Communications in Applied Analysis **19** (2015), 589–604

EXISTENCE OF POSITIVE SOLUTIONS TO A SYSTEM OF HIGHER-ORDER SEMIPOSITONE INTEGRAL BOUNDARY VALUE PROBLEMS

RODICA LUCA[1] AND ALEXANDRU TUDORACHE[2]

[1]Department of Mathematics, Gh. Asachi Technical University
Iasi 700506, Romania
E-mail: rlucatudor@yahoo.com

[2]Faculty of Computer Engineering and Automatic Control
Gh. Asachi Technical University, Iasi 700050, Romania
E-mail: alexandru.tudorache93@gmail.com

ABSTRACT. We investigate the existence of positive solutions for a system of nonlinear higher-order ordinary differential equations with sign-changing nonlinearities, subject to integral boundary conditions.

AMS (MOS) Subject Classification. 34B10, 34B18.

1. INTRODUCTION

We consider the system of nonlinear higher-order ordinary differential equations

$$\begin{cases} u^{(n)}(t) + \lambda f(t, u(t), v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + \mu g(t, u(t), v(t)) = 0, & t \in (0, T), \end{cases} \tag{S}$$

with the integral boundary conditions

$$\begin{cases} u(0) = \int_0^T u(s)dH_1(s), & u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(T) = \int_0^T u(s)dH_2(s), \\ v(0) = \int_0^T v(s)dK_1(s), & v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(T) = \int_0^T v(s)dK_2(s), \end{cases} \tag{BC}$$

where $n, m \in \mathbb{N}$, $n, m \geq 2$. In the case $n = 2$ or $m = 2$ the above conditions are of the form $u(0) = \int_0^T u(s)dH_1(s)$, $u(T) = \int_0^T u(s)dH_2(s)$, or $v(0) = \int_0^T v(s)dK_1(s)$, $v(T) = \int_0^T v(s)dK_2(s)$, respectively, that is, without conditions on the derivatives of u and v in the point 0. The nonlinearities f and g are sign-changing continuous functions (that is, we have a so called system of semipositone boundary value problems), and the integrals from (BC) are Riemann-Stieltjes integrals. These boundary conditions include multi-point and integral boundary conditions and sum of these in a single framework. Integral boundary conditions arise in the thermal conduction problems [2], semiconductor problems [9] and hydrodynamic problems [4].

Received December 15, 2014

By using a nonlinear alternative of Leray-Schauder type, we present intervals for parameters λ and μ such that the above problem (S)–(BC) has at least one positive solution. By a positive solution of problem (S)–(BC) we mean a pair of functions $(u,v) \in C^n([0,T]) \times C^m([0,T])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,T]$ and $u(t) > 0$, $v(t) > 0$ for all $t \in (0,T)$. In the case when f and g are nonnegative functions and, in the boundary conditions (BC), H_1, H_2, K_1, K_2 are scale functions (denoted by $(\widetilde{BC})$), the existence of positive solutions of the above problem ($u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,T]$, $(u,v) \neq (0,0)$) has been studied in [5] and [8] by using the Guo-Krasnosel'skii fixed point theorem. The positive solutions ($u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,T]$, $\sup_{t\in[0,T]} u(t) > 0$, $\sup_{t\in[0,T]} v(t) > 0$) of system (S) with $\lambda = \mu = 1$ and with $f(t,u,v)$ and $g(t,u,v)$ replaced by $\widetilde{f}(t,v)$ and $\widetilde{g}(t,u)$, respectively, ($\widetilde{f}, \widetilde{g}$ nonnegative functions) with the boundary conditions $(\widetilde{BC})$ were investigated in [6] (the nonsingular case) and [7] (the singular case). In [6], the authors obtained the existence and multiplicity of positive solutions by applying some theorems from the fixed point index theory, and in [7], the authors studied the existence of positive solutions by using the Guo-Krasnosel'skii fixed point theorem. We also mention the paper [3], where the authors investigated the existence of positive solutions for the nonlinear nth order differential equation $u^{(n)}(t) + a(t)f(u(t)) = 0$, $t \in (0,1)$, subject to the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $\alpha u(\eta) = u(1)$, with $0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$.

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a boundary value problem for higher-order equations. The main theorem is presented in Section 3, and finally, in Section 4, two examples are given to support the new result.

2. AUXILIARY RESULTS

In this section, we present some auxiliary results related to the following n-order differential equation

$$u^{(n)}(t) + z(t) = 0, \quad t \in (0,T), \tag{2.1}$$

with the integral boundary conditions

$$u(0) = \int_0^T u(s)dH_1(s), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \int_0^T u(s)dH_2(s), \tag{2.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, and $H_1, H_2 : [0,T] \to \mathbb{R}$ are functions of bounded variation. If $n = 2$, the condition (2.2) has the form $u(0) = \int_0^T u(s)dH_1(s)$, $u(T) = \int_0^T u(s)dH_2(s)$.

Lemma 2.1. *If H_1, H_2 are functions of bounded variation, $\Delta_1 = \left(1 - \int_0^T dH_2(s)\right) \times$ $\times \int_0^T s^{n-1}dH_1(s) + \left(1 - \int_0^T dH_1(s)\right)\left(T^{n-1} - \int_0^T s^{n-1}dH_2(s)\right) \neq 0$, and $z \in C([0,T])$,* *then the solution of* (2.1)–(2.2) *is given by*

$$
\begin{aligned}
u(t) = & -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds + \frac{t^{n-1}}{\Delta_1}\left\{\left(1-\int_0^T dH_2(s)\right)\frac{1}{(n-1)!}\right.\\
& \times \int_0^T \left(\int_0^s (s-\tau)^{n-1} z(\tau)d\tau\right) dH_1(s) + \left(1-\int_0^T dH_1(s)\right)\frac{1}{(n-1)!}\\
& \left.\times \left[\int_0^T (T-s)^{n-1} z(s)ds - \int_0^T \left(\int_0^s (s-\tau)^{n-1} z(\tau)d\tau\right) dH_2(s)\right]\right\}\\
& + \frac{1}{\Delta_1}\left\{\left(\int_0^T s^{n-1} dH_1(s)\right)\frac{1}{(n-1)!}\left[\int_0^T (T-s)^{n-1} z(s)ds\right.\right.\\
& \left. - \int_0^T \left(\int_0^s (s-\tau)^{n-1} z(\tau)d\tau\right) dH_2(s)\right] - \left(T^{n-1} - \int_0^T s^{n-1} dH_2(s)\right)\\
& \left.\times \frac{1}{(n-1)!}\int_0^T \left(\int_0^s (s-\tau)^{n-1} z(\tau)d\tau\right) dH_1(s)\right\}.
\end{aligned}
\tag{2.3}
$$

Proof. If $n \geq 3$, then the solution of equation (2.1) is

$$
u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds + At^{n-1} + \sum_{i=1}^{n-2} A_i t^i + B,
$$

with $A, A_i, i = 1, \dots, n-2, B \in \mathbb{R}$. By using the conditions $u'(0) = \cdots = u^{(n-2)}(0) = 0$, we obtain $A_i = 0$ for $i = 1, \dots, n-2$. Then we conclude

$$
u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds + At^{n-1} + B.
$$

If $n = 2$, the solution of our problem is given directly by the above expression where n is replaced by 2.

