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GLOBAL EXISTENCE AND CONVERGENCE ANALYSIS FOR HYBRID DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper some fundamental results concerning the global existence, convergence theorem and continuous dependence on the initial data are proved for the nonlinear hybrid differential equations with a linear perturbation of second type via applications of differential inequalities with a comparison principle. The results of this paper are complementary to the work presented in a recent papers of Dhage and Jadhav (2013) and Dhage (2014).

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1. INTRODUCTION

Given a closed but unbounded interval $J_\infty = [t_0, \infty)$ in $\mathbb{R}$, $\mathbb{R}$ the real line, for some fixed $t_0 \in \mathbb{R}$, let $C(J_\infty, \mathbb{R})$ denote a class of continuous real-valued functions defined on J_∞. Consider an initial value problem of ordinary hybrid differential equations (in short HDE)

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t, x(t))\big] = g(t, x(t)), \quad t \in J_\infty, \\ x(t_0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where, $f, g \in C(J_\infty \times \mathbb{R}, \mathbb{R})$.

By a ***solution*** of the HDE (1.1) we mean a function $x \in C(J_\infty, \mathbb{R})$ such that

(i) the function $t \mapsto x - f(t, x)$ is continuous for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (1.1).

Again, a function $r \in C(J_\infty, \mathbb{R})$ is called a ***maximal solution*** of the HDE (1.1) if for any other solution x defined on J_∞, we have that $x(t) \le r(t)$ for all $t \in J_\infty$.

The HDE (1.1) has been discussed in Dhage and Jadhav [4] for the existence and comparison principle and in Dhage [3] for the existence via monotone iterative technique. However, several other qualitative aspects of the problem such as stability, boundedness of the solutions etc. are still open. In this paper, we shall continue the study of the HDE (1.1) and prove the global existence and convergence theorems under some suitable conditions along the lines of Dhage and Lakshmikantham [5] and

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Lakshmikantham and Leela [8]. The following hypotheses concerning the function f are crucial in the study of HDE (1.1) on J_∞.

(A_0) The function $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for each $t \in J_\infty$.

(A_1) There exists a continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$|f(t,x) - f(t,y)| \le \psi(|x-y|)$$
for all $t \in J_\infty$ and $x, y \in \mathbb{R}$. Moreover, $\psi(r) < r$.

There do exist functions f satisfying the hypotheses (A_0)-(A_1) mentioned above. In fact, the function
$$f(t,x) = x - \tan^{-1} x$$
satisfies (A_0) and the function
$$f(t,x) = \tan^{-1} x$$
satisfies (A_1) with the ψ function given by $\psi(r) = \dfrac{r}{1+\xi^2}$ for $0 < \xi < r$. We note that if $f(t,x) = 0$ on $J_\infty \times \mathbb{R}$, then the HDE (1.1) reduces to the usual nonlinear ordinary differential equation,
$$\left.\begin{aligned} &x'(t) = g(t, x(t)), \quad t \in J_\infty, \\ &x(t_0) = x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.2}$$

We shall also make use of the following result in what follows.

Lemma 1.1. *Assume that hypothesis (A_0) holds. Further, if $g(\cdot, x(\cdot)) \in L^1(J_\infty, \mathbb{R})$ for some $x \in C(J_\infty, \mathbb{R})$, then the HDE* (1.1) *is equivalent to the hybrid integral equation (HIE),*
$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^{t} g(s, x(s))\, ds, \quad t \in J_\infty. \tag{1.3}$$

Proof. Let $g(\cdot, x(\cdot)) \in L^1(J, \mathbb{R}_+)$ for some $x \in C(J_\infty, \mathbb{R})$. Assume first that x is a solution of the HDE (1.1). By definition, the function $t \mapsto x(t) - f(t, x(t))$ is continuous, and so, differentiable there, whence $\dfrac{d}{dt}[x(t) - f(t, x(t))]$ is Lebesgue integrable on J_∞. Applying integration to (1.1) from t_0 to t, we obtain the HIE (1.3) on J_∞.

