Communications in Applied Analysis **19** (2015), 553–564

POSITIVE SOLUTIONS TO SINGULAR HIGHER ORDER BOUNDARY VALUE PROBLEMS ON PURELY DISCRETE TIME SCALES

CURTIS KUNKEL AND ASHLEY MARTIN

[1]Department of Mathematics & Statistics, University of Tennessee Martin
Martin, TN 38238 USA
E-mail: ckunkel@utm.edu *E-mail:* ashnpoor@ut.utm.edu

ABSTRACT. We study singular discrete higher order boundary value problems with mixed boundary conditions of the form

$$u^{\Delta^n}(t_{i-(n-1)}) + f(t_i, u(t_i), \dots, u^{\Delta^{n-1}}(t_{i-(n-1)})) = 0,$$

$$u^{\Delta^{n-1}}(t_0) = u^{\Delta^{n-2}}(t_{N+1}) = u^{\Delta^{n-3}}(t_{N+2}) = \cdots = u^{\Delta}(t_{N+n-2}) = u(t_{N+n-1}) = 0,$$

over a finite discrete interval $\mathbb{T} = \{t_0, t_1, \dots, t_{N+n-2}, t_{N+n-1}\}$. We prove the existence of a positive solution by means of the lower and upper solutions method and the Brouwer fixed point theorem in conjunction with perturbation methods to approximate regular problems.

AMS (MOS) Subject Classification. 39A10, 34B16, 34B18.

1. PRELIMINARIES

This paper is somewhat of an extension of the work done by Rachůnková and Rachůnek [23] and the works done by Kunkel [17], [18]. Rachůnková and Rachůnek studied a second order singular boundary value problem for the discrete p-Laplacian, $\phi_p(x) = |x|^{p-2}x$, $p > 1$. In particular, Rachůnková and Rachůnek dealt with the discrete boundary value problem

$$\Delta\left(\phi_p(\Delta u(t-1))\right) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],$$
$$\Delta u(0) = u(T+2) = 0,$$

in which $f(t, x_1, x_2)$ is singular in x_1. Kunkel's results extended theirs to the third order case, but only for $p = 2$, i.e. $\phi_2(x) = x$. That is, Kunkel's extension focused on the boundary value problem

$$-\Delta^3 u(t-2) + f(t, u(t), \Delta u(t-1), \Delta^2 u(t-2)) = 0, \quad t \in [2, T+1],$$
$$\Delta^2 u(0) = \Delta u(T+2) = u(T+3) = 0.$$

Received December 15, 2014

Kunkel's other work entails an extension to a second order singular discrete boundary value problem with non-uniform step size (what we are calling a purely discrete time scale)

$$u^{\Delta\Delta}(t_{i-1}) + f(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t_i \in \mathbb{T}^{\circ},$$
$$u^{\Delta}(t_0) = u(t_{n+1}) = 0.$$

The methods of this paper rely heavily on upper and lower solutions methods in conjunction with an application of the Brouwer fixed point theorem [26]. We consider only the singular third order boundary value problem, while letting our function range over a discrete interval with non-uniform step size. We will provide definitions of appropriate upper and lower solutions. The upper and lower solutions will be applied to nonsingular perturbations of our nonlinear problem, ultimately giving rise to our boundary value problem by passing to the limit.

Upper and lower solutions have been used extensively in establishing solutions of boundary value problems for finite difference equations. In addition to [11], [17], [23], we mention especially the paper by Jiang, *et al.* [13] in which they dealt with singular discrete boundary value problems using upper and lower solutions methods. For other outstanding results in which upper and lower solutions methods were employed to obtain solutions of boundary value problems for finite difference equations, we refer to [1], [2], [4], [5], [6], [10], [12], [16], [20], [21], [22], [27].

Singular discrete boundary value problems also have received a good deal of attention. For a list of a few representative works, we suggest the references [3], [7], [8], [14], [15], [19], [22], [24], [25], [27],[28].

In this section, we will state the definitions that are used in the remainder of the paper.

Definition 1.1. For $0 \leq i \leq N+n-1$, let $t_i \in \mathbb{R}$, where $t_0 < t_1 < \cdots < t_{N+n-2} < t_{N+n-1}$. Define the discrete intervals

$$\mathbb{T} := [t_0, t_{N+n-1}] = \{t_0, t_1, \ldots, t_{N+n-2}, t_{N+n-1}\},$$

and

$$\mathbb{T}^{\circ} := [t_{n-1}, t_N] = \{t_{n-1}, t_n, \ldots, t_{N-1}, t_N\}.$$

Definition 1.2. For the function $u : \mathbb{T} \to \mathbb{R}$, define the delta derivative [9], u^{Δ}, by

$$u^{\Delta}(t_i) := \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i}, \quad t_i \in \mathbb{T} \backslash t_{N+n-1}.$$

We make note that $u^{\Delta^2}(t_i) = u^{\Delta\Delta}(t_i) = u^{\Delta^{\Delta}}(t_i)$.

