq 1$. Then, there exists $K_3 > 0$ such that*

$$\begin{aligned} N^*_{p,\xi} &:= \frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1\right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \quad (3.18) \\ &\leq K_3 < \infty \end{aligned}$$

for all $\xi \in (0,1]$.

Proof. For all $\nu \in \mathbb{Z}$, we have

$$\frac{\nu^2}{\xi} \geq \frac{|\nu|}{\xi}. \tag{3.19}$$

Therefore,

$$e^{\frac{-\nu^2}{\xi}} \leq e^{\frac{-|\nu|}{\xi}}$$

for all $\nu \in \mathbb{Z}$. Thus

$$\begin{aligned} N^*_{p,\xi} &\leq \sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \\ &\leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \\ &\leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}} \\ &= R_{1.} \end{aligned} \tag{3.20}$$

Hence, by *Proposition* 3.1, we get the desired result. □

We have the following quantitative result.

Theorem 3.8. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$, and the rest as above in this section. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right\|_p \tag{3.21}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(N^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p.$$

Additionally, as $\xi \to 0^+$ we obtain that R.H.S. of (3.21) *goes to zero.*

Proof. By *Theorem* 2.1 and *Proposition* 3.7. □

We have the following result for the special case of $p = 1$.

Theorem 3.9. *Let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right\|_1 \tag{3.22}$$

$$\leq \frac{1}{(n-1)!\,(r+1)} N^*_{1,\xi} \xi \omega_r(f^{(n)}, \xi)_1$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.22) *goes to zero.*

Proof. By *Theorem* 2.2 and *Proposition* 3.7. □

Next, we demonstrate

Proposition 3.10. *Let $0 < \xi \leq 1$, and $1 \leq p < \infty$. Then, there exists $K_4 > 0$ such that*

$$\bar{N}^*_{p,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \leq K_4 < \infty \tag{3.23}$$

for all $\xi \in (0, 1]$.

Proof. By *Proposition* 3.1 and *Proposition* 3.7, for $n \geq 2$, we have

$$\begin{aligned} \bar{N}^*_{p,\xi} &\leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \\ &\leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} |\nu|^{2p-1} e^{\frac{-\nu^2}{\xi}} \\ &\leq R_1 < \infty \end{aligned} \tag{3.24}$$

for all $\xi \in (0, 1]$. ☐

We give the next result for the special case of $n = 0$.

Proposition 3.11. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above in this section. Then*

$$\left\| W^*_{r,\xi}(f; x) - f(x) \right\|_p \leq \left(\bar{N}^*_{p,\xi}\right)^{1/p} \omega_r(f, \xi)_p \tag{3.25}$$

holds. Hence, as $\xi \to 0^+$, we obtain that $W^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

Proof. By *Proposition* 2.3 and *Proposition* 3.10. ☐

Next result is for the special case of $n = 0$ and $p = 1$.

Proposition 3.12. *The inequality*

$$\left\| W^*_{r,\xi}(f; x) - f(x) \right\|_1 \leq \bar{N}^*_{1,\xi} \omega_r(f, \xi)_1 \tag{3.26}$$

holds. Furthermore, we get $W^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

Proof. By *Proposition* 2.4 and *Proposition* 3.10. ☐

Next, we give our results for generalized discrete Poisson-Cauchy operators.

Proposition 3.13. *Let $0 < \xi \leq 1$, $1 \leq p < \infty$, $n \in \mathbb{N}$ such that $np \neq 1$, and $\beta > \frac{p(r+n)+1}{2\alpha}$. Then, there exists $K_5 > 0$ such that*

$$\begin{aligned} S^*_{p,\xi} &:= \frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1\right) |\nu|^{np-1} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \\ &\leq K_5 < \infty \end{aligned} \tag{3.27}$$

for all $\xi \in (0, 1]$.

Proof. For $\nu \geq 1$, we notice that

$$\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}<\nu^{-2\alpha\beta}. \tag{3.28}$$

Then, we observe that

$$\begin{aligned} \xi^{-2\alpha\beta} &< \sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \qquad (3.29)\\ &= \xi^{-2\alpha\beta}+2\sum_{\nu=1}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\\ &< \xi^{-2\alpha\beta}+2\sum_{\nu=1}^{\infty}\nu^{-2\alpha\beta}<\infty. \end{aligned}$$

Therefore, we get

$$\frac{1}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}<\xi^{2\alpha\beta}. \tag{3.30}$$