Conversely, assume that the function $x \in C(J_\infty, \mathbb{R})$ satisfies the HIE (1.2) on J_∞. Then, by direct differentiation we obtain the HDE (1.1). Again, substituting $t = t_0$ in (1.3) yields
$$x(t_0) - f(t_0, x(t_0)) = x_0 - f(t_0, x_0).$$
Since the mapping $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for all $t \in J_\infty$, the mapping $x \mapsto x - f(t_0, x)$ is injective in $\mathbb{R}$, whence $x(t_0) = x_0$. Hence, the proof of the lemma is complete. □

In the following section, we prove a global existence result for the HDE (1.1) via the comparison method.

2. GLOBAL EXISTENCE RESULT

We place and seek the solutions of the HDE (1.1) in the space $C(J_\infty, \mathbb{R})$ of continuous real-valued functions on unbounded interval J_∞. Define a norm $\|\cdot\|$ in $C(J_\infty, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J_\infty} |x(t)|.$$

Clearly, $BC(J_\infty, \mathbb{R})$ is a Banach space with respect to the above supremum norm. The study of hybrid fixed point theorems in Banach space involving the addition of two operators is initiated by Krasnoselskii [7] and some interesting applicable hybrid fixed point theorems may be found in Dhage [1, 2]. We need the following fixed point theorem for proving a main existence result of this paper.

Definition 2.1. A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathcal{D}$-function** if it is an upper semi-continuous and nondecreasing function satisfying $\psi(0) = 0$. A mapping $Q : E \to E$ is called **$\mathcal{D}$-Lipschitz** if there is a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Q\phi - Q\xi\| \leq \psi(\|\phi - \xi\|) \tag{2.1}$$

for all $\phi, \xi \in E$. If $\psi(r) = k\,r$, $k > 0$, then Q is called **Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **contraction** on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called a **nonlinear $\mathcal{D}$-contraction** and the function ψ is called a $\mathcal{D}$-function of Q on X.

The details of different types of contractions appear in the monographs of Dhage [1] and Granas and Dugundji [6]. There do exist $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are

$$\psi(r) = k\,r, \text{ for some constant } k > 0,$$

$$\psi(r) = \frac{L\,r}{K + r}, \text{ for some constants } L > 0, K > 0 \text{ with } L \leq K,$$

$$\psi(r) = \tan^{-1} r,$$

$$\psi(r) = \log(1 + r),$$

$$\psi(r) = e^r - 1.$$

These $\mathcal{D}$-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods. The class of $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Remark 2.2. If $\phi, \psi\ \mathbb{R}_+ \to \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$.

Another notion that we need in the sequel is the following definition.

Definition 2.3. An operator Q on a Banach space E into itself is called **compact** if $Q(E)$ is a relatively compact subset of E. Q is called **totally bounded** if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called **completely continuous** on E.

Theorem 2.4 (Dhage [1]). *Let S be a closed convex and bounded subset of the Banach algebra E and let $A : E \to E$ and $B : S \to E$ be two operators such that*

(a) *A is nonlinear $\mathcal{D}$-contraction,*
(b) *B is compact and continuous, and*
(c) *$x = Ax + By$ for all $y \in S \Longrightarrow x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S.

Now, consider a scaler HDE,

$$\left.\begin{aligned} \frac{d}{dt}\left[p(t) - F(t, p(t))\right] = G(t, p(t)), \quad t \in J_\infty,\\ p(t_0) = p_0 \in \mathbb{R}_+ \end{aligned}\right\} \tag{2.2}$$

where, $F \in C(J_\infty \times \mathbb{R}_+, \mathbb{R}_+)$ and $G \in C(J_\infty \times \mathbb{R}_+, \mathbb{R}_+)$.

We consider the following set of assumptions:

(B_0) The assumption (A_0) holds with f replaced by F.
(B_1) The functions $F(t, p)$ and $G(t, p)$ are nondecreasing in r for each $t \in J_\infty$.
(B_2) The HDE (2.2) has a solution $p(t)$ on J_∞ and $G_p(\cdot) = G(\cdot, p(\cdot)) \in C(J_\infty, \mathbb{R}_+)$.
(B_3) There exists a continuous function $h : J_\infty \to \mathbb{R}$ such that

$$|G(t, x)| \leq h(t), \quad t \in J_\infty,$$

for all $x \in \mathbb{R}$.