Consider the higher order nonlinear discrete dynamic

$$u^{\Delta^n}(t_{i-(n-1)}) + f(t_i, u(t_i), \ldots, u^{\Delta^{n-1}}(t_{i-(n-1)})) = 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{1.1}$$

with mixed boundary conditions

$$u^{\Delta^{n-1}}(t_0) = u^{\Delta^{n-2}}(t_{N+1}) = u^{\Delta^{n-3}}(t_{N+2}) = \cdots = u^{\Delta}(t_{N+n-2}) = u(t_{N+n-1}) = 0. \quad (1.2)$$

Our goal is to prove the existence of a positive solution of problem (1.1), (1.2).

Definition 1.3. By a solution of problem (1.1), (1.2), we mean a function $u : \mathbb{T} \to \mathbb{R}$ such that u satisfies the discrete dynamic (1.1) on $\mathbb{T}^{\circ}$ and the boundary conditions (1.2). If $u(t) > 0$ for $t \in \mathbb{T}^{\circ}$, we say u is a positive solution of the problem (1.1), (1.2).

Definition 1.4. Let $\mathcal{D} \subseteq \mathbb{R}^n$. We say that f is continuous on $\mathbb{T} \times \mathcal{D}$ if $f(t_i, x_1, \ldots, x_n)$ is defined on $\mathbb{T}$ for each $(x_1, \ldots, x_n) \in \mathcal{D}$, and if $f(t_i, x_1, \ldots, x_n)$ is continuous on $\mathcal{D}$ for each $t_i \in \mathbb{T}$.

Definition 1.5. Let $f : \mathbb{T} \times \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}^n$. If $\mathcal{D} = \mathbb{R}^n$, problem (1.1), (1.2) is called regular. If $\mathcal{D} \neq \mathbb{R}^n$ and f has singularities on the boundary of $\mathcal{D}$, then problem (1.1), (1.2) is called singular.

We will assume throughout this paper that the following hold:

(A): $\mathcal{D} = (0, \infty) \times \mathbb{R}^{n-1}$.

(B): f is continuous on $\mathbb{T} \times \mathcal{D}$.

(C): $f(t_i, x_1, \ldots, x_n)$ has a singularity at $x_1 = 0$, i.e. $\limsup_{x_1 \to 0^+} |f(t_i, x_1, \ldots, x_n)| = \infty$ for each $t_i \in \mathbb{T}$ and for some $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$.

2. LOWER AND UPPER SOLUTIONS METHOD FOR REGULAR PROBLEMS

Let us first consider the regular dynamic equation

$$u^{\Delta^n}(t_{i-(n-1)}) + h(t_i, u(t_i), \ldots, u^{\Delta^{n-1}}(t_{i-(n-1)})) = 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{2.1}$$

where h is continuous on $\mathbb{T}^{\circ} \times \mathbb{R}^n$ satisfying the boundary conditions (1.2). We establish a lower and upper solutions method for this regular problem (2.1), (1.2).

Definition 2.1. $\alpha : \mathbb{T} \to \mathbb{R}$ is called a lower solution of (2.1), (1.2) if

$$\alpha^{\Delta^n}(t_{i-(n-1)}) + h(t_i, \alpha(t_i), \ldots, \alpha^{\Delta^{n-1}}(t_{i-(n-1)})) \le 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{2.2}$$

satisfying boundary conditions

$$\begin{array}{rcl} (-1)^{n-1}\alpha^{\Delta^{n-1}}(t_0) & \le & 0, \\ (-1)^{n-2}\alpha^{\Delta^{n-2}}(t_{N+1}) & \le & 0, \\ & \vdots & \\ \alpha^{\Delta^2}(t_{N+n-3}) & \le & 0 \\ \alpha^{\Delta}(t_{N+n-2}) & \ge & 0 \\ \alpha(t_{N+n-1}) & \le & 0. \end{array} \tag{2.3}$$

