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POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR HIGHER-ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the boundary value problem consisting of the higher-order fractional differential equation

$$(-1)^m \left(D_{0+}^{\alpha}\right)^m u = f(t,u), \quad 0 < t < 1,$$

and the boundary conditions

$$\left(\left(D_{0+}^{\alpha}\right)^i u\right)(0) = \left(\left(D_{0+}^{\alpha}\right)^i u\right)(1) = 0, \quad i = 0, 1, \dots, m-1,$$

where $1 < \alpha < 2$, $m \in \mathbb{N}$, D_{0+}^{α} is the Riemann-Liouville fractional differential operator, and $\left(D_{0+}^{\alpha}\right)^{j+1} = D_{0+}^{\alpha}\left(D_{0+}^{\alpha}\right)^j$ for $j = 0, \dots, m-1$, with $\left(D_{0+}^{\alpha}\right)^0 = I$, the identity operator. By finding the Green's function using the iteration method and applying the Krasnosel'skii fixed point theorem, we establish the existence of one, two, any finite number, and even a countably infinite number of positive solutions. Criteria for the nonexistence of positive solutions are also obtained. Our results cover, improve, and complement those by Jiang and Yuan for the case $m = 1$.

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1. INTRODUCTION

Fractional differential equations arise from and have extensive applications in many fields of science and engineering and have been a focus of research in recent years, see [4, 6, 12] and the references therein. For basic knowledge on fractional calculus, the reader is referred to [6, 10, 11].

In this paper, we study the existence of positive solutions of the boundary value problem (BVP) consisting of the $(m\alpha)$-th order fractional differential equation

$$(-1)^m (D_{0+}^{\alpha})^m u = f(t,u) \tag{1.1}$$

and the boundary conditions (BCs)

$$\left((D_{0+}^{\alpha})^i u\right)(0) = \left((D_{0+}^{\alpha})^i u\right)(1) = 0, \quad i = 0, 1, \dots, m-1, \tag{1.2}$$

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where $1 < \alpha < 2$, $m \in \mathbb{N}$, D^{α}_{0+} is the Riemann-Liouville fractional differential operator defined as

$$(D^{\alpha}_{0+}u)(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-s)^{1-\alpha}u(s)ds,$$

and $\left(D^{\alpha}_{0+}\right)^{j+1} = D^{\alpha}_{0+}\left(D^{\alpha}_{0+}\right)^{j}$ for $j = 0, \dots, m-1$, with $\left(D^{\alpha}_{0+}\right)^{0} = I$, the identity operator.

Assume throughout this paper that $f(t,u) \in C([0,1] \times [0,\infty), [0,\infty))$ and there exist $g \in C([0,\infty), [0,\infty))$ and $q_1, q_2 \in C((0,1), (0,\infty))$ such that

$$q_1(t)g(y) \le f(t, t^{\alpha-2}y) \le q_2(t)g(y), \quad t \in (0,1), \quad y \in (0,\infty), \tag{1.3}$$

and

$$\int_0^1 q_i(s)ds < \infty, \quad i = 1, 2. \tag{1.4}$$

We denote

$$g_0 = \lim_{y \to 0^+} \frac{g(y)}{y} \quad \text{and} \quad g_\infty = \lim_{y \to \infty} \frac{g(y)}{y} \tag{1.5}$$

and assume g_0 and g_∞ exist in the generalized sense that $0 \le g_0, g_\infty \le \infty$.

A solution $u(t)$ of BVP (1.1), (1.2) is said to be a positive solution if it satisfies

$$(-1)^k \left(\left(D^{\alpha}_{0+}\right)^k u\right)(t) > 0, \quad t \in (0,1),$$

for $k = 0, \dots, m-1$.

Recently, the special case of BVP (1.1), (1.2) with $m = 1$, i.e., the BVP

$$D^{\alpha}_{0+}u + f(t,u) = 0, \quad u(0) = u(1) = 0, \tag{1.6}$$

has been studied by Bai and Lü [3] and Jiang and Yuan [5]. In fact, Bai and Lü [3] showed that the function

$$G(t,s) = \begin{cases} \frac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1; \end{cases}$$

is the Green's function of BVP (1.6); and based on this, Jiang and Yuan [5] obtained the following criteria for BVP (1.6) to have one and two positive solutions:

Theorem 1.1. *BVP* (1.6) *has at least one positive solution if either* $g_0 = 0$ *and* $g_\infty = \infty$, *or* $g_0 = \infty$ *and* $g_\infty = 0$.

Theorem 1.2. *Let*

$$M_1 = \left(\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\right)^{-1}$$

and

$$M_2 = \left(\int_{1/4}^{3/4} (1/2)^{2-\alpha}G(1/2, s)q_1(s)ds\right)^{-1}.$$

Then BVP (1.6) *has at least two positive solutions if either*

(a) $g_0 = g_\infty = \infty$ *and there exists a* $p > 0$ *such that* $0 \le y \le p$ *implies* $g(y) < M_1 p$, *or*

(b) $g_0 = g_\infty = 0$ *and there exists a* $p > 0$ *such that* $p(\alpha-1)/16 \le y \le p$ *implies* $g(y) > M_2 p$.