Theorem 2.5. *Assume that hypotheses (A_0)-(A_1) and (B_0)-(B_2) hold. Suppose also that the functions f, g and F, G involved respectively in* (1.1) *and* (2.1) *satisfy*

$$\left.\begin{aligned} |f(t, x)| &= F(t, |x|),\\ \text{and} \qquad |g(t, x)| &\leq G(t, |x|), \end{aligned}\right\} \tag{2.3}$$

for all $(t, x) \in J_\infty \times \mathbb{R}$. Then for every initial value x_0 with

$$|x_0 - f(t_0, x_0)| \leq p_0 - F(t_0, p_0),$$

the HDE (1.1) *has a global solution defined on J_∞ satisfying*

$$|x(t)| \leq r(t), \quad t \in J_\infty. \tag{2.4}$$

Proof. Set $E = BC(J_\infty, \mathbb{R})$ and define a subset E_0 of E by

$$E_0 = \big\{x \in E \mid |x(t)| \leq p(t)\big\} \tag{2.5}$$

where, $p(t)$ is a solution of the HDE (2.2) existing on J_∞. Clearly, E_0 is a closed, convex and bounded subset of the Banach algebra E. By hypothesis (B_2) and the condition (2.3) for every $x \in E_0$, the HDE (1.1) is equivalent to the hybrid integral equation,

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^{t} g(s, x(s))\, ds \tag{2.6}$$

for all $t \in J_\infty$.

Define two operators $A, B : E_0 \to E$ by

$$Ax(t) = f(t, x(t)), \quad t \in J_\infty, \tag{2.7}$$

and

$$Bx(t) = x_0 - f(t_0, x_0) + \int_{t_0}^{t} g(s, x(s))\, ds, \quad t \in J_\infty. \tag{2.8}$$

Then, the HIE (2.6) is transformed into an operator equation as

$$Ax(t) + Bx(t) = x(t), \quad t \in J_\infty. \tag{2.9}$$

A solution of the operator equation (2.9) is a solution of the HIE (2.6) and vice versa. We show that the operators A and B satisfy all the conditions of Theorem 2.4. Now, it can be shown as in Dhage and Jadhav [4] that A is a nonlinear $\mathcal{D}$-contraction operator on E into itself and B is a compact and continuous operator on E_0 into E.

Next, we show that hypothesis (c) of Theorem 2.4 is satisfied. Let $x \in E_0$ be arbitrary. Then, by hypotheses (B_0)–(B_2), we obtain

$$\begin{aligned}
|x(t)| &= |Ax(t) + By(t)| \\
&\le |Ax(t)| + |Bx(t)| \\
&\le |x_0 - f(t_0, x_0)| + \big|f(t, x(t))\big| + \Big|\int_{t_0}^{t} |g(s, x(s))|\, ds\Big| \\
&\le |x_0 - f(t_0, x_0)| + F(t, |x(t)|) + \int_{t_0}^{t} G(s, |x(s)|)|\, ds \\
&\le p_0 - F(t_0, p_0) + F(t, p(t)) + \int_{t_0}^{t} G(s, p(s))\, ds \\
&= p(t)
\end{aligned}$$

for all $t \in J_\infty$. It then follows that $x \in E_0$ for all $y \in E_0$.

Finally, by hypothesis (A_1), the operator A is a nonlinear $\mathcal{D}$-contraction with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{M+r}$ and so, hypothesis (a) of Theorem 2.4 is satisfied. Now, we apply Theorem 2.4 to yield that the operator equation $Ax + Bx = x$ has a solution in E_0. As a result the HDE (1.1) has a solution defined on J_∞. This completes the proof. $\square$

Remark 2.6. Note that first equality in the condition (2.3) holds, in particular if the function $f(t,x)$ is positive and even in x, i.e., $f(t,-x) = f(t,x)$ for all $t \in J_\infty$. Then, in this case we identify the function F with f on $J_\infty \times \mathbb{R}_+$.

Below we give a direct proof of the global existence result by assuming the local existence of solution for the HDE (1.1). We need the following result proved in Dhage and Jadhav [4] in what follows.