Definition 2.2. $\beta : \mathbb{T} \to \mathbb{R}$ is called an upper solution of (2.1), (1.2) if

$$\beta^{\Delta^n}(t_{i-(n-1)}) + h(t_i, \beta(t_i), \ldots, \beta^{\Delta^{n-1}}(t_{i-(n-1)})) \geq 0, \quad t_i \in \mathbb{T}^\circ, \tag{2.4}$$

satisfying boundary conditions

$$\begin{aligned} (-1)^{n-1}\beta^{\Delta^{n-1}}(t_0) &\geq 0, \\ (-1)^{n-2}\beta^{\Delta^{n-2}}(t_{N+1}) &\geq 0, \\ &\vdots \\ \beta^{\Delta^2}(t_{N+n-3}) &\geq 0 \\ \beta^{\Delta}(t_{N+n-2}) &\leq 0 \\ \beta(t_{N+n-1}) &\geq 0. \end{aligned} \tag{2.5}$$

Theorem 2.3 (Lower and Upper Solutions Method)**.** *Let α and β be lower and upper solutions of* (2.1), (1.2), *respectively, and $\alpha \leq \beta$ on $\mathbb{T}$. Let $h(t_i, x_1, \ldots, x_n)$ be continuous on $\mathbb{T} \times \mathbb{R}^n$ and nonincreasing in its x_n variable. Then* (2.1), (1.2) *has a solution u satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in \mathbb{T}.$$

Proof. We proceed through a sequence of steps involving modifications of h.
Step 1. For $t_i \in \mathbb{T}^\circ, (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$\widetilde{h}\left(t_i, x_1, \ldots, x_{n-1}, \frac{x_n - x_{n-1}}{t_{i-(n-1)} - t_{i-(n-2)}}\right) =$$
$$\begin{cases} h\left(t_i, \beta(t_i), \ldots, \beta^{\Delta^{n-1}}(t_{i-(n-2)}), \frac{\beta^{\Delta^n}(t_{i-(n-1)}) - \sigma(t_{i-(n-1)}, x_n)}{t_{i-(n-1)} - t_{i-(n-2)}}\right) - \frac{\beta^{\Delta^n}(t_{i-(n-1)}) - x_n}{\beta^{\Delta^n}(t_{i-(n-1)}) - x_n + 1}, \\ \qquad x_n > \beta^{\Delta^n}(t_{i-(n-1)}), \\ h\left(t_i, x_1 \ldots, x_{n-1}, \frac{x_n - \sigma(t_{i-(n-1)}, x_n)}{t_{i-(n-1)} - t_{i-(n-2)}}\right), \qquad \alpha^{\Delta^n}(t_{i-(n-1)}) \leq x_n \leq \beta^{\Delta^n}(t_{i-(n-1)}), \\ h\left(t_i, \alpha(t_i) \ldots, \alpha^{\Delta^{n-1}}(t_{i-(n-2)}), \frac{\alpha^{\Delta^n}(t_{i-(n-1)}) - \sigma(t_{i-(n-1)}, x_n)}{t_{i-(n-1)} - t_{i-(n-2)}}\right) + \frac{x_n - \alpha^{\Delta^n}(t_{i-(n-1)})}{x_n - \alpha^{\Delta^n}(t_{i-(n-1)}) + 1}, \\ \qquad x_n < \alpha^{\Delta^n}(t_{i-(n-1)}), \end{cases}$$

where

$$\sigma(t_{i-(n-1)}, x_n) = \begin{cases} \alpha^{\Delta^n}(t_{i-(n-2)}), & x_n > (-1)^n \alpha^{\Delta^n}(t_{i-(n-1)}), \\ x_n, & (-1)^n \beta^{\Delta^n}(t_{i-(n-1)}) \leq x_n \leq (-1)^n \alpha^{\Delta^n}(t_{i-(n-1)}), \\ \beta^{\Delta^n}(t_{i-(n-2)}), & x_n < (-1)^n \beta^{\Delta^n}(t_{i-(n-1)}). \end{cases}$$

By its construction, $\widetilde{h}$ is continuous on $\mathbb{T}^\circ \times \mathbb{R}^n$, and there exists $M > 0$ so that,

$$|\widetilde{h}(t_i, x_1, \ldots, x_n)| \leq M, \quad t_i \in \mathbb{T}^\circ, (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

We now study the auxiliary equation,

$$u^{\Delta^n}(t_{i-(n-1)}) + \widetilde{h}(t_i, u(t_i), \ldots, u^{\Delta^{n-1}}(t_{i-(n-1)})) = 0, \quad t_i \in \mathbb{T}^\circ, \tag{2.6}$$

satisfying boundary conditions (1.2). Our immediate goal is to prove the existence of a solution of (2.6), (1.2).

