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FIXED POINT RESULTS FOR MAPS WITH WEAKLY SEQUENTIALLY CLOSED GRAPHS

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ABSTRACT. In this paper we present an alternative of Leray-Schauder type and a fixed point result of Furi-Pera type. An application is given to illustrate our theory.

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1. INTRODUCTION

In this paper we first prove an alternative of Leray-Schauder type. This in particular improves a result in [5] where a condition was omitted. Then using this Leray-Schauder alternative we will obtain a new fixed point result of Furi-Pera type. This improves a result in [6] where one of the conditions was incorrectly stated and its proof needs to be adjusted slightly (see Theorem 2.4 below). Our results in particular extend those of [2, 3, 5, 12]. For the remainder of this section we gather some notations and preliminary facts. Let X be a Banach space, let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

Definition 1.1. A function $\psi : \mathcal{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions :

(1) The family $\ker(\psi) = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{co}(M)) = \psi(M)$, where $\overline{co}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0, 1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

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The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty}\psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi(\overline{M^w}) = \psi(M) \tag{1.1}$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \text{ if and only if } M \text{ is relatively weakly compact,} \tag{1.2}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.3}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R}, \tag{1.4}$$

set additive if

$$\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)). \tag{1.5}$$

An important example of a measure of weak noncompactness has been defined by De Blasi [8] as follows :

$$w(M) = \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}, \tag{1.6}$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(\cdot)$ is regular, homogeneous, subadditive and set additive (see [8]).

Let X and Y be topological spaces. A multivalued map $F : X \to 2^Y$ is a point to set function if for each $x \in X$, $F(x)$ is a nonempty subset of Y. For a subset M of X we write $F(M) = \cup_{x\in M}F(x)$ and $F^{-1}(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$. The *graph* of F is the set $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. We say that F is *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for every neighborhood V of $F(x)$ there exists a neighborhood U of x with $F(U) \subseteq V$ (equivalently, $F : X \to 2^Y$ is u.s.c. if for any net $\{x_\alpha\}$ in X and any closed set B in Y with $x_\alpha \to x_0 \in X$ and $F(x_\alpha) \cap B \neq \emptyset$ for all α, we have $F(x_0) \cap B \neq \emptyset$). We say that $F : X \to 2^Y$ is upper semicontinuous if it is upper semicontinuous at every $x \in X$. The function F is lower semicontinuous (l.s.c.) if the set $F^{-1}(B)$ is open for any open set B in Y. If F is l.s.c. and u.s.c., then F is continuous.

If Y is compact, and the images $F(x)$ are closed, then F is upper semicontinuous if and only if F has a closed graph. In this case, if Y is compact, we have that F is upper semicontinuous if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. When X is a Banach space we say that $F : X \to 2^X$ is weakly upper semicontinuous if F is upper semicontinuous in X endowed with the weak topology. Also, $F : X \to 2^X$ is said to have weakly sequentially closed graph if the graph of F is sequentially closed w.r.t. the weak topology of X.

Definition 1.2. Let X be a Banach space and let ψ be a measure of weak noncompactness on X. A multivalued mapping $B : D(B) \subseteq X \to 2^X$ is said to be ψ-condensing if it maps bounded sets into bounded sets and $\psi(B(S)) < \psi(S)$ whenever S is a bounded subset of $D(B)$ such that $\psi(S) > 0$.

The following Sadovskii type fixed point theorem (see [5]) for multivalued mappings with weakly sequentially closed graph will be used in Section 2.

Theorem 1.3. *Let X be a Banach space, ψ a regular set additive measure of weak noncompactness on X and C a nonempty closed convex subset of X. Suppose $F : C \to C(C)$ is ψ-condensing, $F(C)$ is bounded and F has weakly sequentially closed graph; here $C(C)$ denotes the family of nonempty, closed, convex subsets of C. Then F has a fixed point.*

2. FIXED POINT THEOREMS

Our first result is a Leray-Schauder alternative principle.