Lemma 2.7 (Dhage and Jadhav [4])**.** *Let $J = [t_0, t_1]$ be a closed and bounded interval in $\mathbb{R}$ for some $t_0 \leq t_1 < \infty$. Assume that hypotheses (A_0)–(A_1) hold with f replaced by F. Suppose also that hypothesis (B_3) holds. If there exists a function $m \in C(J, \mathbb{R})$ such that*

$$\left.\begin{aligned} \frac{d}{dt}\left[m(t) - F(t, m(t))\right] \leq G(t, m(t)), \quad t \in J,\\ m(t_0) \leq r_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{2.10}$$

then,

$$m(t) \leq r(t), \quad t \in J, \tag{2.11}$$

where r is a maximal solution of the HDE (2.2) *defined on J.*

Theorem 2.8. *Assume that all the hypotheses of Theorem* 2.5 *and condition* (2.3) *hold. Then for any solution x of the HDE* (1.1) *one has*

$$|x(t)| \leq r(t), \tag{2.12}$$

for all $t \in J$, where r is a maximal solution of the HDE (2.2) *on J.*

Proof. Let x be any solution of the HDE (1.1) on J. Take

$$m(t) = |x(t)|, \quad t \in J.$$

Then,

$$\begin{aligned} \frac{d}{dt}\left[m(t) - F(t, m(t))\right] &= \frac{d}{dt}\left[|x(t) - f(t, x(t))|\right] \\ &\leq \left|\frac{d}{dt}\left[x(t) - f(t, x(t))\right]\right| \\ &= |g(t, x(t))| \\ &\leq G(t, |x(t)|) \\ &= G(t, m(t)) \end{aligned}$$

for all $t \in J$ and

$$m(t_0) = |x(t_0)| \leq p_0.$$

Now, a direct application of Lemma 2.7 yields the desired inequality (2.12) on J and the proof is complete. □

Theorem 2.9. *Assume that hypothesis* (A_0) *holds and the function* g *is smooth enough that the HDE* (1.1) *has a local solution. Assume further that the functions* f, g *and* F, G *involved in* (1.1) *and satisfy* (2.3) *on* J_∞*. Suppose further that the maximal solution of the HDE* (2.1) *exists on* J_∞*. Then the solution of the HDE* (1.1) *exists on* J_∞*.*

Proof. Let $x(t)$ be any local solution of the HDE (1.1) existing on the bounded interval $[t_0, t_1)$ for some $t_0 < t_1 < \infty$ such that the value of t_1 cannot be increased.

Now, by Theorem 2.8, we have $|x(t)| \le r(t)$ for all $t \in J$, where r is a maximal solution of the HDE (2.2) defined on J.

Then, for any $t_0 \le \eta < \tau < t_1$, one has

$$\begin{aligned}
\Big|\big(x(\tau) - f(\tau, x(\tau))\big) - \big(x(\eta) - f(\eta, x(\eta))\big)\Big| &= \left|\int_\eta^\tau g(s, x(s))\, ds\right| \\
&\le \int_\eta^\tau |g(s, x(s))|\, ds \\
&\le \int_\eta^\tau G(t, |x(s)|)\, ds \\
&\le \int_\eta^\tau G(t, r(s))\, ds \\
&= \big(r(\tau) - F(\tau, r(\tau))\big) - \big(r(\eta) - F(\eta, r(\eta))\big). \qquad (2.13)
\end{aligned}$$

Since $\lim_{t\to t_1^-} r(t)$ exists and is finite, taking the limit as $\tau, \eta \to t_1^-$ in the inequality (2.13) and using the Cauchy criterion for convergence, it follows that the $\lim_{t\to t_1^-} \big(x(t) - f(t, x(t))\big)$ exists and

$$\lim_{t\to t_1^-} \big[x(t) - f(t, x(t)\big] = \big(x(t_1) - f(t_1, x(t_1))\big).$$

Since (A_0) holds, $\lim_{t\to t_1^-} x(t) = x(t_1)$ and we define $x(t_1) = x_1$. Now, consider the HDE

$$\left.\begin{aligned}
\frac{d}{dt}[x(t) - f(t, x(t))] = g(t, x(t)), \quad t \ge t_1,\\
x(t_1) = x_1 \in \mathbb{R}.
\end{aligned}\right\} \qquad (2.14)$$

By assumed local existence, we find that $x(t)$ can be continued beyond t_1, contradicting to our assumption about its interval of existence. Hence, a solution of the HDE (1.1) with $|x_0| \le r_0$ exists on J_∞ and the proof is complete. □

3. CONVERGENCE THEOREM

In this section we develop a convergence theorem for the HDE (1.1) and show that the sequence of successive approximations defined in a certain way converges to the unique solution of the HDE (1.1) on a closed and bounded interval $J = [t_0, t_0 + a]$. Consider the scalar ODE

$$\left.\begin{aligned} u'(t) &= G(t, u(t)), \quad t \in J, \\ u(t_0) &= u_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{3.1}$$

where $G \in \mathcal{C}(J \times \mathbb{R}_+, \mathbb{R}_+)$.