Step 2. We lay the foundation to use the Brouwer fixed point theorem. To this end, define the Banach space E by

$$\begin{aligned} E = \{u : \mathbb{T} \to \mathbb{R} : u^{\Delta^{n-1}}(t_0) = u^{\Delta^{n-2}}(t_{N+1}) = u^{\Delta^{n-3}}(t_{N+2}) = \cdots \\ = u^{\Delta}(t_{N+n-2}) = u(t_{N+n-1}) = 0\}, \end{aligned}$$

and also define

$$\|u\| = \max\{|u(t_i)| : t_i \in \mathbb{T}\}.$$

Further, we define an operator $\mathcal{T} : E \to E$ by

$$\begin{aligned} (\mathcal{T}u)(t_z) = (-1)^n \sum_{z_{n-2}=z}^{N+n-2} (t_{z_{n-2}+1} - t_{z_{n-2}}) \\ \cdots \sum_{i=n-1}^{z_0+n-1} (t_{i-n+1} - t_{i-n}) \widetilde{h}\left(t_i, u(t_i), \ldots, u^{\Delta^{n-1}}(t_{i-(n-1)})\right). \end{aligned} \tag{2.7}$$

By its construction, $\mathcal{T}$ is a continuous operator.

Moreover, from the bounds placed on $\widetilde{h}$ in Step 1 and from (2.7), if

$$r > (t_{N+n-1} - t_0)^n M,$$

then $\mathcal{T}\left(\overline{B(r)}\right) \subset \overline{B(r)}$, where $B(r) = \{u \in E : \|u\| < r\}$. Therefore, by the Brouwer fixed point theorem [26], there exists $u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

Step 3. We now show that u is a fixed point of $\mathcal{T}$ iff u is a solution of (2.6), (1.2).

First, assume $u = \mathcal{T}u$. Then, $u \in E$, and thus, satisfies (1.2). Furthermore,

$$\begin{aligned} u^{\Delta}(t_z) &= \frac{u(t_{z+1}) - u(t_z)}{t_{z+1} - t_z} \\ &= \frac{1}{t_{z+1} - t_z} \cdot \left[(-1)^n \sum_{z_{n-2}=z+1}^{N+n-2} (t_{z_{n-2}+1} - t_{z_{n-2}}) \cdots \widetilde{h}(t_i, \ldots) \right. \\ &\quad \left. - (-1)^n \sum_{z_{n-2}=z}^{N+n-2} (t_{z_{n-2}+1} - t_{z_{n-2}}) \cdots \widetilde{h}(t_i, \ldots) \right] \\ &= \frac{1}{t_{z+1} - t_z} \cdot \left[-(-1)^n (t_{z+1} - t_z) \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots \widetilde{h}(t_i, \ldots) \right] \\ &= (-1)^{n-1} \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots \widetilde{h}(t_i, \ldots), \end{aligned}$$

and we see that

$$u^{\Delta}(t_z) = (-1)^{n-1} \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots \widetilde{h}(t_i, \ldots).$$

Continuing in this manner,

$$\begin{aligned}
u^{\Delta^n}(t_z) &= \frac{u^{\Delta^{n-1}}(t_{z+1}) - u^{\Delta^{n-1}}(t_z)}{t_{z+1}-t_z} \\
&= \frac{1}{t_{z+1}-t_z} \cdot \Bigg[-\sum_{i=n-1}^{z+n} (t_{i-n+1}-t_{i-n})\widetilde{h}\,(t_i, \dots) \\
&\quad - \sum_{i=n-1}^{z+n-1} (t_{i-n+1}-t_{i-n})\widetilde{h}\,(t_i, \dots) \Bigg] \\
&= \frac{-(t_{z+1}-t_z)\widetilde{h}\,(t_{z+n}, \dots)}{t_{z+1}-t_z} \\
&= -\widetilde{h}\,(t_{z+n}, \dots).
\end{aligned}$$

Thus, we see that $u^{\Delta^n}(t_z) + \widetilde{h}\,(t_{z+n}, \dots) = 0$.

On the other hand, let $u(t)$ solve (2.6), (1.2). Then,

$$\begin{aligned}
u^{\Delta^n}(t_0) &= \frac{u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(t_0)}{t_1 - t_0} \\
&= \frac{u^{\Delta^{n-1}}(t_1)}{t_1 - t_0} \\
&= -\widetilde{h}\left(t_{n-1}, u(t_{n-1}), \dots, u^{\Delta^{n-1}}(t_0)\right).
\end{aligned}$$

This implies that

$$u^{\Delta^{n-1}}(t_1) = -(t_1 - t_0)\widetilde{h}\left(t_{n-1}, u(t_{n-1}), \dots, u^{\Delta^{n-1}}(t_0)\right).$$