Motivated by the iteration method for deriving Greens functions for higher-order BVPs utilized in [1, 2, 7, 8], in this paper, we will further extend the work for BVP (1.6) in [3, 5] to the general BVP (1.1), (1.2). In particular, by finding the Green's function for the higher-order problem and applying the Krasnosel'skii fixed point theorem, we will establish the existence of one, two, any finite number, and even a countably infinite number of positive solutions. Criteria for the nonexistence of positive solutions are also obtained. Our results cover, improve, and complement those in [5] for the case $m = 1$.

This paper is organized as follows: After this introduction, we present our main results in Section 2, followed by several examples for illustrations in Section 3. The proofs of the main results are given in Section 4.

2. MAIN RESULT

To present our main results, we need to introduce the following notation: Let

$$G_1(t,s) = \begin{cases} \frac{(t(1-s))^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.1}$$

and by iteration, we define

$$G_i(t,s) = \int_0^1 G_1(t,\tau)G_{i-1}(\tau,s)d\tau, \quad i = 2,\ldots,m. \tag{2.2}$$

Let

$$G^*(t,s) = t^{2-\alpha}G_m(t,s). \tag{2.3}$$

It is easy to see that $G_i(t,s) > 0$ for $t, s \in (0,1)$ and $i = 1,\ldots,m$. Consequently, $G^*(t,s) > 0$ for $t, s \in (0,1)$. For convenience, we denote

$$M_1 = \left(\left(\frac{\mathcal{B}(\alpha,\alpha)}{\Gamma(\alpha)}\right)^{m-1} \frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\right)^{-1}, \tag{2.4}$$

$$M_2 = \left(\int_{1/4}^{3/4} G^*(1/2,s)q_1(s)ds\right)^{-1},$$

and

$$\beta = \frac{\alpha-1}{16}\left(\frac{\alpha^2-\alpha}{2+4\alpha}\right)^{m-1}, \tag{2.5}$$

where $\mathcal{B}$ is the beta function operator. Clearly, $0 < \beta < 1$.

The first theorem provides conditions for BVP (1.1), (1.2) to have at least one positive solution.

Theorem 2.1. *Assume there exist* $0 < r_* < r^*$ *[respectively,* $0 < r^* < r_*$*] such that*

$$g(y) < M_1 r_* \quad \text{for } y \in [0, r_*] \tag{2.6}$$

and

$$g(y) \geq M_2 r^* \quad \text{for } y \in [\beta r^*, r^*]. \tag{2.7}$$

Then BVP (1.1), (1.2) *has at least one positive solution* $u(t)$ *satisfying that* $r_* < \max_{t\in[0,1]}\{t^{2-\alpha}u(t)\} < r^*$ *[respectively,* $r^* < \max_{t\in[0,1]}\{t^{2-\alpha}u(t)\} < r_*$*].*

From this theorem, we further develop more criteria for the existence of at least one positive solution using g_0 and g_∞ defined by (1.5).

Corollary 2.2. *BVP* (1.1), (1.2) *has at least one positive solution if one of the following conditions is satisfied:*

(a) $g_0 = \infty$ *and* (2.6) *holds for some* $r_* > 0$,
(b) $g_\infty = \infty$ *and* (2.6) *holds for some* $r_* > 0$,
(c) $g_0 = 0$ *and* (2.7) *holds for some* $r^* > 0$,
(d) $g_\infty = 0$ *and* (2.7) *holds for some* $r^* > 0$,
(e) $g_0 = 0$ *and* $g_\infty = \infty$,
(f) $g_0 = \infty$ *and* $g_\infty = 0$.

By combining the results in Corollary 2.2, we also obtain criteria for BVP (1.1), (1.2) to have at least two positive solutions.

Theorem 2.3. *BVP* (1.1), (1.2) *has at least two positive solutions if one of the following conditions is satisfied:*

(a) $g_0 = \infty$, $g_\infty = \infty$, *and* (2.6) *holds for some* $r_* > 0$;
(b) $g_0 = 0$, $g_\infty = 0$, *and* (2.7) *holds for some* $r^* > 0$.

By applying Theorem 2.1 repeatedly, we obtain criteria for BVP (1.1), (1.2) to have multiple positive solutions.

Theorem 2.4. *Let* $\{r_i\}_{i=1}^n \subseteq \mathbb{R}$ *be such that* $0 < r_1 < r_2 < r_3 < \cdots < r_n$. *Assume either*

(a) (2.6) *holds with* $r_* = r_i$ *when* i *is odd, and* (2.7) *holds with* $r^* = r_i$ *when* i *is even; or*
(b) (2.6) *holds with* $r_* = r_i$ *when* i *is even, and* (2.7) *holds with* $r^* = r_i$ *when* i *is odd.*

Then BVP (1.1), (1.2) *has at least* $n - 1$ *positive solutions.*

Theorem 2.5. *Let* $\{r_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ *be such that* $0 < r_1 < r_2 < r_3 < \cdots$. *Assume either*

(a) (2.6) *holds with* $r_* = r_i$ *when* i *is odd, and* (2.7) *holds with* $r^* = r_i$ *when* i *is even; or*
(b) (2.6) *holds with* $r_* = r_i$ *when* i *is even, and* (2.7) *holds with* $r^* = r_i$ *when* i *is odd.*

Then BVP (1.1), (1.2) *has an infinite number of positive solutions.*

The last theorem is on the nonexistence of positive solutions.