We list the following set of assumptions:

(C_1) There exists a constant $M_1 > 0$ such that $0 \le G(t, u) \le M_1$ for all $(t, u) \in J \times \mathbb{R}_+$.
(C_2) $G(t, 0) = 0$ for all $t \in J$ and $G(t, u)$ is nondecreasing in u for each $t \in J$.
(C_3) $u(t) \equiv 0$ is the unique solution of the ODE (3.1) on J with $u(t_0) = 0$.
(C_4) The functions g and G involved in (1.1) and (3.1) satisfy

$$|g(t, x) - g(t, y)| \le G\Big(t, \Big|(x - f(t, x)) - (y - f(t, y))\Big|\Big),$$

for all , $t \in J$, and $x, y \in \mathbb{R}$.

The following result is well-known in the literature.

Lemma 3.1. *Assume that the hypothesis (C_1) holds. If there exists a function $m \in C(J, \mathbb{R}_+)$ such that*

$$\left.\begin{aligned} m'(t) &\le G(t, m(t)), \quad t \in J, \\ m(t_0) &\le u_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{3.2}$$

then,

$$m(t) \le r(t) \tag{3.3}$$

for all $t \in J$, where $r(t)$ is a maximal solution of the HDE (3.1) *defined on J.*

Theorem 3.2. *Assume that the hypotheses (A_0) and (C_1)–(C_4) hold. Then the HDE* (1.1) *has a unique solution x^* and the sequence of successive approximations $\{x_n\}$ defined by*

$$x_{n+1}(t) = x_0 - f(t_0, x_0) + f(t, x_{n+1}(t)) + \int_{t_0}^{t} g(s, x_n(s))\, ds, \quad t \in J, \tag{3.4}$$

converges to the unique solution x^* defined on J.

Proof. It is easy to see, by induction that the successive approximations $\{x_n\}$ given by (3.4) are well defined and continuous on J.

Now, set

$$X_n(t) = x_n(t) - f(t, x_n(t)), \quad t \in J. \tag{3.5}$$

Then, the functions X_n are well defined and continuous real-valued functions defined on J for each $n = 0, \ldots$. Moreover,

$$\|X_n\| \le |x_0 - f(t_0, x_0)| + \|h\|_{L^1} = M$$

for all $n \in \mathbb{N}$. Therefore, $X_n \in \overline{\mathcal{B}}_M(0)$, where $\overline{\mathcal{B}}_M(0)$ is a closed ball in the Banach space $C(J, \mathbb{R})$ centered at origin 0 of radius M.

We shall now define the successive approximations for IVP (3.1) as follows:

$$\left.\begin{array}{rcl} u_0 & = & M(t-t_0), \\ u_{n+1}(t) & = & \displaystyle\int_{t_0}^{t} G(s, u_n(s))ds, \quad t \in J. \end{array}\right\} \tag{3.6}$$

An easy induction proves that the successive approximations (3.6) are well defined and satisfy

$$0 \leq u_{n+1}(t) \leq u_n(t) \quad \text{ on } J$$

for all $n = 0, 1, \ldots$. Since $|u_n'(t)| \leq M_1$, we conclude by Ascoli-Arzelá Theorem and the monotonicity of the sequence $\{u_n(t)\}$ that

$$\lim_{n \to \infty} u_n(t) = u(t)$$

uniformly for $t \in J$. It is clear that $u(t)$ satisfies (3.3) and hence by (C$_3$), $u(t) \equiv 0$ on J by Lemma 3.1. Now,

$$|X_1(t) - X_0| \leq \int_{t_0}^{t} G(s, x_0)\, ds \leq M(t-t_0) = u_0(t).$$

Assume that for some fixed integer k,

$$|X_k(t) - X_{k-1}(t)| \leq u_{k-1}(t).$$

Since

$$|X_{k+1}(t) - X_k(t)| \leq \int_{t_0}^{t} |g(s, x_k)(s) - g(s, x_{k-1})|\, |ds,$$