Theorem 2.1. *Let X be a Banach space and ψ a regular set additive measure of weak noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $0 \in U$. Suppose $F : \overline{U^w} \to C(C)$ has weakly sequentially closed graph, $F(\overline{U^w})$ is bounded and F is a ψ-condensing map. Also assume U is weakly open in C and F transforms relatively weakly compact sets into relatively weakly compact sets. Then either*

$$F \text{ has a fixed point,} \tag{2.1}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{2.2}$$

here $\partial_Q U$ is the weak boundary of U in Q.

Proof. Suppose (2.2) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.1) occurs). Let

$$M = \{x \in \overline{U^w} : x \in \lambda F x \text{ for some } \lambda \in [0,1]\}.$$

The set M is nonempty since $0 \in U$. Also M is weakly sequentially closed. Indeed let (x_n) be sequence of M which converges weakly to some $x \in \overline{U^w}$ and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n)

converges to some $\lambda \in [0,1]$ and without loss of generality assume $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges weakly to some $z \in \overline{U^w}$ with $x = \lambda z$. Since F has weakly sequentially closed graph then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus M is weakly sequentially closed. We now claim that M is relatively weakly compact. Suppose $\psi(M) > 0$. Since $M \subseteq co(F(M) \cup \{0\})$ then

$$\psi(M) \leq \psi(co(F(M) \cup \{0\})) = \psi(F(M)) < \psi(M),$$

which is a contradiction. Hence $\psi(M) = 0$ and therefore $\overline{M^w}$ is weakly compact. This proves our claim. Now let $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact (Eberlein-Šmulian theorem [10 pg. 549]) then there is a sequence (x_n) in M which converges weakly to x. Since M is weakly sequentially closed we have $x \in M$. Thus $\overline{M^w} = M$. Hence M is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \emptyset$. Since X endowed with the weak topology is a locally convex space then there exists a continuous mapping $\rho : \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$. Let

$$J(x) = \begin{cases} \rho(x)F(x), & x \in \overline{U^w}, \\ 0, & x \in C \setminus \overline{U^w}. \end{cases}$$

Clearly $J : C \to C(C)$ has weakly sequentially closed graph since F has sequentially closed graph. Moreover, for any $S \subseteq C$ we have

$$J(S) \subseteq co(J(S \cap U) \cup \{0\}).$$

If $\psi(S \cap U) > 0$ then

$$\psi(J(S)) \leq \psi(co(F(S \cap U) \cup \{0\})) = \psi(F(S \cap U)) < \psi(S \cap U) \leq \psi(S),$$

whereas if $\psi(S \cap U) = 0$ then

$$\psi(J(S)) \leq \psi(F(S \cap U)) = 0 < \psi(S),$$

if $\psi(S) > 0$. Thus $J : C \to C(C)$ is ψ-condensing. From Theorem 1.3 there exists $x \in C$ such that $x \in J(x)$. Now $x \in U$ since $0 \in U$. Consequently $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

Remark 2.2. In Theorem 2.1 above notice $\partial_Q U = \partial_C U$. We note that the condition U is weakly open in C was omitted in Theorem 2.6 in [4] and in Theorem 2.1 (and the other results in Section 2) in [13] and the condition F transforms relatively weakly compact sets into relatively weakly compact sets was omitted in Theorem 2.2 in [5].

Corollary 2.3. *Let X be a Banach space and ψ a regular set additive measure of weak noncompactness on X. Let C be a closed, convex subsets of X. In addition let U be a weakly open subset of C with $0 \in U$. Suppose $F : \overline{U^w} \to C(C)$ has weakly sequentially closed graph, $F(\overline{U^w})$ is bounded and F is a ψ-condensing map. Also* *assume F transforms relatively weakly compact sets into relatively weakly compact sets. Then either*

$$F \text{ has a fixed point,} \tag{2.3}$$

or

$$\text{there is a point } u \in \partial_C U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{2.4}$$

here $\partial_C U$ is the weak boundary of U in C.

Our next result is a Furi-Pera type result.