Also,

$$\begin{aligned}
u^{\Delta^n}(t_1) &= \frac{u^{\Delta^{n-1}}(t_2) - u^{\Delta^{n-1}}(t_1)}{t_2 - t_1} \\
&= \frac{u^{\Delta^{n-1}}(t_2) - -(t_1 - t_0)\widetilde{h}\left(t_{n-1}, u(t_{n-1}), \dots, u^{\Delta^{n-1}}(t_0)\right)}{t_2 - t_1} \\
&= -\widetilde{h}\left(t_n, u(t_n), \dots, u^{\Delta}(t_1)\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
u^{\Delta^{n-1}}(t_2) = &- (t_2 - t_1)\widetilde{h}\left(t_n, u(t_n), \dots, u^{\Delta^{n-1}}(t_1)\right) \\
&+ -(t_1 - t_0)\widetilde{h}\left(t_{n-1}, u(t_{n-1}), \dots, u^{\Delta^{n-1}}(t_0)\right).
\end{aligned}$$

Continuing inductively, we see that

$$u^{\Delta^{n-1}}(t_z) = -\sum_{i=1}^{z}(t_i - t_{i-1})\widetilde{h}\left(t_{i+(n-2)}, u(t_{i+(n-2)}), \dots, u^{\Delta^{n-1}}(t_{i-1})\right), \quad 1 \le z \le N.$$

Now, we use similar techniques to see that

$$\begin{aligned}
u^{\Delta^{n-1}}(t_N) &= \frac{u^{\Delta^{n-2}}(t_{N+1}) - u^{\Delta^{n-2}}(t_N)}{t_{N+1} - t_N} \\
&= \frac{-u^{\Delta^{n-2}}(t_N)}{t_{N+1} - t_N} \\
&= -\sum_{i=1}^{N} (t_i - t_{i-1}) \widetilde{h}\left(t_{i+(n-2)}, u(t_{i+(n-2)}), \dots, u^{\Delta^{n-1}}(t_{i-1})\right).
\end{aligned}$$

and that

$$u^{\Delta^{n-2}}(t_N) = (t_{N+1} - t_N) \sum_{i=1}^{N} (t_i - t_{i-1}) \widetilde{h}\left(t_{i+(n-2)}, u(t_{i+(n-2)}), \dots, u^{\Delta^{n-1}}(t_{i-1})\right).$$

Proceeding through the interval, we see that

$$u^{\Delta^{n-2}}(t_z) = \sum_{z_0=z}^{N} (t_{z_0+1} - t_{z_0}) \sum_{i=1}^{z_0} (t_i - t_{i-1}) \widetilde{h}\left(t_{i+(n-2)}, u(t_{i+(n-2)}), \dots, u^{\Delta^{n-1}}(t_{i-1})\right).$$

In a similar fashion for $j = 3, 4, \dots, n$, we see that

$$\begin{aligned}
u^{\Delta^{n-(j-1)}}(t_{N+j-2}) &= \frac{u^{\Delta^{n-j}}(t_{N+j-1}) - u^{\Delta^{n-j}}(t_{N+j-2})}{t_{N+j-1} - t_{N+j-2}} \\
&= -\frac{u^{\Delta^{n-j}}(t_{N+j-2})}{t_{N+j-1} - t_{N+j-2}} \\
&= (-1)^{j-1} \sum_{z_{j-1}=N+j-2}^{N+j-1} \cdots \widetilde{h}\,(t_i, \dots).
\end{aligned}$$

and that

$$u^{\Delta^{n-j}}(t_{N+j-2}) = (-1)^j (t_{N+j-1} - t_{N+j-2}) \sum_{z_{j-1}=N+j-2}^{N+j-1} \cdots \widetilde{h}\,(t_i, \dots).$$

Proceeding through the interval, similar to before, we conclude that

$$u^{\Delta^{n-j}}(t_z) = (-1)^j \sum_{z_{j-2}=z}^{N+j-2} (t_{z_{j-2}+1} - t_{z_{j-2}}) \cdots \widetilde{h}\,(t_i, \dots).$$

And specifically, for $j = n$,

$$\begin{aligned}
u(t_z) = (-1)^n &\sum_{z_{n-2}=z}^{N+n-2} (t_{z_{n-2}+1} - t_{z_{n-2}}) \\
&\cdots \sum_{i=n-1}^{z_0+n-1} (t_{i-n+1} - t_{i-n}) \widetilde{h}\left(t_i, u(t_i), \dots, u^{\Delta^{n-1}}(t_{i-(n-1)})\right).
\end{aligned}$$

We therefore can conclude that $u = \mathcal{T}u$, and this step of the proof is complete.
Step 4. We now show that solutions $u(t)$ of (2.6), (1.2) satisfy