using the nondecreasing nature of $G(t, u)$ in u and the assumption (C$_3$), we get

$$|X_{k+1}(t) - X_k(t)| \leq \int_{t_0}^{t} G(s, u_{k-1}(s))ds = u_k(t),$$

in view of (3.6). Thus, by principle of induction, the inequality

$$|X_{n+1}(t) - X_n(t)| \leq u_n(t), \quad t \in [t_0, t_0 + \alpha] \tag{3.7}$$

is true for all n. Also, we have

$$\begin{array}{rcl} |X_{n+1}'(t) - X_n'(t)| & \leq & |g(s, x_n(s)) - g(s, x_{n-1}(s))| \\ & \leq & G(t, |X_n(t) - X_{n-1}(t)|) \\ & \leq & G(t, u_{n-1}(t)), \end{array} \tag{3.8}$$

because of (3.7) and the nondecreasing character of $G(t,u)$. Let $m \geq n$. Then, one can easily obtain using (3.8),

$$\begin{aligned}|X_n'(t) - X_m'(t)| &= |g(t,x_{n-1}(t)) - g(t,x_{m-1}(t))| \\ &\leq |g(t,x_n(t)) - g(t,x_{n-1}(t))| + |g(t,x_m(t)) - g(t,x_{m-1}(t))| \\ &\quad + |g(t,x_n(t)) - g(t,x_m(t))| \\ &\leq G(t,u_{n-1}(t)) + G(t,u_{m-1}(t)) + G(t,|X_n(t) - X_m(t)|). \end{aligned} \tag{3.9}$$

Since $u_{n+1}(t) \leq u_n(t)$, it follows that

$$\frac{d}{dt}\Big[|X_n(t) - X_m(t)|\Big] \leq G(t,|X_n(t) - X_m(t)|) + 2G(t,u_{n-1}(t)). \tag{3.10}$$

An application of the comparison theorem given in Lemma 3.1 yields

$$|X_n(t) - X_m(t)| \leq r_n(t), \quad t \in J,$$

where $r_n(t)$ is the maximal solution of

$$v_n' = G(t,v_n) + 2G(t,u_{n-1}(t)), \quad v_n(t_0) = 0,$$

for each n. Since $G(t,u_{n-1}(t)) \to 0$, as $n \to \infty$, uniformly on J, it follows by Lemma 3.1 that $r_n(t) \to 0$ uniformly on J. This implies that $X_n(t)$ converges uniformly to $X(t)$ on J. Therefore, we obtain

$$\lim_{n\to\infty}\Big(x_n(t) - f(t,x_n(t))\Big) = \lim_{n\to\infty} x_n(t) - f\left(t, \lim_{n\to\infty} x_n(t)\right) = x(t) - f(t,x(t))$$

uniformly on J. Since, (A_0) holds, $\lim_{n\to\infty} x_n(t) = x(t)$. Now using the standard arguments, it is proved that $x(t)$ is a solution of the HDE (1.1). To show that the solution is unique, let $y(t)$ be another solution of HDE (1.1) existing on J. Define $m(t) = |(x(t) - f(t,x(t))) - (y(t) - f(t,y(t)))|$. Note that $m(t_0) = 0$. Then, by hypothesis (C_4),

$$\begin{aligned} m'(t) &\leq \left|\frac{d}{dt}\left[x(t) - f(t,x(t))\right] - \frac{d}{dt}\left[y(t) - f(t,y(t))\right]\right| \\ &= |g(t,x(t)) - g(t,y(t))| \\ &\leq G\Big(t, |(x(t) - f(t,x(t))) - (y(t) - f(t,y(t)))|\Big) \\ &= G(t,m(t)), \end{aligned}$$

whenever $x, y \in C(J,\mathbb{R})$, using assumption C_4). Again, applying Lemma 3.1, we get

$$m(t) \leq r(t), \quad t \in J,$$

where $r(t)$ is the maximal solution of the HDE (3.1). But by assumption (C_3), $r(t) \equiv 0$ and this proves that $x(t) \equiv y(t)$. Hence, the limit of the successive approximations is the unique solution of HDE (1.1) and the proof is complete. □

4. CONTINUOUS DEPENDENCE

Finally, in this section, we discuss the continuity of solutions of the HDE (1.1) with respect to initial data t_0, x_0. For this purpose, we need the following result.

