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NEW OSCILLATION CRITERIA FOR FOURTH ORDER NEUTRAL DYNAMIC EQUATIONS

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ABSTRACT. In this paper, the oscillation of a class of fourth order nonlinear neutral functional dynamic equations of the form

$$\left(r(t)\left((y(t)+p(t)y(\alpha(t)))^{\Delta^2}\right)\right)^{\Delta^2}+q(t)f(y(\beta(t)))=0$$

is studied on an arbitrary time scale $\mathbb{T}$, under the assumption

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty, t_0>0,$$

for various ranges of $p(t)$.

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1. INTRODUCTION

The study of Functional differential and difference equations is growing due to the development in science and technology and the varied applications in many areas. For examples, equations involving delay and those involving advance and a combination of both arise in nerve conduction (Life Sciences), organizational behaviour (Social sciences), signal processing pantograph equations (mechanical engineering), to mention a few (see for e.g [3, 6, 9, 15]). Study of such equations has been an active area of research for many researchers and recently an importance is given to the unification of continuous and discrete aspects of analysis on time scales.

It was Stefan Hilger [10], who has developed the time scales in his Ph.D work and recently has received a lot of attention for the researchers. The purpose of the time scales was not only to unify the study of continuous and discrete aspects of mathematics but also some cases in between. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and allows

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us to avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of the real numbers $\mathbb{R}$. In this way the results in this paper not only apply to the set of real numbers or set of integers, but also to more general time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k,\ k \in \mathbb{N}_0\}$ with $q > 1$ (which has important applications in quantum theory [11]), $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$ etc. For basic notations on time scale calculus, we refer the reader to the monographs [4, 5], the survey paper [1], and the references cited therein.

In this work, an attempt is made to study the oscillatory behaviour of solutions of nonlinear delay dynamic equations of the form

$$\Big(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2}\Big)^{\Delta^2} + q(t)f(y(\beta(t))) = 0, \tag{1.1}$$

where $q, r \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$, $\alpha, \beta \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\alpha(t) \le t$, $\beta(t) \le t$, and $\lim_{t\to\infty} \alpha(t) = \infty = \lim_{t\to\infty} \beta(t)$, $f \in C(\mathbb{R}, \mathbb{R})$ is a continuous function with the property $uf(u) > 0$ for $u \neq 0$, and $p \in C_{rd}(\mathbb{T}, \mathbb{R})$, under the assumption

(H_0) $\int\limits_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty$, $t_0 > 0$.

If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, then (1.1) reduces to

$$(r(t)((y(t) + p(t)y(\alpha(t)))'')'' + q(t)f(y(\beta(t))) = 0 \tag{1.2}$$

and

$$\Delta^2 \Big(r(n)(\Delta^2(y(n) + p(n)y(\alpha(n))))\Big) + q(n)f(y(\beta(n))) = 0 \tag{1.3}$$

respectively.

In the sequel, we assume the following hypotheses on f, α and β:

(H_1) $f(uv) = f(u)f(v)$, for $u, v \in \mathbb{R}$ and $u, v > 0$,

(H_2) $f(-u) = -f(u)$, for $u \in \mathbb{R}$,

(H_3) there exist $\lambda > 0$, such that $f(u) + f(v) \ge \lambda f(u + v)$, for $u, v \in \mathbb{R}$ and $u, v > 0$,

(H_4) α and β are bijective functions satisfying the properties:
$\alpha(\beta(t)) = \beta(\alpha(t))$, $\beta^{-1}(\alpha^{-1}(t)) = \alpha^{-1}(\beta^{-1}(t))$, $\beta(\alpha^{-1}(t)) = \alpha^{-1}(\beta(t))$,
$\alpha^{-1}(t) \ge t$, $\beta^{-1}(t) \ge t$, for every right-scattered point $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 \ge 0$.