Theorem 2.4. *Let X be a Banach space and ψ a regular and set additive measure of weak noncompactness on X. Let C be a closed convex subset of X and Q a closed convex subset of C with $0 \in Q$. Assume the weak topology on C is metrizable. Also, assume $F : Q \to C(C)$ has weakly sequentially closed graph, F is ψ-condensing map, $F(Q)$ bounded and F transforms relatively weakly compact sets into relatively weakly compact sets. In addition, assume that the following conditions are satisfied:*

- (i) *there exists a weakly continuous retraction $r : X \to Q$, with $r(D) \subseteq \overline{co}(D \cup \{0\})$ for any bounded subset D of X and $r(x) = x$ for $x \in Q$.*
- (ii) *if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial Q$, $\lambda_j \to \lambda$ and $x \in \lambda F(x), 0 \le \lambda < 1$, then $\lambda_j F(x_j) \subseteq Q$ for j sufficiently large; here ∂Q is the weak boundary of Q relative to C.*

Then F has a fixed point in Q.

Proof. Let r be as described in (i) and let

$$B = \{x \in X \,:\, x \in Fr(x)\}.$$

We first show that $B \neq \emptyset$. To see this, consider $Fr : C \to C(C)$. Clearly Fr has weakly sequentially closed graph, since F has weakly sequentially closed graph and r is weakly continuous. Now we show that Fr is a ψ-condensing map. To see this, let A be a bounded subset of C and $\psi(A) > 0$. Now

$$F\,r(A) \subseteq F\,\overline{co}(A \cup \{0\}).$$

Note $\psi(\overline{co}(A \cup \{0\})) = \psi(A) > 0$ so

$$\psi(F\,r(A)) < \psi(\overline{co}(A \cup \{0\})) = \psi(A).$$

Thus Fr is a ψ-condensing map. Now Theorem 1.3 guarantees that there exists $y \in C$ with $y \in Fr(y)$. Thus $y \in B$ and $B \neq \emptyset$. Note B is weakly sequentially closed, since Fr has weakly sequentially closed graph. Moreover, we claim that B is weakly compact. To see this, first note

$$B \subseteq F\,r\,(B) \subseteq F\,\overline{co}(B \cup \{0\}).$$

If $\psi(B) > 0$ then since $\psi(\overline{co}(B \cup \{0\})) = \psi(B) > 0$ we have

$$\psi(B) \leq \psi(F\,\overline{co}(B \cup \{0\})) < \psi(\overline{co}(B \cup \{0\})) = \psi(B),$$

a contradiction. Thus, $\psi(B) = 0$ and so B is relatively weakly compact. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact then there is a sequence (x_n) of elements of B which converges weakly to some x. Since B is weakly sequentially closed then $x \in B$. Thus, $\overline{B^w} = B$. This implies that B is weakly compact.

We now show that $B \cap Q \neq \emptyset$. Suppose $B \cap Q = \emptyset$. From our assumption the weak topology on C is metrizable, let d^* denote the metric. With respect to (C, d^*) note Q is closed, B is compact, $B \cap Q = \emptyset$ so there exists $\epsilon > 0$ with

$$d^*(B, Q) = \inf\{d^*(x, y) : x \in B, y \in Q\} > \epsilon.$$

For $i \in \{1, 2 \ldots\}$, let

$$U_i = \left\{x \in C : d^*(x, Q) < \frac{\epsilon}{i}\right\}.$$

For each $i \in \{1, 2 \ldots\}$ fixed, U_i is open with respect to d^* and so U_i is weakly open in C. Also

$$\overline{U_i^w} = \overline{U_i^{d^*}} = \left\{x \in C : d^*(x, Q) \leq \frac{\epsilon}{i}\right\} \quad \text{and} \quad \partial U_i = \left\{x \in C \,:\, d^*(x, Q) = \frac{\epsilon}{i}\right\}.$$