Lemma 4.1. *Let $G : J \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying*

$$|g(t,x)| \leq G\Big(t, |(x - f(t,x)) - (x_0 - f(t_0,x_0))|\Big) \tag{4.1}$$

for all $t \in J$ and $x \in \mathbb{R}$. Assume that $x(t) = x(t,t_0,x_0)$ be any solution of the HDE (1.1) *on J. Then,*

$$|(x(t) - f(t,x(t))) - (x_0 - f(t_0,x_0))| \leq r^*(t,t_0,0) \tag{4.2}$$

for all $t \in J$, where $r^*(t,t_0,0)$ is the maximal solution of the differential equation

$$\left.\begin{aligned} u'(t) &= G(t,u(t)), \quad t \in J, \\ u(t_0) &= 0. \end{aligned}\right\} \tag{4.3}$$

Proof. Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = |(x(t) - f(t,x(t))) - (x_0 - f(t_0,x_0))| \,. \tag{4.4}$$

Then,

$$\begin{aligned} m'(t) &\leq \left| \frac{d}{dt}\big[x(t) - f(t,x(t))\big] \right| \\ &= |g(t,x(t))| \\ &\leq G\Big(t, |(x(t) - f(t,x(t))) - (x_0 - f(t_0,x_0))|\Big) \\ &= G(t,m(t)) \end{aligned}$$

for all $t \in J$ and $m(t_0) = 0$. Now an application of Lemma 3.1 yields that

$$|(x(t,t_0,x_0) - f(t,x(t,t_0,x_0))) - (x_0 - f(t_0,x_0))| \leq r^*(t,t_0,0).$$

□

Theorem 4.2. *Assume that hypotheses (A_0) and (C_1)–(C_4) hold. Further, if the solutions of the differential equations* (3.1) *are continuous with respect to* (t_0, x_0)*, then the solutions of the HDE* (1.1) *are unique and continuous with respect to the initial values* (t_0, x_0).

Proof. Since the uniqueness of the solution follows from Theorem 3.2, we only prove the continuity part of the theorem. To this end, let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,y_0)$ be two solutions of the HDE (1.1) through (t_0,x_0) and (t_0,y_0) respectively.

Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = |(x(t) - f(t,x(t))) - (y(t) - f(t,y(t)))| \,. \tag{4.5}$$

Then,

$$m'(t) \le G(t, m(t)), \quad t \in J,$$

and hence by the comparison theorem

$$m(t) \le r\left(t, t_0, |(x_0 - f(t_0, x_0)) - (y_0 - f(t_0, y_0))|\right)$$

where, $r(t, t_0, u_0)$ with $u_0 = \big|(x_0 - f(t_0, x_0)) - (y_0 - f(t_0, y_0))\big|$ is a maximal solution of the ODE (3.1) defined on J.

Since the solutions of ODE (3.1) are continuous with respect to initial values, it follows that

$$\lim_{x_0 \to y_0} r(t, t_0, u_0) = r(t, t_0, 0)$$

and by hypothesis $r(t, t_0, 0) = 0$. This, in view of definition of $m(t)$, it shows that

$$\lim_{x_0 \to y_0} x(t, t_0, x_0) = y(t, t_0, y_0)$$

whence the continuity of the solution relative to x_0 follows.

Next, we shall prove the continuity of the solutions with respect to t_0. If $x(t, t_0, x_0)$ and $y(t, t_1, x_0)$, $t_1 > t_0$, are any two solutions of the HDE (1.1) through (t_0, x_0) and (t_1, x_0) respectively, then, as before, we obtain the inequality

$$m'(t) \le G(t, m(t)), \quad t \in [t_1, t_0 + a],$$

where,

$$m(t) = \left|(x(t, t_0, x_0) - f(t, x(t, t_0, x_0))) - (y(t, t_1, x_0) - f(t, y(t, t_1, x_0)))\right|.$$

Thus,

$$m(t_1) = \left|(x(t_1, t_0, x_0) - f(t_1, x(t_1, t_0, x_0))) - (x_0 - f(t_0, x_0))\right|.$$

Hence, by Lemma 4.1,

$$m(t_1) \le r^*(t_1, t_0, 0)$$

which further implies that

$$m(t) \le \hat{r}(t), \quad t > t_1,$$

where $\hat{r}(t) = r(t, t_1, r^*(t_1, t_0, 0))$ is a maximal solution of (4.3) through $(t_1, r^*(t_1, t_0, 0))$.