Therefore, for a general $n \geq 2$, by using the conditions $u(0) = \int_0^T u(s)dH_1(s)$ and $u(T) = \int_0^T u(s)dH_2(s)$, we deduce

$$
\begin{cases}
B = \displaystyle\int_0^T \left[-\int_0^s \frac{(s-\tau)^{n-1}}{(n-1)!} z(\tau)d\tau + As^{n-1} + B\right] dH_1(s),\\
\quad -\displaystyle\int_0^T \frac{(T-s)^{n-1}}{(n-1)!} z(s)ds + AT^{n-1} + B\\
\quad = \displaystyle\int_0^T \left[-\int_0^s \frac{(s-\tau)^{n-1}}{(n-1)!} z(\tau)d\tau + As^{n-1} + B\right] dH_2(s),
\end{cases}
$$

or

$$
\begin{cases}
A\displaystyle\int_0^T s^{n-1} dH_1(s) + B\left(\int_0^T dH_1(s) - 1\right)\\
\qquad = \displaystyle\int_0^T \left(\int_0^s \frac{(s-\tau)^{n-1}}{(n-1)!} z(\tau)d\tau\right) dH_1(s),\\
A\left(T^{n-1} - \displaystyle\int_0^T s^{n-1} dH_2(s)\right) + B\left(1 - \int_0^T dH_2(s)\right)\\
\qquad = \displaystyle\int_0^T \frac{(T-s)^{n-1}}{(n-1)!} z(s)ds - \int_0^T \left(\int_0^s \frac{(s-\tau)^{n-1}}{(n-1)!} z(\tau)d\tau\right) dH_2(s).
\end{cases}
\tag{2.4}
$$

The above system with the unknown A and B has the determinant

$$\Delta_1=\left(1-\int_0^T dH_2(s)\right)\int_0^T s^{n-1}dH_1(s)\\+\left(1-\int_0^T dH_1(s)\right)\left(T^{n-1}-\int_0^T s^{n-1}dH_2(s)\right)\neq 0,$$

by using the assumptions of this lemma. Hence, the system (2.4) has a unique solution, namely

$$A=\frac{1}{\Delta_1}\left\{\left(1-\int_0^T dH_2(s)\right)\frac{1}{(n-1)!}\int_0^T\left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)\right.\\+\left(1-\int_0^T dH_1(s)\right)\frac{1}{(n-1)!}\left[\int_0^T (T-s)^{n-1}z(s)ds\right.\\\left.\left.-\int_0^T\left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]\right\},$$

$$B=\frac{1}{\Delta_1}\left\{\left(\int_0^T s^{n-1}dH_1(s)\right)\frac{1}{(n-1)!}\left[\int_0^T (T-s)^{n-1}z(s)ds\right.\right.\\\left.-\int_0^T\left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]-\left(T^{n-1}-\int_0^T s^{n-1}dH_2(s)\right)\\\left.\times\frac{1}{(n-1)!}\int_0^T\left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)\right\}.$$

Therefore, we obtain the expression (2.3) for the solution $u(t)$ of problem (2.1)–(2.2). □

Lemma 2.2. *Under the assumptions of Lemma* 2.1, *the solution of problem* (2.1)–(2.2) *can be expressed as* $u(t)=\int_0^T G_1(t,s)z(s)ds$, *where the Green's function* G_1 *is defined by*

$$G_1(t,s)=g_1(t,s)+\frac{1}{\Delta_1}\left[\left(T^{n-1}-t^{n-1}\right)\left(1-\int_0^T dH_2(\tau)\right)\right.\\\left.+\int_0^T\left(T^{n-1}-\tau^{n-1}\right)dH_2(\tau)\right]\int_0^T g_1(\tau,s)dH_1(\tau)\\+\frac{1}{\Delta_1}\left[t^{n-1}\left(1-\int_0^T dH_1(\tau)\right)+\int_0^T \tau^{n-1}dH_1(\tau)\right]\int_0^T g_1(\tau,s)dH_2(\tau), \tag{2.5}$$

for all $(t,s)\in[0,T]\times[0,T]$, *and*

$$g_1(t,s)=\frac{1}{(n-1)!T^{n-1}}\begin{cases} t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}, & 0\le s\le t\le T,\\ t^{n-1}(T-s)^{n-1}, & 0\le t\le s\le T.\end{cases} \tag{2.6}$$

Proof. By Lemma 2.1 and relation (2.3), we conclude

$$u(t)=\frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t\left[t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}\right]z(s)ds\right.$$

$$
\begin{aligned}
&+\int_t^T t^{n-1}(T-s)^{n-1}z(s)ds-\int_0^T t^{n-1}(T-s)^{n-1}z(s)ds\\
&+\frac{T^{n-1}t^{n-1}}{\Delta_1}\left\{\left(1-\int_0^T dH_2(s)\right)\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)\,d\tau\right)dH_1(s)\right.\\
&\left.+\left(1-\int_0^T dH_1(s)\right)\left[\int_0^T(T-s)^{n-1}z(s)ds-\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]\right\}\\
&+\frac{T^{n-1}}{\Delta_1}\left\{\left(\int_0^T s^{n-1}dH_1(s)\right)\left[\int_0^T(T-s)^{n-1}z(s)ds\right.\right.\\
&\left.-\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]-\left(T^{n-1}-\int_0^T s^{n-1}dH_2(s)\right)\\
&\left.\left.\times\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)\right\}\right\}\\
&=\frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t\left[t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}\right]z(s)\,ds\right.\\
&+\int_t^T t^{n-1}(T-s)^{n-1}z(s)ds-\frac{1}{\Delta_1}\left[\left(1-\int_0^T dH_2(s)\right)\left(\int_0^T s^{n-1}dH_1(s)\right.\right.\\
&\left.+\left(1-\int_0^T dH_1(s)\right)\left(T^{n-1}-\int_0^T s^{n-1}dH_2(s)\right)\right]\int_0^T t^{n-1}(T-s)^{n-1}z(s)\,ds\\
&+\frac{T^{n-1}t^{n-1}}{\Delta_1}\left\{\left(1-\int_0^T dH_2(s)\right)\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)\right.\\
&\left.+\left(1-\int_0^T dH_1(s)\right)\left[\int_0^T(T-s)^{n-1}z(s)ds-\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]\right\}\\
&+\frac{T^{n-1}}{\Delta_1}\left\{\left(\int_0^T s^{n-1}dH_1(s)\right)\left[\int_0^T(T-s)^{n-1}z(s)ds\right.\right.\\
&\left.-\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_2(s)\right]-\left(T^{n-1}-\int_0^T s^{n-1}dH_2(s)\right)\\
&\left.\left.\times\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)\right\}\right\}
\end{aligned}
$$