Note $\overline{U_i^w} \cap B = \emptyset$, so Corollary 2.3 (with $F = Fr$, $U = U_i$) guarantees that there exists $y_i \in \partial U_i$ and $\lambda_i \in (0, 1)$ with $y_i \in \lambda_i Fr(y_i)$; note $F\,r$ transforms relatively weakly compact sets into relatively weakly compact sets since r is weakly continuous and F transforms relatively weakly compact sets into relatively weakly compact sets and note also (see above) that $F\,r$ is a ψ-condensing map. Note since $y_i \in \partial U_i$ that $\lambda_i Fr(y_i) \not\subseteq Q$. We now consider

$$D = \{x \in X : x \in \lambda Fr(x),\ \text{for some } \lambda \in [0, 1]\}.$$

Note

$$D \subseteq \overline{co}(F\,r\,D \cup \{0\}) \subseteq \overline{co}(F(\overline{co}(D \cup \{0\})) \cup \{0\})$$

so if $\psi(D) > 0$ then since $\psi(\overline{co}(D \cup \{0\})) = \psi(D)$ we have

$$\psi(D) \leq \psi(\overline{co}(F(\overline{co}(D \cup \{0\})) \cup \{0\})) = \psi(F(\overline{co}(D \cup \{0\}))) < \psi(\overline{co}(D \cup \{0\})) = \psi(D),$$

a contradiction. Thus $\psi(D) = 0$ so D is relatively weakly compact. The reasoning above implies that D is weakly compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0, 1]$ and $y_i \rightharpoonup y^* \in Q$. Since F has weakly sequentially closed graph then $y^* \in \lambda^* Fr(y^*)$. Note $\lambda^* \neq 1$ since $B \cap Q = \emptyset$. From assumption (ii) it follows that $\lambda_i Fr(y_i) \subseteq Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e. $x \in Fx$. □

Remark 2.5. One of the conditions in Theorem 2.3 in [6] was stated incorrectly and the proof has to be adjusted slightly (i.e. modify slightly the proof of Theorem 2.4 above).

Next we establish an existence principle for the operator equation

$$y(t) \in N\,y(t), \quad t \in [0,T] \quad (T > 0 \ \text{ fixed}) \tag{2.5}$$

in $C([0,T],\mathbf{R}^n)$. Our result extends a result in [12, Theorem 3.9] and in [3, Theorem 2.8] (we note that one of the assumption in [12] was stated incorrectly). Recall $W^{k,p}([0,T],\mathbf{R}^n)$, $1 \le p < \infty$, denotes the space of functions $u : [0,T] \to \mathbf{R}^n$ with $u^{(k-1)} \in AC[0,T]$ and $u^{(k)} \in L^p[0,T]$. Note $W^{k,p}([0,T],\mathbf{R}^n)$ is reflexive if $1 < p < \infty$. Also we let $\|\cdot\|_\infty$ denote the usual supremum norm and $\|\cdot\|_2$ the usual L^2 norm.

Theorem 2.6. *Suppose $N : W^{1,2}([0,T],\mathbf{R}^n) \to K(W^{1,2}([0,T],\mathbf{R}^n))$ has weakly sequentially closed graph; here $K(W^{1,2}([0,T],\mathbf{R}^n))$ denotes the family of nonempty, convex, weakly closed subsets of $W^{1,2}([0,T],\mathbf{R}^n)$. In addition assume the following two conditions hold:*

$$\begin{cases} \exists\ M_0 > 0 \textit{ such that if } u \in W^{1,2}([0,T],\mathbf{R}^n) \textit{ satisfies} \\ u \in \lambda N u \textit{ for } 0 < \lambda < 1, \ \textit{then } \|u\|_\infty \neq M_0 \end{cases} \tag{2.6}$$

and

$$\begin{cases} \exists\, N_0 \ge M_0, \ \textit{and } \exists\, N_1 > 0 \textit{ such that if } u \in W^{1,2}([0,T],\mathbf{R}^n) \\ \textit{with } \|u\|_\infty \le M_0 \textit{ and } \|u'\|_2 \le N_1, \ \textit{then } \|N\,u\|_\infty \le N_0 \\ \textit{and } \|N\,u\|_2 \le N_1. \end{cases} \tag{2.7}$$

Then (2.5) *has a solution in* $W^{1,2}([0,T],\mathbf{R}^n)$.