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in \mathbb{T}.$$

Consider the case of obtaining $u(t) \le \beta(t)$. Let $v(t) = u(t) - \beta(t)$. For the sake of establishing a contradiction, assume that

$$\max\{v(t) : t \in \mathbb{T}\} := v(t_l) > 0.$$

From the boundary conditions (1.2) and (2.5), we see that $t_l \in \mathbb{T}^\circ$. Thus, $v(t_{l-1}) \le v(t_l)$ and $v(t_{l+1}) \le v(t_l)$. Consequently, $v^\Delta(t_l) \le 0$ and $v^\Delta(t_{l-1}) \ge 0$. This in turn implies that $v^{\Delta\Delta}(t_{l-1}) \le 0$. Continuing in this manner we see that

$$v^{\Delta^n}(t_l) \le 0.$$

On the other hand, since h is nonincreasing in its x_n variable, we have from (2.1) that

$$\begin{aligned}
v^{\Delta^n}(t_l) &= u^{\Delta^n}(t_l) - \beta^{\Delta^n}(t_l) \\
&\ge -\widetilde{h}\left(t_{l+n}, u(t_{l+n}), \ldots, u^{\Delta^{n-1}}(t_l)\right) - (-)\widetilde{h}\left(t_{l+n}, \beta(t_{l+n}), \ldots, \beta^{\Delta^{n-1}}(t_l)\right) \\
&= -\widetilde{h}(t_{l+n}, \beta(t_{l+n}), \ldots, \beta^{\Delta^{n-1}}(t_l)) + \frac{\beta^{\Delta^n}(t_l) - u^{\Delta^n}(t_l)}{\beta^{\Delta^n}(t_l) - u^{\Delta^n}(t_l) + 1} \\
&\quad + \widetilde{h}(t_{l+n}, \beta(t_{l+n}), \ldots, \beta^{\Delta^{n-1}}(t_l)) \\
&= \frac{\beta^{\Delta^n}(t_l) - u^{\Delta^n}(t_l)}{\beta^{\Delta^n}(t_l) - u^{\Delta^n}(t_l) + 1} \\
&> 0.
\end{aligned}$$

But this is a contradiction. Therefore, $v(l) \le 0$. Which means that $u(t) \le \beta(t)$ for all $t \in \mathbb{T}$. A similar argument shows that $\alpha(t) \le u(t)$ for all $t \in \mathbb{T}$.

Thus, the conclusion of the theorem holds and our proof is complete. □

3. EXISTENCE RESULT

In this section, we make use of Theorem 2.3 to obtain positive solutions of the singular problem (1.1), (1.2). In particular, in applying Theorem 2.3, we deal with a sequence of regular perturbations of (1.1), (1.2). Ultimately, we obtain a desired solution of (1.1), (1.2) by passing to the limit on a sequence of solutions for the perturbations.

Theorem 3.1. *Assume conditions (A), (B), and (C) hold, along with the following:*

(D): *There exists $c \in (0, \infty)$ so that $f(t_i, c, 0, \cdots, 0) \le 0$ for all $t \in \mathbb{T}^\circ$.*

(E): *$f(t_i, x_1, x_2, x_3, \ldots, x_n)$ is nonincreasing in its x_n variable for $t_i \in \mathbb{T}$ and $x_1 \in (0, c]$.*

(F): $\lim_{x_1 \to 0^+} f(t_i, x_1, \ldots, x_n) = \infty$ *for* $t_i \in \mathbb{T}$.

Then, (1.1), (1.2) *has a solution* u *satisfying*

$$0 < u(t) \leq c, \quad t \in \mathbb{T}^\circ.$$

Proof. Again for the proof, we proceed through a sequence of steps. Step 1. For $k \in \mathbb{N},\ t \in \mathbb{T}^\circ,\ (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$f_k(t_i, x_1, \ldots, x_n) = \begin{cases} f\left(t_i, |x_1|, x_2, \ldots, x_n\right), & |x_1| \geq \frac{1}{k} \\ f\left(t_i, \frac{1}{k}, x_2, \ldots, x_n\right), & |x_1| < \frac{1}{k} \end{cases}$$

Then, f_k is continuous on $\mathbb{T}^\circ \times \mathbb{R}^n$ and nonincreasing for $t_i \in \mathbb{T}^\circ,\ x_1 \in [-c, c]$.