Theorem 2.6. *BVP* (1.1), (1.2) *has no positive solution if either*

(a) $g(y)/y < M_1$ *for all* $y \in (0, \infty)$*; or*
(b) $g(y)/y > \beta^{-1} M_2$ *for all* $y \in (0, \infty)$.

3. EXAMPLES

In this section we give several examples to demonstrate the applications of the criteria obtained in Section 2. All the examples below are for BVP (1.1), (1.2), where α is given in the equation, and β, M_1, and M_2 are given in Section 2.

Example 3.1. Let $f(t, u) = t^{k(2-\alpha)} u^k$. Then $f(t, t^{\alpha-2} y) = y^k$ and

$$q_1(t) g(y) \leq f(t, t^{\alpha-2} y) \leq q_2(t) g(y)$$

with $q_1(t) = q_2(t) = 1$ and $g(y) = y^k$.

(a) If $k > 1$, then $g_0 = 0$ and $g_\infty = \infty$; i.e., condition (e) of Corollary 2.2 is satisfied. By Corollary 2.2, BVP (1.1), (1.2) has at least one positive solution.
(b) If $0 < k < 1$, then $g_0 = \infty$ and $g_\infty = 0$; i.e., condition (f) of Corollary 2.2 is satisfied. By Corollary 2.2, BVP (1.1), (1.2) has at least one positive solution.

Example 3.2. Let $f(t, u) = c\left(t^{k_1(2-\alpha)} u^{k_1} + t^{k_2(2-\alpha)} u^{k_2}\right)$ with $c > 0$ and $0 < k_1 < 1 < k_2 < \infty$. Then $f(t, t^{\alpha-2} y) = c\left(y^{k_1} + y^{k_2}\right)$ and

$$q_1(t) g(y) \leq f(t, t^{\alpha-2} y) \leq q_2(t) g(y)$$

with $q_1(t) = q_2(t) = 1$ and $g(y) = c\left(y^{k_1} + y^{k_2}\right)$. Let $r = \left(\frac{1-k_1}{k_2-1}\right)^{\frac{1}{k_2-k_1}}$. Then

(a) BVP (1.1), (1.2) has at least two positive solutions when $0 < c < r\left(r^{k_1} + r^{k_2}\right)^{-1} M_1$;
(b) BVP (1.1), (1.2) has no positive solution when $c > r\left(r^{k_1} + r^{k_2}\right)^{-1} M_2$.

In fact, it is easy to see that g is strictly increasing, $g_0 = \infty$ and $g_\infty = \infty$, and $g(y)/y$ assumes its minimum at $y = r$ on $[0, \infty)$.

When $0 < c < r\left(r^{k_1} + r^{k_2}\right)^{-1} M_1$ and $y \in [0, r]$, we have $g(y) < g(r) \leq r M_1$. Thus, (2.6) holds holds with $r_* = r$ and hence condition (a) of Theorem 2.3 is satisfied. By Theorem 2.3, BVP (1.1), (1.2) has at least two positive solutions.

When $c > r\left(r^{k_1} + r^{k_2}\right)^{-1} M_2$, $g(y)/y \ge g(r)/r > M_2$ on $(0,\infty)$ and hence condition (b) of Theorem 2.6 is satisfied. Then by Theorem 2.6, BVP (1.1), (1.2) has no positive solution.

Example 3.3. Let $f(t,u) = \dfrac{ct^{4-2\alpha}u^2}{1+t^{4-2\alpha}u^2}$ with $c > 0$. Then $f(t,t^{\alpha-2}y) = \dfrac{cy^2}{1+y^2}$ and

$$q_1(t)g(y) \le f(t,t^{\alpha-2}y) \le q_2(t)g(y)$$

with $q_1(t) = q_2(t) = 1$ and $g(y) = cy^2/(1+y^2)$. Then

(a) BVP (1.1), (1.2) has at least two positive solutions when $c \ge \beta^{-2}\left(1+\beta^2\right)M_2$;
(b) BVP (1.1), (1.2) has no positive solution when $0 < c < 2M_1$.

In fact, it is easy to see that g is strictly increasing, $g_0 = 0$ and $g_\infty = 0$, and $g(y)/y$ assumes its maximum $c/2$ at $y = 1$ on $(0,\infty)$.

When $c \ge \beta^{-2}\left(1+\beta^2\right)M_2$ and $y \in [\beta, 1]$, we have $g(y) \ge g(\beta) = M_2$. Thus, (2.7) holds for $r^* = 1$ and hence condition (b) of Theorem 2.3 is satisfied. By Theorem 2.3, BVP (1.1), (1.2) has at least two positive solutions.

When $0 < c < 2M_1$, $g(y)/y \le g(r^*)/r^* < M_1$ on $(0,\infty)$ and hence condition (a) of Theorem 2.6 is satisfied. Then by Theorem 2.6, BVP (1.1), (1.2) has no positive solution.

4. PROOFS

The proof of Theorem 2.1 utilizes the following Krasnosel'skii's fixed point theorem from [9].