Therefore, we deduce

$$
\begin{aligned}
u(t)=&\frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t[t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}]z(s)ds\right.\\
&+\int_t^T t^{n-1}(T-s)^{n-1}z(s)ds+\frac{t^{n-1}}{\Delta_1}\left[-\left(1-\int_0^T dH_2(\tau)\right)\left(\int_0^T s^{n-1}dH_1(s)\right)\right.\\
&\times\left(\int_0^T(T-\tau)^{n-1}z(\tau)d\tau\right)-\left(1-\int_0^T dH_1(\tau)\right)\left(\int_0^T T^{n-1}(T-s)^{n-1}z(s)ds\right)\\
&+\left(1-\int_0^T dH_1(\tau)\right)\left(\int_0^T s^{n-1}dH_2(s)\right)\left(\int_0^T(T-\tau)^{n-1}z(\tau)d\tau\right)\\
&+T^{n-1}\left(1-\int_0^T dH_2(\tau)\right)\int_0^T\left(\int_0^s(s-\tau)^{n-1}z(\tau)d\tau\right)dH_1(s)
\end{aligned}
$$

$$+ T^{n-1}\left(1-\int_0^T dH_1(\tau)\right)\int_0^T (T-s)^{n-1}z(s)ds$$

$$- T^{n-1}\left(1-\int_0^T dH_1(\tau)\right)\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_2(s)\Bigg]$$

$$+ \frac{T^{n-1}}{\Delta_1}\left[\left(\int_0^T s^{n-1}dH_1(s)\right)\int_0^T (T-\tau)^{n-1}z(\tau)d\tau - \left(\int_0^T s^{n-1}dH_1(s)\right)\right.$$

$$\times \int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_2(s) - T^{n-1}\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)$$

$$\left. + \left(\int_0^T s^{n-1}dH_2(s)\right)\left(\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)\right)\right]\Bigg\}$$

$$= \frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t [t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}]z(s)ds\right.$$

$$+ \int_t^T t^{n-1}(T-s)^{n-1}z(s)ds + \frac{t^{n-1}}{\Delta_1}\left\{\left(1-\int_0^T dH_2(\tau)\right)\right.$$

$$\times \int_0^T \left(\int_0^s T^{n-1}(s-\tau)^{n-1}z(\tau)d\tau - \int_0^T s^{n-1}(T-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)$$

$$+ \left(1-\int_0^T dH_1(\tau)\right)\left[\int_0^T \left(\int_0^T s^{n-1}(T-\tau)^{n-1}z(\tau)d\tau\right) dH_2(s)\right.$$

$$\left.\left. - T^{n-1}\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_2(s)\right]\right\}$$

$$+ \frac{T^{n-1}}{\Delta_1}\left[\int_0^T s^{n-1}\left(\int_0^T (T-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s) - \left(\int_0^T \tau^{n-1}dH_1(\tau)\right)\right.$$

$$\times \left(\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_2(s)\right) - \int_0^T \left(\int_0^s T^{n-1}(s-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)$$

$$\left.\left. + \left(\int_0^T \tau^{n-1}dH_2(\tau)\right)\left(\int_0^T \left(\int_0^s (s-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)\right)\right]\right\}$$

Hence, we obtain

$$u(t) = \frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t [t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}]z(s)ds\right.$$

$$+ \int_t^T t^{n-1}(T-s)^{n-1}z(s)ds + \frac{t^{n-1}}{\Delta_1}\left\{\left(1-\int_0^T dH_2(\tau)\right)\int_0^T \left[\int_0^s \left(T^{n-1}(s-\tau)^{n-1}\right.\right.\right.$$

$$\left.\left. - s^{n-1}(T-\tau)^{n-1}\right) z(\tau)d\tau - \int_s^T s^{n-1}(T-\tau)^{n-1}z(\tau)d\tau\right] dH_1(s)$$

$$+ \left(1-\int_0^T dH_1(\tau)\right)\int_0^T \left[\int_0^s (s^{n-1}(T-\tau)^{n-1} - T^{n-1}(s-\tau)^{n-1})z(\tau)d\tau\right.$$

$$\left.\left. + \int_s^T s^{n-1}(T-\tau)^{n-1}z(\tau)d\tau\right] dH_2(s)\right\} + \frac{T^{n-1}}{\Delta_1}\left[\int_0^T \left(\int_0^s \left(s^{n-1}(T-\tau)^{n-1}\right.\right.\right.$$

$$\left.\left.\left. - T^{n-1}(s-\tau)^{n-1}\right) z(\tau)d\tau + \int_s^T s^{n-1}(T-\tau)^{n-1}z(\tau)d\tau\right) dH_1(s)\right]$$

$$
\begin{aligned}
&-\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_1(\tau)\right)\left(\int_0^T\left(\int_0^s T^{n-1}(s-\tau)^{n-1} z(\tau) d\tau\right) dH_2(s)\right) \\
&+\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_2(\tau)\right)\left(\int_0^T\left(\int_0^s T^{n-1}(s-\tau)^{n-1} z(\tau) d\tau\right) dH_1(s)\right) \\
&+\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_1(\tau)\right)\left(\int_0^T\left(\int_0^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right) dH_2(s)\right) \\
&\left.-\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_2(\tau)\right)\left(\int_0^T\left(\int_0^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right) dH_1(s)\right)\right\} \\
&=\frac{1}{(n-1)!T^{n-1}}\left\{\int_0^t [t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}] z(s) ds\right. \\
&+\int_t^T t^{n-1}(T-s)^{n-1} z(s) ds+\frac{t^{n-1}}{\Delta_1}\left\{-\left(1-\int_0^T dH_2(\tau)\right)\int_0^T\left[\int_0^s \left(s^{n-1}(T-\tau)^{n-1}\right.\right.\right. \\
&\left.\left.-T^{n-1}(s-\tau)^{n-1}\right) z(\tau) d\tau+\int_s^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right] dH_1(s) \\
&+\left(1-\int_0^T dH_1(\tau)\right)\int_0^T\left[\int_0^s (s^{n-1}(T-\tau)^{n-1}-T^{n-1}(s-\tau)^{n-1}) z(\tau) d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.\left.+\int_s^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right] dH_2(s)\right\}+\frac{T^{n-1}}{\Delta_1}\left[\int_0^T\left(\int_0^s \left(s^{n-1}(T-\tau)^{n-1}\right.\right.\right. \\
&\left.\left.\left.-T^{n-1}(s-\tau)^{n-1}\right) z(\tau) d\tau+\int_s^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right) dH_1(s)\right] \\
&+\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_1(\tau)\right)\left[\int_0^T\left(\int_0^s \left[s^{n-1}(T-\tau)^{n-1}\right.\right.\right. \\
&\left.\left.\left.-\ T^{n-1}(s-\tau)^{n-1}\right] z(\tau) d\tau+\int_s^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right) dH_2(s)\right] \\
&-\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_2(\tau)\right)\left[\int_0^T\left(\int_0^s \left[s^{n-1}(T-\tau)^{n-1}\right.\right.\right. \\
&\left.\left.\left.\left.-\ T^{n-1}(s-\tau)^{n-1}\right] z(\tau) d\tau+\int_s^T s^{n-1}(T-\tau)^{n-1} z(\tau) d\tau\right) dH_1(s)\right]\right\}.
\end{aligned}
$$