Assumption (F) implies that there exists k_0 such that for all $k \geq k_0$,

$$f_k(t_i, 0, \ldots, 0) = f\left(t_i, \frac{1}{k}, 0, \ldots, 0\right) > 0, \quad t_i \in \mathbb{T}^\circ.$$

Consider, for each $k \geq k_0$,

$$u^{\Delta^n}(t_{i-(n-1)}) + f_k(t_i, u(t_i), \ldots, u^{\Delta^{n-1}}(t_{i-(n-1)})) = 0, \quad t_i \in \mathbb{T}^\circ. \tag{3.1k}$$

Define $\alpha(t) = 0$ and $\beta(t) = c$. Then, α and β are lower and upper solutions for (3.1k), (1.2), and $\alpha(t) \leq \beta(t)$ on $\mathbb{T}^\circ$. Thus, by Theorem 2.3, there exists u_k a solution of (3.1k), (1.2) satisfying $0 \leq u_k(t_i) \leq c$, $t_i \in \mathbb{T}$, $k \geq k_0$. Consequently,

$$|u_k^\Delta(t_i)| \leq \frac{c}{(t_{i+1} - t_i)}, \quad t_i \in \mathbb{T}^\circ. \tag{3.2}$$

Step 2. Let $k \in \mathbb{N}$, $k \geq k_0$. Since $u_k(t)$ solves (3.1k), we get from work similar to that exhibited in Theorem 2.3,

$$u_k^\Delta(t_z) = (-1)^{n-1} \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots f_k\left(t_i, \ldots\right), \quad t_z \in \mathbb{T}^\circ. \tag{3.3}$$

By assumption (F), there exists $\varepsilon_1 \in \left(0, \frac{1}{k_0}\right)$ such that if $k \geq \frac{1}{\varepsilon_1}$,

$$f_k(t_1, x_1, \ldots, x_n) > \frac{c}{(t_2 - t_1)^n}, \quad x_1 \in (0, \varepsilon_1]. \tag{3.4}$$

For the sake of establishing a contradiction, assume that for $k \geq \frac{1}{\varepsilon_1}$, we have $u_k(t_1) < \varepsilon_1$. Then, by (3.3) and (3.4),

$$\begin{aligned} u_k^{\Delta}(t_1) &= (-1)^{n-1} \sum_{z_{n-3}=1}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots f_k\left(t_i, \dots\right) \\ &= (-1)^{n-1} \left((t_2 - t_1)^{n-1} f_k(t_i, \dots) + \sum_{z_{n-3}=2}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots f_k\left(t_i, \dots\right) \right) \\ &\geq (t_2 - t_1)^{n-1} f_k(t_i, \dots) \\ &> \frac{c}{(t_2 - t_1)}. \end{aligned}$$

But this contradicts (3.2). Hence $u_k(t_1) \geq \varepsilon_1$ for all $k \geq \frac{1}{\varepsilon_1}$.

Proceeding across the interval, we get a sequence of epsilons where $0 < \varepsilon_{N+n-1} < \cdots < \varepsilon_2 < \varepsilon_1$ such that $u_k(t_i) \geq \varepsilon_{N+n-1}$ for $t_i \in \mathbb{T}$. Hence $u_k(t_{N+n-1}) \geq \varepsilon_{N+n-1}$ for all $k \geq \frac{1}{\varepsilon_{N+n-1}}$. Therefore, by letting $\varepsilon = \frac{\varepsilon_{N+n-1}}{2}$, we get

$$0 < \varepsilon \leq u_k(t_i) \leq c, \quad t_i \in \mathbb{T}^{\circ}, \quad k \geq \frac{1}{\varepsilon}. \tag{3.5}$$

Since $u_k(t_i)$ satisfies (3.5) and (1.2), we can choose a subsequence $\{u_{k_l}(t)\} \subset \{u_k(t)\}$ such that $\lim\limits_{l\to\infty} u_{k_l}(t) = u(t),\ t \in \mathbb{T}^{\circ},\ u(t) \in E$, where E is the Banach space as defined in Step 2 of Theorem 2.3.

Moreover, (3.3) yields for each sufficiently large l,

$$u_{k_l}^{\Delta}(t_z) = (-1)^{n-1} \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots f_k\left(t_i, \dots\right),$$

and so by letting $l \to \infty$ and from the continuity of f, we get that

$$u^{\Delta}(t_z) = (-1)^{n-1} \sum_{z_{n-3}=z}^{N+n-3} (t_{z_{n-3}+1} - t_{z_{n-3}}) \cdots f\left(t_i, \dots\right).$$

Consequently, we also get, via similar methods exhibited in Step 3 of Theorem 2.3,

$$u^{\Delta^n}(t_z) = -f\left(t_{z+n}, u(t_{z+n}), \dots, u^{\Delta^{n-1}}(t_z)\right).$$

Therefore, u solves (1.1), (1.2), and by (3.5), our theorem holds. $\square$

REFERENCES

[1] R. P. Agarwal, A. Cabada and V. Otero-Espinar, Existence and uniqueness results for n-th order nonlinear difference equations in presence of lower and upper solutions, *Arch. Inequal. Appl.* **1** (2003), no. 3-4, 421–431.