Lemma 4.1. *Let X be a Banach space and $K \subset X$ a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$\Gamma : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(a) $\|\Gamma u\| \le \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|\Gamma u\| \ge \|u\|$ *for any* $u \in K \cap \partial\Omega_2$*; or*
(b) $\|\Gamma u\| \ge \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|\Gamma u\| \le \|u\|$ *for any* $u \in K \cap \partial\Omega_2$*.*

Then Γ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We then discuss the Green's functions which are related to BVP (1.1), (1.2).

Lemma 4.2. *For $k = 1, 2, \dots, m$, the function $G_k(t,s)$ given by (2.2) is the Green's function for the BVP consisting of the equation*

$$(-1)^k\left((D_{0+}^{\alpha})^k u\right)(t) = 0, \quad 0 < t < 1, \tag{4.1}$$

and the BC

$$\left(\left(D_{0+}^{\alpha}\right)^i u\right)(0) = \left(\left(D_{0+}^{\alpha}\right)^i u\right)(1) = 0, \quad i = 0, 1, \dots, k-1; \tag{4.2}$$

i.e., for any $h \in C([0,1],\mathbb{R})$

$$u(t) = \int_0^1 G_k(t,s)h(s)ds \tag{4.3}$$

is the unique solution of the BVP consisting of the equation

$$(-1)^k \left((D_{0+}^{\alpha})^k u\right)(t) = h(t), \quad 0 < t < 1, \tag{4.4}$$

and BC (4.2).

Proof. From [5, Lemma 2.3], the conclusion holds for $k = 1$. Assume it holds for some $k \le m-1$. Then the solution $u(t)$ of BVP (4.4), (4.2) with k replaced by $k+1$ satisfies that

$$(-1)\left(\left(D_{0+}^{\alpha}\right)u\right)(t) = \int_0^1 G_k(t,s)h(s)ds$$

and BC $u(0) = u(1) = 0$. By the conclusion for $k = 1$,

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,\tau)\left(\int_0^1 G_k(\tau,s)h(s)ds\right)d\tau \\ &= \int_0^1 \left(\int_0^1 G_1(t,\tau)G_k(\tau,s)d\tau\right)h(s)ds \\ &= \int_0^1 G_{k+1}(t,s)h(s)ds. \end{aligned}$$

This means that $G_{k+1}(t,s)$ is the Green's function for BVP (4.1), (4.2) with k replaced by $k+1$. □

In the sequel, we denote $G(t,s) = G_m(t,s)$ for simplicity. Recall that $G^*(t,s)$ is defined by (2.3).

Lemma 4.3. *Let*

$$A = \frac{\alpha-1}{\Gamma(\alpha)}\mathcal{B}(\alpha+1,\alpha+1) \quad \text{and} \quad B = \frac{1}{\Gamma(\alpha)}\mathcal{B}(\alpha,\alpha). \tag{4.5}$$

Then for $t,s \in [0,1]$,

$$G(t,s) = G(1-s,1-t) \tag{4.6}$$

and

$$\frac{A^{m-1}(\alpha-1)}{\Gamma(\alpha)}t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le G(t,s) \le \frac{B^{m-1}}{\Gamma(\alpha)}t^{\alpha-1}(1-t)(1-s)^{\alpha-2}. \tag{4.7}$$

Furthermore, for $t,s \in [0,1]$,

$$\frac{A^{m-1}(\alpha-1)}{\Gamma(\alpha)}t(1-t)s(1-s)^{\alpha-1} \le G^*(t,s) \le \frac{B^{m-1}}{\Gamma(\alpha)}s(1-s)^{\alpha-1}. \tag{4.8}$$

Proof. We first show that

$$G_k(t,s) = G_k(1-s, 1-t), \quad t,s \in [0,1] \tag{4.9}$$

for $k = 1, 2, \ldots, m$. In fact, by [5, Theorem 1.1], (4.9) holds for $k = 1$. Assume (4.9) holds for some $k \le m-1$. Note from (2.2) we have

$$\int_0^1 G_1(t,r)G_k(r,s)dr = \int_0^1 \cdots \int_0^1 G_1(t,r_1)G_1(r_1,r_2)\cdots G_1(r_k,s)dr_k \cdots dr_2 dr_1$$

and

$$\int_0^1 G_k(t,r)G_1(r,s)dr = \int_0^1 \cdots \int_0^1 G_1(t,r_1)\cdots G_1(r_{k-1},r_k)G_1(r_k,s)dr_1 \cdots dr_{k-1}dr_k.$$

Thus,

$$\begin{aligned}
G_{k+1}(t,s) &= \int_0^1 G_1(t,\tau)G_k(\tau,s)d\tau = \int_0^1 G_k(t,\tau)G_1(\tau,s)d\tau \\
&= \int_0^1 G_k(1-\tau,1-t)G_1(1-s,1-\tau)d\tau = \int_0^1 G_1(1-s,1-\tau)G_k(1-\tau,1-t)d\tau \\
&= \int_0^1 G_1(1-s,\tau)G_k(\tau,1-t)d\tau = G_{k+1}(1-s,1-t),
\end{aligned}$$

i.e., $G_{k+1}(t,s) = G_{k+1}(1-s,1-t)$. As a result, (4.6) holds.