Then, the solution u of problem (2.1)–(2.2) is

$$
\begin{aligned}
u(t) &= \int_0^T g_1(t,s) z(s) ds+\frac{t^{n-1}}{\Delta_1}\left[-\left(1-\int_0^T dH_2(\tau)\right)\right. \\
&\times \int_0^T\left(\int_0^T g_1(s,\tau) z(\tau)\right) dH_1(s)+\left(1-\int_0^T dH_1(\tau)\right) \\
&\left.\times \int_0^T\left(\int_0^T g_1(s,\tau) z(\tau) d\tau\right) dH_2(s)\right]+\frac{T^{n-1}}{\Delta_1}\int_0^T\left(\int_0^T g_1(s,\tau) z(\tau) d\tau\right) dH_1(s) \\
&+\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_1(\tau)\right)\int_0^T\left(\int_0^T g_1(s,\tau) z(\tau) d\tau\right) dH_2(s) \\
&-\frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1} dH_2(\tau)\right)\int_0^T\left(\int_0^T g_1(s,\tau) z(\tau) d\tau\right) dH_1(s)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^T g_1(t,s)z(s)ds + \frac{t^{n-1}}{\Delta_1}\left[-\left(1-\int_0^T dH_2(\tau)\right)\right. \\
&\quad \times \int_0^T \left(\int_0^T g_1(\tau,s)dH_1(\tau)\right)z(s)ds + \left(1-\int_0^T dH_1(\tau)\right) \\
&\quad \left.\times \int_0^T \left(\int_0^T g_1(\tau,s)dH_2(\tau)\right)z(s)ds\right] + \frac{T^{n-1}}{\Delta_1}\int_0^T \left(\int_0^T g_1(\tau,s)dH_1(\tau)\right)z(s)ds \\
&\quad + \frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1}dH_1(\tau)\right)\int_0^T \left(\int_0^T g_1(\tau,s)dH_2(\tau)\right)z(s)ds \\
&\quad - \frac{1}{\Delta_1}\left(\int_0^T \tau^{n-1}dH_2(\tau)\right)\int_0^T \left(\int_0^T g_1(\tau,s)dH_1(\tau)\right)z(s)ds,
\end{aligned}
$$

where g_1 is given in (2.6).

Therefore, we conclude

$$
\begin{aligned}
u(t) &= \int_0^T g_1(t,s)z(s)ds + \frac{1}{\Delta_1}\left[-t^{n-1}\left(1-\int_0^T dH_2(\tau)\right) + T^{n-1} - \int_0^T \tau^{n-1}dH_2(\tau)\right] \\
&\quad \times \int_0^T \left(\int_0^T g_1(\tau,s)dH_1(\tau)\right)z(s)ds + \frac{1}{\Delta_1}\left[t^{n-1}\left(1-\int_0^T dH_1(\tau)\right) + \int_0^T \tau^{n-1}dH_1(\tau)\right] \\
&\quad \times \int_0^T \left(\int_0^T g_1(\tau,s)dH_2(\tau)\right)z(s)ds \\
&= \int_0^T \left\{g_1(t,s) + \frac{1}{\Delta_1}\left[\left(T^{n-1}-t^{n-1}\right)\left(1-\int_0^T dH_2(\tau)\right) + \int_0^T \left(T^{n-1}-\tau^{n-1}\right)dH_2(\tau)\right]\right. \\
&\quad \times \int_0^T g_1(\tau,s)dH_1(\tau) + \frac{1}{\Delta_1}\left[t^{n-1}\left(1-\int_0^T dH_1(\tau)\right) + \int_0^T \tau^{n-1}dH_1(\tau)\right] \\
&\quad \left.\times \int_0^T g_1(\tau,s)dH_2(\tau)\right\}z(s)ds = \int_0^T G_1(t,s)z(s)ds
\end{aligned}
$$

where G_1 is given in (2.5). □

Using similar arguments as those used in the proof of Lemma 2.2 from [10], we obtain the following lemma.

Lemma 2.3. *For any $n \geq 2$, the function g_1 given by* (2.6) *has the properties:*

a) $g_1 : [0,T]\times[0,T] \to \mathbb{R}_+$ is a continuous function, $g_1(t,s) \geq 0$ for all $(t,s) \in [0,T]\times[0,T]$, $g_1(t,s) > 0$ for all $(t,s) \in (0,T)\times(0,T)$.

b) $g_1(t,s) \leq h_1(s)$ for all $(t,s) \in [0,T]\times[0,T]$, where $h_1(s) = \frac{s(T-s)^{n-1}}{(n-2)!T}$.

c) $g_1(t,s) \geq k_1(t)h_1(s)$ for all $(t,s) \in [0,T]\times[0,T]$, where

$$
k_1(t) = \min\left\{\frac{(T-t)t^{n-2}}{(n-1)T^{n-1}}, \frac{t^{n-1}}{(n-1)T^{n-1}}\right\} = \begin{cases} \dfrac{t^{n-1}}{(n-1)T^{n-1}}, & 0 \leq t \leq T/2, \\ \dfrac{(T-t)t^{n-2}}{(n-1)T^{n-1}}, & T/2 \leq t \leq T. \end{cases}
$$

Lemma 2.4. *Assume that $H_1, H_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$ and $H_2(T) - H_2(0) < 1$. Then the Green's function G_1 of problem (2.1)–(2.2) given by (2.3) is continuous on $[0,T] \times [0,T]$ and satisfies $G_1(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$, $G_1(t,s) > 0$ for all $(t,s) \in (0,T) \times (0,T)$. Moreover, if $z \in C([0,T])$ satisfies $z(t) \geq 0$ for all $t \in [0,T]$, then the unique solution u of problem (2.1)–(2.2) satisfies $u(t) \geq 0$ for all $t \in [0,T]$.*

Proof. By using the assumptions of this lemma, Lemma 2.2 and Lemma 2.3, we obtain $\Delta_1 > 0$, $G_1(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$, $G_1(t,s) > 0$ for all $(t,s) \in (0,T) \times (0,T)$, and so $u(t) \geq 0$ for all $t \in [0,T]$. □