Remark 1.1. *(H_1) and (H_2) implies that $f(-1) = -f(1)$.*

In [14, 17], Parhi and Tripathy have considered the equations (1.2) and (1.3) when $\alpha(t) = t - \alpha$ and $\beta(t) = t - \beta$, and established the sufficient results for oscillation and asymptotic behaviour of solutions, under the assumptions

$$\int\limits_0^{\infty} \frac{t}{r(t)} dt = \infty,$$

and its discrete analogue

$$\sum_{n=0}^{\infty} \frac{n}{r(n)} = \infty$$

respectively. It is interesting to see the unification of continuous and discrete aspects (1.2) and (1.3) through the dynamic equations on time scales in [13]. But, the problem lies there in the works [13], [14] and [17] concerning an *all solution oscillatory*.

The objective of this work is to establish the sufficient conditions for oscillation of all solutions of (1.1) under the assumption (H_0) on an arbitrary time scale $\mathbb{T}$.

Since we are interested in the oscillatory behaviour of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$. Let $t_{-1} = inf_{t \in [t_0,\infty)_{\mathbb{T}}} \{\alpha(t), \beta(t)\}$.

By a solution of (1.1) we mean a nontrivial real valued function y on $[T_y, \infty)_{\mathbb{T}}$ such that $(y(t) + p(t)y(\alpha(t))) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$, $\left(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2}\right) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$ and satisfies (1.1), for $T_y \geq t_{-1} > t_0 > 0$. In this paper, we do not consider the solutions that eventually vanish identically. A solution y of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative and it is *nonoscillatory* otherwise.

We may note that, (1.1) includes a class of differential or difference equations with delay argument of neutral type. In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of different classes of neutral dynamic equations. We refer the reader to some of the works [2, 7, 8, 12, 18, 19, 20], and the references cited therein.

2. PRELIMINARY RESULTS

We define the quasi-operators as follows:

$$\begin{aligned} &L_0 z(t) = z(t), L_1 z(t) = L_0^{\Delta} z(t), \\ &L_2 z(t) = r(t) L_1^{\Delta} z(t), L_3 z(t) = L_2^{\Delta} z(t), \\ &L_4 z(t) = L_3^{\Delta} z(t), \end{aligned}$$

where $z(t) = y(t) + p(t)y(\alpha(t))$. We need the following results for our use in the sequel:

Lemma 2.1 ([13])**.** *Let (H_0) hold. Let u be a real valued function on $[t_0, \infty)_{\mathbb{T}}$ such that $L_4 u(t) \leq 0$ for large t. If $u(t) > 0$ ultimately, then one of cases (a) and (b) holds for large t, and if $u(t) < 0$ ultimately, then one of cases (b), (c), (d) and (e) holds for large t, where*

(a) $L_1 u(t) > 0$, $L_2 u(t) > 0$ *and* $L_3 u(t) > 0$,
(b) $L_1 u(t) > 0$, $L_2 u(t) < 0$ *and* $L_3 u(t) > 0$,
(c) $L_1 u(t) < 0$, $L_2 u(t) < 0$ *and* $L_3 u(t) > 0$,

(d) $L_1u(t) < 0$, $L_2u(t) < 0$ *and* $L_3u(t) < 0$,
(e) $L_1u(t) < 0$, $L_2u(t) > 0$ *and* $L_3u(t) > 0$.

Lemma 2.2 ([13]). *Let the conditions of Lemma 2.1 hold. If $u(t) > 0$ ultimately, then $u(t) > R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta} = R_T(t)L_3u(t)$, $t \geq T \geq t_0$, where*

$$R_T(t) = \int_T^{\rho(t)} \frac{(s-T)(t-\sigma(s))}{r(s)}\Delta s.$$

Lemma 2.3 ([16]). *Assume that $p(t) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$. If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1,$$

then the inequality

$$x^{\Delta} + p(t)x(\tau(t)) \leq 0 (\geq 0)$$

doesn't admit any eventually positive (negative) solution.