Now we show that

$$\frac{A^{k-1}(\alpha-1)}{\Gamma(\alpha)} t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le G_k(t,s) \le \frac{B^{k-1}}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2} \tag{4.10}$$

holds for $k = 1, \ldots, m$. It has been shown in [5] that for all $t,s \in [0,1]$,

$$\frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le G_1(t,s) \le \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}.$$

Thus, (4.10) holds for $k = 1$. Assume (4.10) holds for some $k \le m-1$; i.e., for every $t,s \in [0,1]$,

$$\frac{A^{k-1}(\alpha-1)}{\Gamma(\alpha)} t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le G_k(t,s) \le \frac{B^{k-1}}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}.$$

Then

$$\begin{aligned}
G_{k+1}(t,s) &= \int_0^1 G_1(t,\tau)G_k(\tau,s)d\tau \\
&\le \int_0^1 \left(\frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-\tau)^{\alpha-2}\right)\left(\frac{B^{k-1}}{\Gamma(\alpha)} \tau^{\alpha-1}(1-\tau)(1-s)^{\alpha-2}\right) d\tau \\
&= \frac{B^{k-1}}{\Gamma^2(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\alpha-1}d\tau \\
&= \frac{B^{k-1}}{\Gamma^2(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}\mathcal{B}(\alpha,\alpha) \\
&= \frac{B^k}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}
\end{aligned}$$

and

$$\begin{aligned}
G_{k+1}(t,s) &= \int_0^1 G_1(t,\tau)G_k(\tau,s)d\tau \\
&\geq \int_0^1 \left(\frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)\tau(1-\tau)^{\alpha-1}\right)\left(\frac{A^{k-1}(\alpha-1)}{\Gamma(\alpha)}(1-\tau)\tau^{\alpha-1}s(1-s)^{\alpha-1}\right)d\tau \\
&= A^{k-1}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^2 t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\int_0^1 \tau^{\alpha}(1-\tau)^{\alpha}d\tau \\
&= A^{k-1}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^2 t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\mathcal{B}(\alpha+1,\alpha+1) \\
&= A^{k}\frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}.
\end{aligned}$$

The combination of the above two inequalities shows that (4.10) holds for $k+1$. As a result, (4.7) holds.

Inequality (4.8) follows from the definition of G^* and the first part of this lemma. □

We denote by $E = C([0,1])$ the Banach space of continuous functions on $[0,1]$ endowed with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$. We define a cone K in E by

$$K = \{y \in E \mid y(t) \geq 16\beta t(1-t)\|y\|, t \in [0,1]\}, \tag{4.11}$$

where β is given by (2.5), and an operator $T:\, E \to E$ by

$$(Ty)(t) = \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))\,ds, \quad t \in [0,1]. \tag{4.12}$$

Lemma 4.4. *The operator T satisfies that $T(E) \subset K$ and is completely continuous on K.*

Proof. Due to the nonnegativity of f and G^*, we have $(Ty)(t) \geq 0$ for $y \in E$ and $t \in [0,1]$. By applying (4.8) to (4.12) we have

$$\begin{aligned}
(Ty)(t) &= \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds \\
&\geq \frac{A^{m-1}(\alpha-1)}{\Gamma(\alpha)}t(1-t)\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds
\end{aligned} \tag{4.13}$$

and

$$\|Ty\| \leq \frac{B^{m-1}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds. \tag{4.14}$$

We note from (4.5) that

$$\frac{A}{B} = \frac{(\alpha-1)\mathcal{B}(1+\alpha,1+\alpha)}{\mathcal{B}(\alpha,\alpha)} = \frac{(\alpha-1)\left(\frac{(\Gamma(1+\alpha))^2}{\Gamma(2+2\alpha)}\right)}{\left(\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\right)}$$

$$= (\alpha-1)\left(\frac{(\Gamma(1+\alpha))^2}{(\Gamma(\alpha))^2}\right)\left(\frac{\Gamma(2\alpha)}{\Gamma(2+2\alpha)}\right) = \frac{(\alpha-1)\alpha^2}{\alpha(2+4\alpha)} = \frac{\alpha^2-\alpha}{2+4\alpha}.$$

Thus, by (4.13), (4.14), and (2.5) we see that for $t \in [0,1]$,

$$(Ty)(t) \geq \left(\frac{A}{B}\right)^{m-1}(\alpha-1)t(1-t)\|Ty\|$$

$$= (\alpha-1)\left(\frac{\alpha^2-\alpha}{2+4\alpha}\right)^{m-1} t(1-t)\|Ty\| = 16\beta t(1-t)\|Ty\|.$$

Hence, $T(E) \subset K$.

Let $D \subseteq K$ be bounded. Then there exists $r > 0$ such that $\|y\| \leq r$ for all $y \in D$. Let

$$M = \max_{0\leq y\leq r} |g(y)|.$$

Then from (4.8), for all $y \in D$, we have

$$\begin{aligned} 0 \leq (Ty)(t) &\leq \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds \\ &\leq \int_0^1 G^*(t,s)q_2(s)g(y(s))ds \\ &\leq \frac{MB^{m-1}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds < \infty. \end{aligned} \tag{4.15}$$

Thus $T(D)$ is bounded. This means that T is a bounded operator on K.