Lemma 2.5. *Assume that $H_1, H_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$ and $H_2(T) - H_2(0) < 1$. Then the Green's function G_1 of problem (2.1)–(2.2) satisfies the inequalities*

a) $G_1(t,s) \leq J_1(s)$, $\forall\, (t,s) \in [0,T] \times [0,T]$, *where* $J_1(s) = \tau_1 h_1(s)$, $s \in [0,T]$ *and*

$$\tau_1 = 1 + \frac{1}{\Delta_1}\left[T^{n-1}(1 - H_2(T) + H_2(0)) + \int_0^T (T^{n-1} - \tau^{n-1})dH_2(\tau)\right] \tag{2.7}$$
$$\times (H_1(T) - H_1(0)) + \frac{1}{\Delta_1}\left[T^{n-1}(1 - H_1(T) + H_1(0)) + \int_0^T \tau^{n-1}dH_1(\tau)\right]$$
$$\times (H_2(T) - H_2(0)).$$

b) $G_1(t,s) \geq \gamma_1(t) J_1(s)$, $\forall\, (t,s) \in [0,T] \times [0,T]$, *where*

$$\gamma_1(t) = \frac{1}{\tau_1}\Bigg\{k_1(t) + \frac{1}{\Delta_1}\Bigg[(T^{n-1} - t^{n-1})(1 - H_2(T) + H_2(0)) \tag{2.8}$$
$$+ \int_0^T (T^{n-1} - \tau^{n-1})dH_2(\tau)\Bigg] \int_0^T k_1(\tau)dH_1(\tau)$$
$$+ \frac{1}{\Delta_1}\left[t^{n-1}(1 - H_1(T) + H_1(0)) + \int_0^T \tau^{n-1}dH_1(\tau)\right] \int_0^T k_1(\tau)dH_2(\tau)\Bigg\}.$$

Proof. a) We have

$$G_1(t,s) \leq h_1(s) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1 - \int_0^T dH_2(\tau)\right) + \int_0^T \left(T^{n-1} - \tau^{n-1}\right) dH_2(\tau)\right]$$
$$\times \int_0^T h_1(s)dH_1(\tau) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1 - \int_0^T dH_1(\tau)\right) + \int_0^T \tau^{n-1}dH_1(\tau)\right]$$
$$\times \int_0^T h_1(s)dH_2(\tau) = \tau_1 h_1(s) = J_1(s), \quad \forall\, (t,s) \in [0,T] \times [0,T],$$

where τ_1 is given in (2.7).

b) For the second inequality, we obtain

$$G_1(t,s) \geq k_1(t)h_1(s) + \frac{1}{\Delta_1}\left[\left(T^{n-1} - t^{n-1}\right)\left(1 - \int_0^T dH_2(\tau)\right)\right.$$

$$
\begin{aligned}
&+ \int_0^T \left(T^{n-1} - \tau^{n-1}\right) dH_2(\tau)\Big] \int_0^T k_1(\tau) h_1(s) dH_1(\tau) \\
&+ \frac{1}{\Delta_1}\left[t^{n-1}\left(1 - \int_0^T dH_1(\tau)\right) + \int_0^T \tau^{n-1} dH_1(\tau)\right] \int_0^T k_1(\tau) h_1(s) dH_2(\tau) \\
= &\frac{1}{\tau_1}(\tau_1 h_1(s)) \left\{ k_1(t) + \frac{1}{\Delta_1}\left[\left(T^{n-1} - t^{n-1}\right)\left(1 - \int_0^T dH_2(\tau)\right)\right.\right. \\
&+ \int_0^T \left(T^{n-1} - \tau^{n-1}\right) dH_2(\tau)\Big] \int_0^T k_1(\tau) dH_1(\tau) + \frac{1}{\Delta_1}\left[t^{n-1}\left(1 - \int_0^T dH_1(\tau)\right)\right. \\
&\left.\left. + \int_0^T \tau^{n-1} dH_1(\tau)\right] \int_0^T k_1(\tau) dH_2(\tau)\right\} = \gamma_1(t) J_1(s), \quad \forall\, (t,s) \in [0,T] \times [0,T],
\end{aligned}
$$

where $\gamma_1(t)$ is defined in (2.8). □

Lemma 2.6. *Assume that $H_1, H_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$, $H_2(T) - H_2(0) < 1$, $z \in C([0,T])$, $z(t) \geq 0$ for all $t \in [0,T]$. Then the solution $u(t)$, $t \in [0,T]$ of problem (2.1)–(2.2) satisfies the inequality $u(t) \geq \gamma_1(t) \max_{t' \in [0,T]} u(t')$ for all $t \in [0,T]$.*

Proof. For $t \in [0,T]$, we deduce

$$
\begin{aligned}
u(t) &= \int_0^T G_1(t,s) z(s) ds \geq \int_0^T \gamma_1(t) J_1(s) z(s) ds = \gamma_1(t) \int_0^T J_1(s) z(s) ds \\
&\geq \gamma_1(t) \int_0^T G_1(t',s) z(s) ds = \gamma_1(t) u(t'), \quad \forall t' \in [0,T].
\end{aligned}
$$

Therefore, we conclude that $u(t) \geq \gamma_1(t) \max_{t' \in [0,T]} u(t')$ for all $t \in [0,T]$. □

We can also formulate similar results as Lemmas 2.1–2.6 above for the ordinary differential equation

$$
v^{(m)}(t) + \widetilde{z}(t) = 0, \quad 0 < t < T, \tag{2.9}
$$

with the integral boundary conditions

$$
v(0) = \int_0^T v(s) dK_1(s), \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \int_0^T v(s) dK_2(s), \tag{2.10}
$$

where $m \in \mathbb{N}$, $m \geq 2$, $K_1, K_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions and $\widetilde{z} \in C([0,T])$. In the case $m = 2$, the boundary conditions have the form $v(0) = \int_0^T v(s) dK_1(s)$, $v(T) = \int_0^T v(s) dK_2(s)$. We denote by $\Delta_2, g_2, G_2, h_2, k_2, \tau_2, J_2$ and γ_2 the corresponding constants and functions for problem (2.9)–(2.10) defined in a similar manner as $\Delta_1, g_1, G_1, h_1, k_1, \tau_1, J_1$ and γ_1, respectively.

In the proof of our main result, we shall use the following nonlinear alternative of Leray-Schauder type (see [1]).

Theorem 2.7. *Let X be a Banach space with $\Omega \subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0 \in U$, and let $S : \bar{U} \to \Omega$ be a completely continuous operator (continuous and compact). Then either*

1) S has a fixed point in $\bar{U}$, or

2) there exists $u \in \partial U$ and $\nu \in (0,1)$ such that $u = \nu S u$.

3. MAIN RESULT

In this section, we investigate the existence of positive solutions for our problem (S)–(BC). We present now the assumptions that we shall use in the sequel

$(H1)$ $H_1, H_2, K_1, K_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$, $H_2(T) - H_2(0) < 1$, $K_1(T) - K_1(0) < 1$ and $K_2(T) - K_2(0) < 1$.