[2] R. P. Agarwal, A. Cabada, V. Otero-Espinar and S. Dontha, Existence and uniqueness of solutions for anti-periodic difference equations, *Arch. Inequal. Appl.* **2** (2004), no. 4, 397–411.

[3] R. P. Agarwal, D. O'Regan and S. Stanêk, An existence principle for nonlocal difference boundary value problems with φ-Laplacian and its application to singular problems, *Adv. Difference Equ.* **2008** (2008), Art. ID 154302, 14 pp.

[4] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic Publishers Group, Dordrecht, 1999.

[5] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, Existence of constant-sign solutions to a system of difference equations: the semipositone and singular case, *J. Difference Equ. Appl.* **11** (2005), no. 2, 151–171.

[6] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers Group, Dordrecht, 1997.

[7] E. Akın-Bohner, F. M. Atıcı and B. Kaymakçalan, Lower and upper solutions of boundary value problems, in: *Advances in Dynamic Equations on Time Scales,* (Ed: M. Bohner and A. Peterson), Birkhäuser, Boston, 2003.

[8] F. M. Atıcı, A. Cabada and V. Otero-Espinar, Criteria for existence and nonexistence of positive solutions to a discrete periodic boundary value problem, *J. Difference Equ. Appl.* **9** (2003), no. 9, 765–775.

[9] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[10] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* **2011** (2011), Art. ID 893753, 18 pp.

[11] J. Henderson and C. Kunkel, Singular discrete higher order boundary value problems, *Int. J. Difference Equ.* **1** (2006), no. 1, 119–133.

[12] J. Henderson and H. B. Thompson, Existence of multiple solutions for second-order discrete boundary value problems, *Comput. Math. Appl.* **43** (2002), no. 10-11, 1239–1248.

[13] D. Q. Jiang, D. O'Regan and R. P. Agarwal, A generalized upper and lower solution method for singular discrete boundary value problems for the one-dimensional p-Laplacian, *J. Appl. Anal.* **11** (2005), no. 1, 35–47.

[14] L. Jodar, Singular bilateral boundary value problems for discrete generalized Lyapunov matrix equations, *Stochastica* **11** (1987), no. 1, 45–52.

[15] L. Jodar, E. Navarro and J. L Morera, A closed-form solution of singular regular higher-order difference initial and boundary value problems, *Appl. Math. Comput.* **48** (1992), no. 2-3, 153–166.

[16] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Harcourt/Academic Press, San Diego, 2001.

[17] C. Kunkel, Singular discrete third order boundary value problems, *Comm. Appl. Nonlinear Anal.* **13** (2006), no. 3, 27–38.

[18] C. Kunkel, Singular second order boundary value problems on purely discrete time scales, *J. Difference Equ. Appl.* **14** (2008), no. 4, 411–420.

[19] D. S. Naidu and A. K. Rao, Singular perturbation methods for a class of initial- and boundary value problems in discrete systems, *Internat. J. Control* **36** (1982), no. 1, 77–94.

[20] D. O'Regan and M. El-Gebeily, Existence, upper and lower solutions and quasilinearization for singular differential equations, *IMA J. Appl. Math.* **73** (2008), no. 2, 323–344.

[21] C. V. Pao, Monotone iterative methods for finite difference system of reaction-diffusion equations, *Numer. Math.* **46** (1985), no. 4, 571–586.

[22] A. C. Peterson, Y. N. Raffoul, and C. C. Tisdell, Three point boundary value problems on time scales, *J. Difference Equ. and Appl.* **10** (2004), no. 9, 843–849.

[23] I. Rachůnková and L. Rachůnek, Singular discrete second order BVPs with p-Laplacian, *J. Difference Equ. Appl.* **12** (2006), no. 8, 811–819.

[24] I. Rachůnková and L. Rachůnek, Singular discrete and continuous mixed boundary value problems, *Math. Comput. Modelling* **49** (2009), no. 3-4, 413–422.

[25] C. Yuan, D. Jiang, and Y. Zhang, Existence and uniqueness of solutions for singular higher order continuous and discrete boundary problems, *Bound. Value Probl.*, **2008** (2008), Art. ID 123823, 11 pp.

[26] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

[27] B. Zhang, L. Kong, Y. Sun and X. Deng, Existence of positive solutions for BVPs of fourth-order difference equations, *Appl. Math. Comput.* **131** (2002), no 2-3, 583–591.

[28] B. Zheng, H. Xiao, and H. Shi, Existence of positive, negative, and sign-changing solutions to discrete boundary value problems, *Bound. Value Probl.* **2011** (2011), Art. ID 172818, 19 pp.