Finally, we show that T is equicontinuous. Since $G^*(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for any $\epsilon > 0$, $y \in D$, and $t_1, t_2 \in [0,1]$, there exists a $\delta > 0$, such that if $|t_2 - t_1| < \delta$, then

$$|G^*(t_2,s) - G^*(t_1,s)| < \frac{\epsilon}{M\int_0^1 q_2(s)ds}.$$

By (1.3),

$$\begin{aligned} |(Ty)(t_2)-(Ty)(t_1)| &= \left|\int_0^1 \left(G^*(t_2,s)-G^*(t_1,s)\right) f(s,s^{\alpha-2}y(s))ds\right| \\ &\leq \int_0^1 \left|G^*(t_2,s)-G^*(t_1,s)\right| f(s,s^{\alpha-2}y(s))ds \\ &< \left(\frac{\epsilon}{M\int_0^1 q_2(s)ds}\right)\int_0^1 q_2(s)g(y(s))ds \\ &\leq \left(\frac{\epsilon}{M\int_0^1 q_2(s)ds}\right) M\int_0^1 q_2(s)ds = \epsilon. \end{aligned}$$

Thus, T is equicontinuous on K. By the Arzela-Arscoli Theorem, T is completely continuous on K. $\square$

In the following, for $r > 0$ we define

$$\Omega_r = \{y \in E \,|\, \|y\| < r\} \quad \text{and} \quad \partial\Omega_r = \{y \in E \,|\, \|y\| = r\}.$$

Proof of Theorem 2.1. Without loss of generality, we assume $0 < r_* < r^*$. Let $y \in K \cap \partial\Omega_{r_*}$. Then by Lemma 4.3 and (1.3)

$$\int_0^1 G^*(t,s) f(s, s^{\alpha-2} y(s)) ds \le \frac{B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s, s^{\alpha-2} y(s)) ds$$
$$\le \frac{B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds.$$

Since $\|y\| = r_*$, we have $0 \le y(t) \le r_*$ for $t \in [0,1]$ and hence by (2.6), $g(y(t)) < M_1 r_*$ for $t \in [0,1]$. It follows from (2.4) and (4.5) that for $t \in [0,1]$

$$\begin{aligned}(Ty)(t) &\le \frac{B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds \\ &< \frac{B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) M_1 r_* ds \\ &= r_* \left(\frac{M_1 B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \right) = r_* = \|y\|. \end{aligned} \tag{4.16}$$

This implies that $\|Ty\| < \|y\|$ for all $y \in K \cap \partial\Omega_{r_*}$.

Let $y \in K \cap \partial\Omega_{r^*}$. Since $\|y\| = r^*$, by (4.11) we see that $\beta r^* \le y(t) \le r^*$ for $1/4 \le t \le 3/4$ and hence by (2.7), $g(y(t)) \ge M_2 r^*$ for $1/4 \le t \le 3/4$. It follows from (1.3) that

$$\begin{aligned}\|Ty\| &\ge (Ty)(\frac{1}{2}) > \int_{1/4}^{3/4} G^*(\frac{1}{2}, s) q_1(s) g(y(s)) ds \\ &\ge \int_{1/4}^{3/4} G^*(\frac{1}{2}, s) q_1(s) M_2 r^* ds = r^* \left(M_2 \int_{1/4}^{3/4} G^*(\frac{1}{2}, s) q_1(s) ds \right) \\ &= r^* = \|y\|. \end{aligned} \tag{4.17}$$

This implies $\|Ty\| > \|y\|$ for all $y \in K \cap \partial\Omega_{r^*}$.

By Lemma 4.1, the operator T has a fixed point y in $K \cap \left(\overline{\Omega}_{r^*} \backslash \Omega_{r_*}\right)$. Note from (1.3) and (4.17) we have $r_* < \|y\| < r^*$. It follows that

$$y(t) = \int_0^1 G^*(t,s) f(s, s^{\alpha-2} y(s)) ds, \quad t \in [0,1].$$

Define $u(t) = t^{\alpha-2} y(t)$ for $t \in (0,1]$. Then by (2.3), we have for $t \in (0,1]$

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds. \tag{4.18}$$

To confirm that $u(t)$ is a solution of BVP (1.1), (1.2), we need to show $u(t)$ is defined and continuous at 0, and hence $f(t, u(t))$ is continuous on $[0, 1]$. In fact, since $y(t) \le r^*$ for $t \in (0, 1]$, we have

$$\begin{aligned}
0 \le \lim_{t\to 0^+} u(t) &= \lim_{t\to 0^+} \int_0^1 G(t,s) f(s,u(s))ds \\
&= \lim_{t\to 0^+} \int_0^1 G(t,s) f(s, s^{\alpha-2} y(s))ds \\
&\le \lim_{t\to 0^+} \int_0^1 G(t,s) q_2(s) g(y(s))ds \\
&\le \lim_{t\to 0^+} \int_0^1 G(t,s) q_2(s) ds \left(\max_{0\le y\le r^*} g(y) \right) = 0.
\end{aligned}$$

Hence, we may define $u(0) = 0$ to make $u(t)$ continuous at 0. It is easy to see that (4.18) holds for $t \in [0, 1]$. By Lemma 4.2, $u(t)$ is a solution of BVP (1.1), (1.2).