$(H2)$ The functions $f, g \in C([0,T] \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$ and there exist functions $p_1, p_2 \in C([0,T], (0,\infty))$ such that $f(t,u,v) \geq -p_1(t)$ and $g(t,u,v) \geq -p_2(t)$ for any $t \in [0,T]$ and $u, v \in [0,\infty)$.

$(H3)$ $f(t,0,0) > 0$, $g(t,0,0) > 0$ for all $t \in [0,T]$.

We consider the system of nonlinear ordinary differential equations

$$\begin{cases} x^{(n)}(t) + \lambda(f(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_1(t)) = 0, & 0 < t < T, \\ y^{(m)}(t) + \mu(g(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_2(t)) = 0, & 0 < t < T, \end{cases} \tag{3.1}$$

with the integral boundary conditions

$$\begin{cases} x(0) = \int_0^T x(s)dH_1(s), \quad x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(T) = \int_0^T x(s)dH_2(s), \\ y(0) = \int_0^T y(s)dK_1(s), \quad y'(0) = \cdots = y^{(m-2)}(0) = 0, \quad y(T) = \int_0^T y(s)dK_2(s), \end{cases} \tag{3.2}$$

where

$$z(t)^* = \begin{cases} z(t), & z(t) \geq 0, \\ 0, & z(t) < 0. \end{cases}$$

Here q_1 and q_2 are given by $q_1(t) = \lambda\int_0^T G_1(t,s)p_1(s)ds$ and $q_2(t) = \mu\int_0^T G_2(t,s)p_2(s)ds$, that is, they are the solutions of the problems

$$\begin{cases} q_1^{(n)}(t) + \lambda p_1(t) = 0, \quad t \in (0,T), \\ q_1(0) = \int_0^T q_1(s)dH_1(s), \quad q_1'(0) = \cdots = q_1^{(n-2)}(0) = 0, \quad q_1(T) = \int_0^T q_1(s)dH_2(s), \end{cases} \tag{3.3}$$

and

$$\begin{cases} q_2^{(m)}(t) + \mu p_2(t) = 0, \quad t \in (0,T), \\ q_2(0) = \int_0^T q_2(s)dK_1(s), \quad q_2'(0) = \cdots = q_2^{(m-2)}(0) = 0, \quad q_2(T) = \int_0^T q_2(s)dK_2(s), \end{cases} \tag{3.4}$$

respectively. If $n = 2$ or $m = 2$ then the above conditions do not contain the conditions on the derivatives in the point 0. By $(H1)$–$(H2)$ and Lemma 2.4, we have $q_1(t) \geq 0$, $q_2(t) \geq 0$ for all $t \in [0,T]$, and $q_1(t) > 0$, $q_2(t) > 0$ for all $t \in (0,T)$.

We shall prove that there exists a solution (x, y) for the boundary value problem (3.1)–(3.2) with $x(t) \geq q_1(t)$ and $y(t) \geq q_2(t)$ for all $t \in [0, T]$. In this case, the pair of functions (u, v) with $u(t) = x(t) - q_1(t)$ and $v(t) = y(t) - q_2(t)$, $t \in [0, T]$ represents a positive solution (nonnegative on $[0, T]$ and positive on $(0, T)$) of the boundary value problem (S)–(BC). Indeed, by (3.1)–(3.2) and (3.3)–(3.4), we have

$$\begin{aligned} u^{(n)}(t) &= x^{(n)}(t) - q_1^{(n)}(t) = -\lambda f(t, [x(t) - q_1(t)]^*, [y(t) - q_2(t)]^*) \\ &\quad - \lambda p_1(t) + \lambda p_1(t) = -\lambda f(t, u(t), v(t)), \quad \forall\, t \in (0, T), \\ v^{(m)}(t) &= y^{(m)}(t) - q_2^{(m)}(t) = -\mu g(t, [x(t) - q_1(t)]^*, [y(t) - q_2(t)]^*) \\ &\quad - \mu p_2(t) + \mu p_2(t) = -\mu g(t, u(t), v(t)), \quad \forall\, t \in (0, T), \end{aligned}$$

and

$$\begin{aligned} &u(0) = x(0) - q_1(0) = \int_0^T u(s)dH_1(s), \\ &u'(0) = x'(0) - q_1'(0) = 0, \ldots, u^{(n-2)}(0) = x^{(n-2)}(0) - q_1^{(n-2)}(0) = 0, \\ &u(T) = x(T) - q_1(T) = \int_0^T u(s)dH_2(s), \\ &v(0) = y(0) - q_2(0) = \int_0^T v(s)dK_1(s), \\ &v'(0) = y'(0) - q_2'(0) = 0, \ldots, v^{(m-2)}(0) = y^{(m-2)}(0) - q_2^{(m-2)}(0) = 0, \\ &v(T) = y(T) - q_2(T) = \int_0^T v(s)dK_2(s). \end{aligned}$$

Therefore, in what follows, we shall investigate the boundary value problem (3.1)–(3.2).

By using Lemma 2.2, the system (3.1)–(3.2) is equivalent to the system

$$\begin{cases} x(t) = \lambda \displaystyle\int_0^T G_1(t, s) \left(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s)\right) ds, & t \in [0, T], \\ y(t) = \mu \displaystyle\int_0^T G_2(t, s) \left(g(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s)\right) ds, & t \in [0, T]. \end{cases}$$

We consider the Banach space $X = C([0, T])$ with supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X$ with the norm $\|(x, y)\|_Y = \|x\| + \|y\|$. We also define the cones

$$\begin{aligned} P_1 &= \{x \in X, \quad x(t) \geq \gamma_1(t)\|x\|, \quad \forall\, t \in [0, T]\} \subset X, \\ P_2 &= \{y \in X, \quad y(t) \geq \gamma_2(t)\|y\|, \quad \forall\, t \in [0, T]\} \subset X, \end{aligned}$$

and $P = P_1 \times P_2 \subset Y$.

For $\lambda, \mu > 0$, we define the operator $Q : P \to Y$ by $Q(x,y) = (Q_1(x,y), Q_2(x,y))$ with

$$Q_1(x,y)(t) = \lambda \int_0^T G_1(t,s) \left(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s) \right) ds, 0 \le t \le T,$$
$$Q_2(x,y)(t) = \mu \int_0^T G_2(t,s) (g(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s)) \, ds, 0 \le t \le T.$$

Lemma 3.1. *If $(H1)$–$(H2)$ hold, then the operator $Q : P \to P$ is a completely continuous operator.*

Proof. The operators Q_1, Q_2 are well-defined. For every $(x,y) \in P$, by Lemma 2.5 a), we have $Q_1(x,y)(t) < \infty$ and $Q_2(x,y)(t) < \infty$ for all $t \in [0,T]$. Then, by Lemma 2.6 we obtain

$$Q_1(x,y)(t) \ge \gamma_1(t) \sup_{t' \in [0,T]} Q_1(x,y)(t'), \quad Q_2(x,y)(t) \ge \gamma_2(t) \sup_{t' \in [0,T]} Q_2(x,y)(t'),$$

for all $t \in [0,T]$. Therefore, we conclude

$$Q_1(x,y)(t) \ge \gamma_1(t) \|Q_1(x,y)\|, \quad Q_2(x,y)(t) \ge \gamma_2(t) \|Q_2(x,y)\|, \quad \forall \, t \in [0,T],$$

and $Q(x,y) = (Q_1(x,y), Q_2(x,y)) \in P$.