Thus, by (3.30), we have

$$\begin{aligned} S_{p,\xi}^{*} &\leq \xi^{2\alpha\beta}\left[\sum_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}-1\right)|\nu|^{np-1}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\right] \qquad (3.31)\\ &\leq \xi^{2\alpha\beta}\left[\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}|\nu|^{np-1}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\right]\\ &:= R_2. \end{aligned}$$

Moreover, by (3.28), we obtain

$$\begin{aligned} R_2 &= \sum_{\nu=-\infty}^{\infty}\left(\xi^{\frac{2\alpha\beta}{rp+1}}+\xi^{\frac{2\alpha\beta}{rp+1}-1}|\nu|\right)^{rp+1}|\nu|^{np-1}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \qquad (3.32)\\ &\leq 2\sum_{\nu=1}^{\infty}(1+\nu)^{rp+1}\nu^{np-1-2\alpha\beta}\\ &\leq 2^{rp+2}\sum_{\nu=1}^{\infty}\frac{\nu^{rp+1}}{\nu^{2\alpha\beta-np+1}}\\ &= 2^{rp+2}\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}\right)^{2\alpha\beta-p(r+n)}\\ &< \infty \end{aligned}$$

for all $\xi \in (0,1]$ since $2\alpha\beta - p(r+n) > 1$. ☐

We have the following quantitative result

Theorem 3.14. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $n\in\mathbb{N}$, $\beta>\frac{p(r+n)+1}{2\alpha}$, and the rest as above in this section. Then*

$$\left\|\Theta_{r,\xi}^{*}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_p \tag{3.33}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}\left(S_{p,\xi}^{*}\right)^{\frac{1}{p}}\xi^{\frac{1}{p}}\omega_r(f^{(n)},\xi)_p.$$

Additionally, as $\xi \to 0^+$, *we obtain that R.H.S. of* (3.33) *goes to zero.*

Proof. By *Theorem* 2.1 and *Proposition* 3.13. □

We have the following result for the special case of $p = 1$.

Theorem 3.15. *Let* $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, $\beta > \frac{r+n+1}{2\alpha}$, *and* $n \in \mathbb{N} - \{1\}$. *Then*

$$\left\|\Theta_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_1 \tag{3.34}$$

$$\leq \frac{1}{(n-1)!\,(r+1)}S_{1,\xi}^{*}\xi\omega_r(f^{(n)},\xi)_1$$

holds. Hence, as $\xi \to 0^+$, *we obtain that R.H.S. of* (3.34) *goes to zero.*

Proof. By *Theorem* 2.2 and *Proposition* 3.13. □

Next, we demonstrate

Proposition 3.16. *Let* $0 < \xi \leq 1$, $\beta > \frac{p(r+2)+1}{2\alpha}$, *and* $1 \leq p < \infty$. *Then there exist* $K_6 > 0$ *such that*

$$\bar{S}_{p,\xi}^{*} := \frac{\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{rp}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq K_6 < \infty \tag{3.35}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\begin{aligned} \bar{S}_{p,\xi}^{*} &\leq \xi^{2\alpha\beta}\left[\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{rp}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\right] \\ &\leq \xi^{2\alpha\beta}\left[\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}|\nu|^{2p-1}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\right] \\ &\leq R_2 < \infty, \end{aligned} \tag{3.36}$$

for all $n \geq 2$. Therefore, by *Proposition* 3.13, we get the desired result. □

We give the next result for the special case of $n = 0$.

Proposition 3.17. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+2)+1}{2\alpha}$, *and the rest as above in this section. Then*

$$\left\|\Theta_{r,\xi}^{*}(f;x) - f(x)\right\|_p \leq \left(\bar{S}_{p,\xi}^{*}\right)^{1/p}\omega_r(f,\xi)_p \tag{3.37}$$

holds. Hence, as $\xi \to 0^+$, *we obtain that* $\Theta_{r,\xi}^{*} \to$ *unit operator* I *in the* L_p *norm for* $p > 1$.

Proof. By *Proposition* 2.3 and *Proposition* 3.16. □

Next result is for the special case of $n = 0$ and $p = 1$.

Proposition 3.18. *Let $\beta > \frac{r+3}{2\alpha}$ and the rest as above in this section. The inequality*

$$\left\|\Theta^*_{r,\xi}(f;x) - f(x)\right\|_1 \leq \bar{S}^*_{1,\xi}\omega_r(f,\xi)_1 \tag{3.38}$$

holds. Furthermore, we get $\Theta^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

Proof. By *Proposition* 2.4 and *Proposition* 3.16. □

Next, we give our results for the error quantities $E_{0,P}(f,x)$, $E_{0,W}(f,x)$ and the errors $E_{n,P}(f,x)$, $E_{n,W}(f,x)$.