Note that for $k = 0, 1, \ldots, m-1$ and $t \in [0, 1]$,

$$(-1)^k \left(D_{0+}^{\alpha} u \right)(t) = \int_0^1 G_k(t,s) f(s,u(s))ds > 0.$$

Thus, $u(t)$ is a positive solution of the BVP (1.1), (1.2). It is easy to see that $r_* < \max_{t\in[0,1]}\{t^{2-\alpha}u(t)\} < r^*$. □

Proof of Corollary 2.2. (a) Since $g_0 = \infty$, there exists some $r^* \in (0, r_*)$ such that, $g(y)/y \ge \beta^{-1} M_2$ for $y \in [\beta r^*, r^*]$. This implies $g(y) \ge \beta^{-1} M_2 y \ge M_2 r^*$. Hence condition (2.7) holds. Since (2.6) also holds, by Theorem 2.1, BVP (1.1), (1.2) has at least one positive solution u. Note that $\max_{t\in[0,1]}\{t^{2-\alpha}u(t)\} < r_*$.

(b) Since $g_\infty = \infty$, there exists some $r^* > r_*$ such that $g(y)/y \ge \beta^{-1} M_2$ for $y \in [\beta r^*, r^*]$. This implies $g(y) \ge \beta^{-1} M_2 y \ge M_2 r^*$. Hence condition (2.7) holds. Since (2.6) also holds, by Theorem 2.1, BVP (1.1), (1.2) has at least one positive solution u. Note that $\max_{t\in[0,1]}\{t^{2-\alpha}u(t)\} > r_*$.

(c) Since $g_0 = 0$, there exists some $r_* \in (0, r^*)$ such that $g(y)/y < M_1$ for $y \in [0, r_*]$. This implies $g(y) < M_1 y \le M_1 r_*$, for $y \in [0, r_*]$. Hence condition (2.6) holds. Since (2.7) also holds, by Theorem 2.1 BVP (1.1), (1.2) has at least one positive solution.

(e) As above, $g_0 = 0$ and $g_\infty = \infty$ imply (2.6) and (2.7) hold for some $0 < r_* < r^*$. Thus, the conclusion follows from Theorem 2.1.

The proofs of parts (d) and (f) involve more technical arguments. In both cases we have $g_\infty = 0$. Thus, there exists $r > 0$ such that

$$g(y) \le \epsilon y \text{ for } y \ge r, \tag{4.19}$$

where $\epsilon > 0$ is chosen to satisfy that

$$B^{m-1}\frac{\epsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds < 1.$$

Hence, $B^{1-m}\Gamma(\alpha) - \epsilon\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds > 0$. Let R be any constant such that

$$R \geq \frac{\max_{0\leq y\leq r}\{g(y)\}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds}{B^{1-m}\Gamma(\alpha) - \epsilon\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds}. \tag{4.20}$$

Then for $y \in K \cap \partial\Omega_R$,

$$\begin{aligned}(Ty)(t) &= \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds \\ &\leq \frac{B^{m-1}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds \\ &\leq \frac{B^{m-1}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds.\end{aligned}$$

Let $I_1 = \{s \in [0,1] \,|\, y(s) \in [0,r]\}$ and $I_2 = \{s \in [0,1] \,|\, y(s) \in [r,R]\}$. Then

$$\begin{aligned}&\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds \\ &= \int_{I_1} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds + \int_{I_2} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds.\end{aligned}$$

By (4.20),

$$\begin{aligned}&\int_{I_1} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds \leq \max_{0\leq y\leq r}\{g(y)\}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \\ &\leq R\left(B^{1-m}\Gamma(\alpha) - \epsilon\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\right) \\ &= RB^{1-m}\Gamma(\alpha) - \epsilon R\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds.\end{aligned}$$

By (4.19),

$$\begin{aligned}&\int_{I_2} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds \leq \max_{r\leq y\leq R}\{g(y)\}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \\ &\leq \epsilon R\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds.\end{aligned}$$

Combining the above inequalities we have

$$\begin{aligned}(Ty)(t) \leq &\frac{B^{m-1}}{\Gamma(\alpha)}\left[\int_{I_1} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\right. \\ &\left.+\int_{I_2} s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\right] \\ \leq &\frac{B^{m-1}}{\Gamma(\alpha)}RB^{1-m}\Gamma(\alpha) = R = \|y\|.\end{aligned}$$

Thus, for $y \in K \cap \partial\Omega_R$, we have $\|Ty\| \le \|y\|$.

(d) As shown in the proof of Theorem 2.1, (2.7) implies that $\|Ty\| > \|y\|$ for all $y \in K \cap \partial\Omega_{r^*}$. Now we choose $R > r^*$. By Lemma 4.1, the operator T has a fixed point y in $K \cap \left(\overline{\Omega}_R \backslash \Omega_{r^*}\right)$. The rest of the proof is the same as that of Theorem 2.1 and hence is omitted.