By using standard arguments, we deduce that the operator $Q : P \to P$ is a completely continuous operator (a compact operator, that is it maps bounded sets into relatively compact sets, and continuous). □

Then $(x,y) \in P$ is a solution of problem (3.1)–(3.2) if and only if (x,y) is a fixed point of operator Q.

Theorem 3.2. *Assume that $(H1)$–$(H3)$ hold. Then there exist constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, the boundary value problem (S)–(BC) has at least one positive solution.*

Proof. Let $\delta \in (0,1)$ be fixed. From $(H3)$, there exists $R_0 > 0$ such that

$$f(t,u,v) \ge \delta f(t,0,0) > 0, \quad g(t,u,v) \ge \delta g(t,0,0) > 0, \tag{3.5}$$

for all $t \in [0,T]$ and $u, v \in [0, R_0]$.

We define

$$\bar{f}(R_0) = \max_{0 \le t \le T, 0 \le u,v \le R_0} \{f(t,u,v) + p_1(t)\} \ge \max_{0 \le t \le T} \{\delta f(t,0,0) + p_1(t)\} > 0,$$
$$\bar{g}(R_0) = \max_{0 \le t \le T, 0 \le u,v \le R_0} \{g(t,u,v) + p_2(t)\} \ge \max_{0 \le t \le T} \{\delta g(t,0,0) + p_2(t)\} > 0,$$

$$c_1 = \int_0^T J_1(s) ds > 0, \quad c_2 = \int_0^T J_2(s) ds > 0,$$
$$\lambda_0 = \frac{R_0}{4 c_1 \bar{f}(R_0)} > 0, \qquad \mu_0 = \frac{R_0}{4 c_2 \bar{g}(R_0)} > 0.$$

We will show that for any $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, problem (3.1)-(3.2) has at least one positive solution.

So, let $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$ be arbitrary, but fixed for the moment. We define the set $U = \{(x, y) \in P, \|(x, y)\|_Y < R_0\}$. We suppose that there exist $(x, y) \in \partial U$ ($\|(x, y)\|_Y = R_0$ or $\|x\| + \|y\| = R_0$) and $\nu \in (0, 1)$ such that $(x, y) = \nu Q(x, y)$ or $x = \nu Q_1(x, y)$, $y = \nu Q_2(x, y)$.

We deduce that

$$
\begin{aligned}
&[x(t) - q_1(t)]^* = x(t) - q_1(t) \le x(t) \le R_0, \quad \text{if} \quad x(t) - q_1(t) \ge 0,\\
&[x(t) - q_1(t)]^* = 0, \quad \text{for} \quad x(t) - q_1(t) < 0, \quad \forall\, t \in [0, T],\\
&[y(t) - q_2(t)]^* = y(t) - q_2(t) \le y(t) \le R_0, \quad \text{if} \quad y(t) - q_2(t) \ge 0,\\
&[y(t) - q_2(t)]^* = 0, \quad \text{for} \quad y(t) - q_2(t) < 0, \quad \forall\, t \in [0, T].
\end{aligned}
$$

Then, for all $t \in [0, T]$, we obtain

$$
\begin{aligned}
x(t) &= \nu Q_1(x, y)(t) \le Q_1(x, y)(t)\\
&= \lambda \int_0^T G_1(t, s)\, (f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*)\\
&\qquad + p_1(s))\, ds \le \lambda \int_0^T G_1(t, s) \bar{f}(R_0) ds\\
&\le \lambda \int_0^T J_1(s) \bar{f}(R_0) ds \le \lambda_0 c_1 \bar{f}(R_0) = R_0/4.
\end{aligned}
$$

In a similar manner we conclude $y(t) \le \mu_0 c_2 \bar{g}(R_0) = R_0/4$, for all $t \in [0, T]$.

Hence $\|x\| \le R_0/4$ and $\|y\| \le R_0/4$. Then $R_0 = \|(x, y)\|_1 = \|x\| + \|y\| \le \frac{R_0}{4} + \frac{R_0}{4} = \frac{R_0}{2}$, which is a contradiction.

Therefore, by Theorem 2.7 (with $\Omega = P$), we deduce that Q has a fixed point $(x, y) \in \bar{U} \cap P$. That is $(x, y) = Q(x, y) \Leftrightarrow x = Q_1(x, y)$, $y = Q_2(x, y)$, and $\|x\| + \|y\| \le R_0$, with $x(t) \ge \gamma_1(t)\|x\| \ge 0$ and $y(t) \ge \gamma_2(t)\|y\| \ge 0$ for all $t \in [0, T]$.

Moreover, by (3.5), we obtain

$$
\begin{aligned}
x(t) = Q_1(x, y)(t) &= \lambda \int_0^T G_1(t, s)\, (f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds\\
&\ge \lambda \int_0^T G_1(t, s)(\delta f(s, 0, 0) + p_1(s)) ds > \lambda \int_0^T G_1(t, s) p_1(s) ds = q_1(t) > 0,
\end{aligned}
$$

for all $t \in (0, T)$. In a similar manner, we have $y(t) > q_2(t) > 0$ for all $t \in (0, T)$.

Let $u(t) = x(t) - q_1(t) \ge 0$ and $v(t) = y(t) - q_2(t) \ge 0$ for all $t \in [0, T]$, with $u(t) > 0$, $v(t) > 0$ on $(0, T)$. Then, (u, v) is a positive solution of the boundary value problem (S)–(BC). $\square$

4. EXAMPLES

Let $T=1$, $n=3$, $m=4$, $H_1(t)=\frac{t^4}{3}$, $K_2(t)=t^3/2$, and

$$H_2(t)=\begin{cases} 0, & t\in[0,1/3),\\ 1/3, & t\in[1/3,2/3),\\ 5/6, & t\in[2/3,1],\end{cases}\qquad K_1(t)=\begin{cases} 0, & t\in[0,1/2),\\ 1/2, & t\in[1/2,1].\end{cases}$$

Then, we have $\int_0^1 u(s)dH_1(s)=\frac{4}{3}\int_0^1 s^3u(s)ds$, $\int_0^1 u(s)dH_2(s)=\frac{1}{3}u\left(\frac{1}{3}\right)+\frac{1}{2}u\left(\frac{2}{3}\right)$, $\int_0^1 v(s)dK_1(s)=\frac{1}{2}v\left(\frac{1}{2}\right)$, $\int_0^1 v(s)dK_2(s)=\frac{3}{2}\int_0^1 s^2v(s)ds$.