Since $r^*(t_1, t_0, 0) = 0$, we have

$$\lim_{t_1 \to t_0} = \hat{r}(t, t_1, r^*(t_1, t_0, 0)) = \hat{r}(t, t_0, 0)$$

and by hypothesis, $\hat{r}(t, t_0, 0) = 0$. Now, from definition of $m(t)$ and the hypothesis (A_0), it follows that

$$\lim_{t_1 \to t_0} x(t, t_1, x_0) = x(t, t_0, x_0).$$

This proves the desired continuity of the solutions $x(t, t_0, x_0)$ of the HDE (1.1) with respect to t_0 and the proof of the theorem is complete. □

Next, we shall now prove the continuous dependence of solutions with respect to a parameter involved in the HDE (1.1). Consider the HDE,

$$\left.\begin{aligned} \frac{d}{dt}\left[x(t) - f(t, x(t))\right] = g(t, x(t), \mu), \quad t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{4.6}$$

where, $g \in C(J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R})$.

Let $x_0(t) = x(t, t_0, x_0, \mu_0)$ denote the solution of the HDE (4.6) corresponding to the parameter $\mu = \mu_0$. We consider the following hypotheses in what follows.

(D$_1$) $\lim_{\mu \to \mu_0} g(t, x, \mu) = g(t, x, \mu_0)$ uniformly in $(t, x) \in J \times \mathbb{R}$.

(D$_2$) There exists a function $G \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ such that

$$|g(t, x, \mu) - g(t, y, \mu)| \leq G\Big(t, |(x(t) - f(t, x(t))) - (y(t) - f(t, y(t)))|\Big), \quad t \in J,$$

for all $x, y \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$.

Theorem 4.3. *Assume that hypotheses (A$_0$)–(A$_2$), (C$_3$) and (D$_1$)–(D$_2$) hold. Then, for given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that the HDE* (4.6) *admits a unique solution $x(t) = x(t, t_0, x_0, \mu)$ satisfying*

$$|x(t) - x_0(t)| < \epsilon \tag{4.7}$$

for all $t \in J$, whenever $|\mu - \mu_0| < \delta(\epsilon)$.

Proof. The uniqueness of the solutions is obvious from Theorem 3.2. From the assumption that $u(t) \equiv 0$ is the only solution of the DE (4.3) existing on J, it follows by Theorem 3.2 that for given $\epsilon > 0$ there exists a number $\eta = \eta(\epsilon) > 0$ such that the maximal solution $r(t, t_0, x_0, \eta)$ of

$$u'(t) = G(t, u(t)) + \eta, \quad t \in J, \tag{4.8}$$

exists on J and satisfies

$$r(t, t_0, 0, \eta) < \epsilon \tag{4.9}$$

for all $t \in J$. Also, because of hypothesis (D$_1$), there exists a $\delta = \delta(\eta) > 0$ such that

$$|g(t, x, \mu) - g(t, x, \mu_0)| < \eta \tag{4.10}$$

whenever $|\mu - \mu_0| < \delta$. Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = |(x(t) - f(t, x(t))) - (x_0(t) - f(t, x_0(t)))|$$

where, $x(t)$ and $x_0(t)$ are the solutions of the HDE (4.6) corresponding to the values of parameter μ and μ_0 respectively. Then, using hypothesis (D$_2$), we get

$$m'(t) \leq G(t, m(t)) + \eta, \quad t \in J, \tag{4.11}$$

and by a comparison result(Lemma 3.1),

$$m(t) \leq r(t, t_0, 0, \eta)$$

for all $t \in J$. Hence, if $|\mu - \mu_0| < \delta$, then we have $m(t) < \epsilon$ for all $t \in J$. This further in view of hypothesis (A_0) together with continuity of the function $x \mapsto x - f(t,x)$ for all $t \in J$ implies that

$$|x(t, t_0, x_0, \mu) - x(t, t_0, x_0, \mu_0)| < \epsilon,$$

whenever $|\mu - \mu_0| < \delta$. This completes the proof. □

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