(f) Since $g_0 = \infty$, there exists some $r^* \in (0, R)$ such that $g(y)/y \ge \beta^{-1}M_2$ for $y \in [\beta r^*, r^*]$. This implies that $g(y) \ge \beta^{-1}M_2 y \ge M_2 r^*$ for $y \in [\beta r^*, r^*]$. As in the proof Theorem 2.1 we have $\|Ty\| > \|y\|$ for all $y \in K \cap \partial\Omega_{r^*}$. By Lemma 4.1, the operator T has a fixed point y in $K \cap \left(\overline{\Omega}_R \backslash \Omega_{r^*}\right)$. The rest of the proof is the same as that of Theorem 2.1 and hence is omitted. □

Proof of Theorem 2.3. (a) By Corollary 2.2, Parts (a) and (b) we see that that BVP (1.1), (1.2) has at least two positive solution u_1 and u_2. From the proofs of Corollary 2.2, Parts (a) and (b), we also see that

$$\max_{t\in[0,1]}\{t^{2-\alpha}u_1(t)\} < r_* < \max_{t\in[0,1]}\{t^{2-\alpha}u_2(t)\}.$$

This means that u_1 and u_2 are distinct.

(b) By Corollary 2.2, Parts (c) and (e), we see that that BVP (1.1),(1.2) has at least two positive solutions u_1 and u_2. As shown in Part (a), the two solutions are distinct. □

Proof of Theorem 2.4. By Theorem 2.1, for $k = 1, \dots, n-1$, BVP (1.1),(1.2) has a solution u_k satisfying that

$$r_k < \max_{t\in[0,1]}\{t^{2-\alpha}u_k(t)\} < r_{k+1}.$$

Hence the solutions u_k, $k = 1, \dots, n-1$, are distinct. □

The proof of Theorem 2.5 is essentially the same as that of Theorem 2.4 and hence is omitted.

Proof of Theorem 2.6. (a) Assume BVP (1.1), (1.2) has a positive solution $u(t)$. Then $y(t) = t^{2-\alpha}u(t)$ is a fixed point of the operator T defined by (4.12). Let $\|y\| = r$. Then by the condition, $g(y(t)) < M_1 y(t) \le M_1 r$. Hence by (1.3), (4.7), (2.4), and (4.5), we have that for $t \in [0,1]$

$$\begin{aligned} y(t) = (Ty)(t) &= \int_0^1 G^*(t,s) f(s, s^{\alpha-2}y(s))ds \\ &\le \int_0^1 G^*(t,s) q_2(s) g(y(s))ds \le \frac{B^{m-1}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s))ds \\ &< \frac{B^{m-1}}{\Gamma(\alpha)} \left(\int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \right) M_1 r = r. \end{aligned}$$

This contradicts the assumption that $\|y\| = r$.

(b) Again, assume BVP (1.1), (1.2) has a positive solution $u(t)$, then $y(t) = t^{2-\alpha}u(t)$ is a fixed point of the operator T defined by (4.12). Let $\|y\| = r$. By Lemma 4.4, $Ty \in K$, and so is y. This implies that $\beta r \le y(t) \le r$ for $1/4 \le t \le 3/4$. Let $1/4 \le t \le 3/4$. By the condition, $g(y(t)) > \beta^{-1}M_2 y(t) \ge M_2 r$, and hence

$$\begin{aligned}\|Ty\| \ge (Ty)(\frac{1}{2}) &= \int_0^1 G^*(\frac{1}{2}, s) f(s, s^{\alpha-2}y(s))ds \\ &> \int_{1/4}^{3/4} G^*(\frac{1}{2}, s) q_1(s) g(y(s))ds \\ &\ge \left(\int_{1/4}^{3/4} G^*(\frac{1}{2}, s) q_1(s)ds \right) M_2 r = r.\end{aligned}$$

This also contradicts the assumption that $\|y\| = r$. □

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REFERENCES

[1] J. Ehme, P. W. Eloe, and J. Henderson, Existence of solutions $2n$th order fully nonlinear generalized Sturm-Liouville boundary value problems, *Math. Inequal. Appl.*, **4** (2001), 247–255.

[2] J. Ehme, P. W. Eloe, and J. Henderson, Upper and lower solution methods for fully nonlinear boundary value problems, *J. Differential Equations*, **180** (2001), 51–64.

[3] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495–505.

[4] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[5] D. Jiang and C. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its applicaiton, *Nonlinear Anal.*, **72** (2010), 710–719.

[6] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier, New York, 2006.

[7] L. Kong and Q. Kong, Positive solutions of higher order boundary value problems, *Proc. Edinburgh Math. Soc.*, **48** (2005), 445–464.

[8] Q. Kong, M. Wang, Positive solutions of even order periodic boundary value problems, *Rocky Mountain J. Math.*, **41** (2011), 1907–1930.

[9] M. A, Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Oxford-London-New York-Paris: Pergamon Press, 1964.

[10] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, vol. 198, Academic Press, New York, London, Toronto, 1999.

[11] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integral and Derivatives (Theory and Application)*, Gordon and Breach, Switzerland, 1993.

[12] V. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer-Verlag, New York, 2011.