We consider the system of differential equations

$$\begin{cases} u^{(3)}(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1),\\ v^{(4)}(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1),\end{cases}\tag{S_0}$$

with the boundary conditions

$$\begin{cases} u(0)=\frac{4}{3}\int_0^1 s^3u(s)ds, \quad u'(0)=0, \quad u(1)=\frac{1}{3}u\left(\frac{1}{3}\right)+\frac{1}{2}u\left(\frac{2}{3}\right),\\ v(0)=\frac{1}{2}v\left(\frac{1}{2}\right), \quad v'(0)=v''(0)=0, \quad v(1)=\frac{3}{2}\int_0^1 s^2v(s)ds.\end{cases}\tag{BC_0}$$

Then, we obtain $H_1(1)-H_1(0)=\frac{1}{3}<1$, $H_2(1)-H_2(0)=\frac{5}{6}<1$, $K_1(1)-K_1(0)=\frac{1}{2}<1$ and $K_2(1)-K_2(0)=\frac{1}{2}<1$.

We also deduce

$$g_1(t,s)=\frac{1}{2}\begin{cases} t^2(1-s)^2-(t-s)^2, & 0\le s\le t\le 1,\\ t^2(1-s)^2, & 0\le t\le s\le 1,\end{cases}$$
$$g_2(t,s)=\frac{1}{6}\begin{cases} t^3(1-s)^3-(t-s)^3, & 0\le s\le t\le 1,\\ t^3(1-s)^3, & 0\le t\le s\le 1,\end{cases}$$

$\Delta_1=\frac{43}{81}$, $\Delta_2=\frac{13}{32}$, $\tau_1=\frac{123}{43}$, $\tau_2=\frac{34}{13}$, $h_1(s)=s(1-s)^2$, $h_2(s)=\frac{1}{2}s(1-s)^3$, $J_1(s)=\frac{123}{43}s(1-s)^2$, $J_2(s)=\frac{17}{13}s(1-s)^3$, $s\in[0,1]$, $c_1=\int_0^1 J_1(s)ds=\frac{123}{516}$, $c_2=\int_0^1 J_2(s)ds=\frac{17}{260}$.

Example 1. We consider the functions

$$f(t,u,v)=(u-1)(u-2)+\cos(3v), \quad g(t,u,v)=(v-2)(v-3)+\sin(2u),$$

for $t\in[0,1]$ and $u,v\ge 0$. There exists $M_0>0$ ($M_0=\frac{5}{4}$) such that $f(t,u,v)+M_0\ge 0$, $g(t,u,v)+M_0\ge 0$, ($p_1(t)=p_2(t)=M_0$, $\forall\, t\in[0,1]$) for all $t\in[0,1]$ and $u,v\ge 0$.

Let $\delta=\frac{1}{4}<1$ and $R_0=\frac{1}{2}$. Then

$$f(t,u,v)\ge \delta f(t,0,0)=\frac{3}{4}, \quad g(t,u,v)\ge \delta g(t,0,0)=\frac{3}{2}, \quad \forall\, t\in[0,1], u,v\in[0,1/2].$$

Besides

$$\bar f(R_0)=\max_{0\le t\le 1, 0\le u,v\le R_0}\{f(t,u,v)+p_1(t)\}=4.25,$$
$$\bar g(R_0)=\max_{0\le t\le 1, 0\le u,v\le R_0}\{g(t,u,v)+p_2(t)\}=7.25+\sin 1.$$

Then $\lambda_0 = \frac{1}{34c_1} \approx 0.12338594$ and $\mu_0 = \frac{1}{8c_2(7.25+\sin 1)} \approx 0.23626911$. By Theorem 3.2, for any $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, we conclude that problem $(S_0) - (BC_0)$ has a positive solution (u, v) with $\|(u, v)\| \leq 1/2$.

Example 2. We consider the functions

$$f(t, u, v) = v^3 + \cos(2u), \quad g(t, u, v) = u^{1/4} + \cos(3v), \quad t \in [0, 1], \quad u, v \geq 0.$$

There exists $M_0 > 0$ ($M_0 = 1$) such that $f(t, u, v) + M_0 \geq 0$, $g(t, u, v) \geq 0$, ($p_1(t) = p_2(t) = M_0, \quad \forall\, t \in [0, 1]$) for all $t \in [0, 1]$ and $u, v \geq 0$.

Let $\delta = \frac{1}{2} < 1$ and $R_0 = \frac{\pi}{9}$. Then

$$f(t, u, v) \geq \delta f(t, 0, 0) = \frac{1}{2}, \quad g(t, u, v) \geq \delta g(t, 0, 0) = \frac{1}{2}, \quad \forall\, t \in [0, 1], \quad u, v \in [0, \pi/9].$$

Besides $\bar{f}(R_0) = \frac{\pi^3}{81} + 2$, $\bar{g}(R_0) = \left(\frac{\pi}{9}\right)^{1/4} + 2$. Then $\lambda_0 \approx 0.15364044$, $\mu_0 \approx 0.48206348$.

By Theorem 3.2, for any $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, we deduce that problem $(S_0) - (BC_0)$ has a positive solution (u, v) with $\|(u, v)\| \leq \pi/9$.

ACKNOWLEDGMENTS

This work was supported by the CNCS grant PN-II-ID-PCE-2011-3-0557, Romania.

REFERENCES

[1] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2001.

[2] J. R. Cannon, The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **21** (1963), 155–160.

[3] P. W. Eloe, B. Ahmad, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Appl. Math. Lett.* **18** (2005), 521–527.

[4] R. Yu. Chegis, Numerical solution of a heat conduction problem with an integral boundary condition, *Litovsk. Mat. Sb.* **24** (1984), 209–15.

[5] J. Henderson, R. Luca, Existence of positive solutions for a system of higher-order multi-point boundary value problems, *Appl. Math. Comput.* **219** (2012), no. 8, 3709–3720.

[6] J. Henderson, R. Luca, Existence and multiplicity of positive solutions for a system of higher-order multi-point boundary value problems, *Adv. Dyn. Syst. Applic.* **8** (2013), no. 2, 233–245.

[7] J. Henderson, R. Luca, Positive solutions for systems of singular higher-order multi-point boundary value problems, *Brit. J. Math. Comput. Sc.* **4** (2014), no. 4, 460–473.

[8] J. Henderson, R. Luca, Positive solutions for systems of multi-point nonlinear boundary value problems, *Comm. Appl. Nonlinear Anal.* **21** (2014), no. 3, 1–2.

[9] N. I. Ionkin, Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions, *Differ. Equ.* **13** (1977), 294–304.

[10] D. Xie, C. Bai, Y. Liu, C. Wang, Positive solutions for nonlinear semipositone nth-order boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* (2008), no. 7, 12 pp.