Theorem 3.19. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ such that $np \neq 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\begin{aligned} &\left\|E_{n,P}(f,x)\right\|_p \\ &\leq \frac{\xi^{\frac{1}{p}}\omega_r(f^{(n)},\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \\ &\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1\right)|\nu|^{np-1} e^{\frac{-|\nu|}{\xi}}\right)}{1+2\xi e^{-\frac{1}{\xi}}}\right]^{\frac{1}{p}} \\ &+ \left\|f(x)\right\|_p |m_{\xi,P} - 1| \end{aligned} \tag{3.39}$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (3.39) *goes to zero.*

Proof. By (2.37), (2.45), (3.2), (3.11), and *Theorem* 3.2. □

For the special case of $p = 1$, we have the following result

Theorem 3.20. *Let $f \in C^n(\mathbb{R})$, $f \in L_1(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\begin{aligned} \left\|E_{n,P}(f,x)\right\|_1 &\leq \frac{\xi\omega_r(f^{(n)},\xi)_1}{(n-1)!\,(r+1)} \\ &\times \left[\frac{\sum_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1} - 1\right)|\nu|^{n-1} e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right] \\ &+ \left\|f(x)\right\|_1 |m_{\xi,P} - 1| \end{aligned} \tag{3.40}$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (3.40) *goes to zero.*

Proof. By (2.37), (2.45), (3.2), (3.11), and *Theorem* 3.3. □

For the special case of $n = 0$, we have the following result

Proposition 3.21. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\|E_{0,P}(f,x)\|_p \leq \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}} \times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \right)^{1/p}}{1+2\xi e^{-\frac{1}{\xi}}} \right] + \|f(x)\|_p \left|m_{\xi,P} - 1\right| \tag{3.41}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.41) *goes to zero.*

Proof. By (2.37), (2.43), (3.15), and *Proposition* 3.5. □

Next, we demonstrate the special case of $n = 0$ and $p = 1$

Proposition 3.22. *The inequality*

$$\|E_{0,P}(f,x)\|_1 \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{r} e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) \omega_r(f,\xi)_1 + \|f(x)\|_1 \left|m_{\xi,P} - 1\right| \tag{3.42}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.42) *goes to zero.*

Proof. By (2.37), (2.43), (3.15), and *Proposition* 3.6. □

Next, we have the following quantitative result for $E_{n,W}(f,x)$

Theorem 3.23. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ such that $np \neq 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\|E_{n,W}(f,x)\|_p \leq \frac{\xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)}{\sqrt{\pi\xi}\left(1 - erf\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]^{\frac{1}{p}} + \|f(x)\|_p \left|m_{\xi,W} - 1\right| \tag{3.43}$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (3.43) *goes to zero.*

Proof. By (2.38), (2.46), (3.20), and *Theorem* 3.8. □

For the special case of $p = 1$, we have the following result

Theorem 3.24. *Let $f \in C^n(\mathbb{R}), f \in L_1(\mathbb{R}),\ f^{(n)} \in L_1(\mathbb{R}),$ and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| E_{n,W}(f,x) \right\|_1 \leq \frac{\xi \omega_r(f^{(n)},\xi)_1}{(n-1)!\,(r+1)} \times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - erf\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \left\| f(x) \right\|_1 |m_{\xi,W} - 1| \tag{3.44}$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (3.44) *goes to zero.*

Proof. By (2.38), (2.46), (3.20), and *Theorem* 3.9. □

For the special case of $n = 0$, we have the following result

Proposition 3.25. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\left\| E_{0,W}(f,x) \right\|_p \leq \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} \times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi}\left(1 - erf\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \left\| f(x) \right\|_p |m_{\xi,W} - 1| \tag{3.45}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.45) *goes to zero.*

Proof. By (2.38), (2.44), (3.24), and *Proposition* 3.11. □

Next, we demonstrate the special case of $n = 0$ and $p = 1$.

Proposition 3.26. *The inequality*

$$\left\| E_{0,W}(f,x) \right\|_1 \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{r} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - erf\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) \omega_r(f,\xi)_1 + \left\| f(x) \right\|_1 |m_{\xi,W} - 1| \tag{3.46}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (3.46) *goes to zero.*

Proof. By (2.38), (2.44), (3.24), and *Proposition* 3.12. □

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