Proof. Let $E = W^{1,2}([0,T],\mathbf{R}^n)$,

$$C = \left\{ u \in W^{1,2}([0,T],\mathbf{R}^n) : \ \|u\|_\infty \le N_0 \ \text{ and } \ \|u'\|_2 \le N_1 \right\}$$

and

$$U = \left\{ u \in W^{1,2}([0,T],\mathbf{R}^n) : \ \|u\|_\infty < M_0 \ \text{ and } \ \|u'\|_2 \le N_1 \right\}.$$

Notice C is a convex, closed, bounded subset of E. We first show U is weakly open in C. To do this we will show that $C\backslash U$ is weakly closed. Let $x \in \overline{C\backslash U}^w$. Then there exists $x_n \in C\backslash U$ (see [7 pp. 81, 9 pp. 93]) with $x_n \rightharpoonup x$ (here $W^{1,2}([0,T],\mathbf{R}^n)$ is endowed with the weak topology and $\rightharpoonup$ denotes weak convergence). We must show $x \in C\backslash U$. Now since the embedding $j : W^{1,2}([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is completely continuous [1], there is a subsequence S of integers with

$$x_n \to x \ \text{ in } \ C([0,T],\mathbf{R}^n) \ \text{ and } \ x_n' \rightharpoonup x' \ \text{ in } \ L^2([0,T],\mathbf{R}^n)$$

as $n \to \infty$ in S. Also

$$\|x\|_\infty = \lim_{n\to\infty} \|x_n\|_\infty \ \text{ and } \ \|x'\|_2 \le \liminf \|x_n'\|_2 \le N_1.$$

Note $M_0 \leq \|x\|_\infty \leq N_0$ since $M_0 \leq \|x_n\|_\infty \leq N_0$ for all n. As a result $x \in C \backslash U$, so $\overline{C \backslash U}^w = C \backslash U$. Thus U is weakly open in C. Also

$$\partial U = \{u \in C : \|u\|_\infty = M_0\} \quad \text{and} \quad \overline{U^w} = \{u \in C : \ \|u\|_\infty \leq M_0\}.$$

To see this let $x \in \overline{U^w}$. Then [7 pp. 81] guarantees that there exists $x_n \in U$ with $x_n \rightharpoonup x$. Essentially the same reasoning as above yields $\|x\|_\infty \leq M_0$ and $\|x'\|_2 \leq N_1$, so $\overline{U^w} \subseteq \{u \in C : \|u\|_\infty \leq M_0\}$. On the other hand if $x \in A = \{u \in C : \|u\|_\infty \leq M_0\}$ (note A is closed), then there exists $x_n \in U$ with $x_n \to x$ in $W^{1,2}([0,T], \mathbf{R}^n)$, so in particular $x_n \rightharpoonup x$ in $W^{1,2}([0,T], \mathbf{R}^n)$. Thus $x \in \overline{U^w}$, so $\overline{U^w} = \{u \in C : \|u\|_\infty \leq M_0\}$.

Next note C is weakly compact (note $W^{1,2}([0,T], \mathbf{R}^n)$ is reflexive), (2.7) guarantees that $N : \overline{U^w} \to C(C)$ and N transforms relatively weakly sets into relatively weakly compact sets (note $N(\overline{U^w}) \subseteq C$ and C is weakly compact). Also (2.6) guarantees that (2.4) is not true (note if there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda N x$ then $\|x\|_\infty = M_0$ since $x \in \partial U$ and $\|x\|_\infty \neq M_0$ from (2.6)). Corollary 2.3 guarantees that N has a fixed point in $\overline{U^w}$. □

Remark 2.7. In Theorem 2.6 it is enough to assume $N : \overline{U^w} \to K(C)$ has weakly sequentially closed graph; here U and C are as described in the proof.

Remark 2.8. Indeed it is clear that there is an analogue of Theorem 2.6 where $W^{1,2}([0,T], \mathbf{R}^n)$ is replaced by $W^{k,p}([0,T], \mathbf{R}^n)$, here $1 < p < \infty$.

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