Proof. The proof of the lemma follows from the proof of Theorem 2.4 [16]. Hence the details are omitted. □

3. NEW OSCILLATION CRITERIA

This section deals with the new oscillation criteria for (1.1). Before stating our main results, we assume the following hypotheses for our use in the next discussion:

$$A[s,v] = \int_v^s (s-\sigma(t))\frac{(t-v)}{r(t)}\Delta t, \quad s > \sigma(t) > t > v,$$
$$B[v,u] = \int_u^v (\sigma(u)-u)\frac{(u-t)}{r(t)}\Delta t, \quad v > \sigma(t) > t > u,$$
$$C[v,u] = \int_u^v (\sigma(t)-u)\frac{(u-t)}{r(t)}\Delta t, \quad v > \sigma(t) > t > u;$$

(H_5) $Q(t) = \min\{q(t),\, q(\alpha(t))\}$, for $t \geq t_0$,
(H_6) $\frac{f(u)}{u} \geq M_1 > 0$, for $u \neq 0$,
(H_7) $\limsup_{s\to\infty} \int_{\alpha(s)}^{s} Q(\theta)f[A(\beta(\theta),\, \beta(s))]\Delta\theta > \frac{1+f(a)}{\lambda M_1}$, $a > 0$,
(H_8) $\limsup_{\theta\to\infty} \int_{\alpha(\theta)}^{\theta} Q(v)f[C(\beta(v),\, \beta(\theta))]\Delta v > \frac{1+f(a)}{\lambda M_1}$, $a > 0$,
(H_9) $\limsup_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta)f[A(\beta(\theta),\, \beta(s))]\Delta\theta > \frac{1}{f(1-a)M_1}$, $0 < a < 1$,
(H_{10}) $\limsup_{\theta\to\infty} \int_{\alpha(\theta)}^{\theta} q(v)f[C(\beta(v),\, \beta(\theta))]\Delta v > \frac{1}{f(1-a)M_1}$, $0 < a < 1$,
(H_{11}) $\limsup_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta)f[A(\beta(\theta),\, \beta(s))]\Delta\theta > \frac{1}{M_1}$,
(H_{12}) $\limsup_{\theta\to\infty} \int_{\alpha(\theta)}^{\theta} q(v)f[C(\beta(v),\, \beta(\theta))]\Delta v > \frac{1}{M_1}$,
(H_{13}) $\tau^n(t) = \tau(\tau^{n-1}(t))$, $\lim_{n\to\infty} \tau^n(t) < \infty$,
(H_{14}) $\limsup_{v\to\infty} \int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u)f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))])\Delta u > \frac{1}{M_1 f(\frac{1}{b})}$, $b > 0$,
(H_{15}) $\int_{t_0}^{\infty} q(t)\Delta t = +\infty$, $t_0 > 0$,
(H_{16}) $\limsup_{s\to\infty} \int_{\beta(s)}^{\alpha^{-1}(\beta(s))} q(\theta)f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))])\Delta\theta > \frac{1}{M_1 f(\frac{1}{b})}$,

(H_{17}) $\limsup_{s\to\infty}\int_{\alpha(s)}^{s} Q(t)f(R_T(\beta(t)))\Delta t > \frac{1+f(a)}{\lambda M_1}$, $a>0$,
(H_{18}) $\limsup_{s\to\infty}\int_{\alpha(s)}^{s} q(t)f(R_T(\beta(t)))\Delta t > \frac{1}{M_1}$,
(H_{19}) $\limsup_{t\to\infty}\int_{\beta(\alpha^{-1}(t)}^{\sigma(t)}(\sigma(s)-s)^3\frac{q(s)}{r(s)}\Delta s > \frac{1}{M_1 f(b^{-1})}$.

Theorem 3.1. *Let* $0 \le p(t) \le a < \infty$ *and* $\beta(t) \le \alpha^2(t)$, *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If* (H_0)*–*(H_8) *hold, then* (1.1) *is oscillatory.*

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large such that $y(t)>0, y(\alpha(t)), y(\beta(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$. From (1.1), we have

$$L_4 z(t) = -q(t)f(y(\beta(t))) \le 0. \tag{3.1}$$

Hence, we can find a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $L_i z(t)$, $i=1,2,3$ are eventually of one sign on $[t_2,\infty)_{\mathbb{T}}$. In what follows, we consider *Cases (a)* and *(b)* of Lemma 2.1.

Case (a) For $u \ge v > t_2$,

$$L_2 z(u) - L_2 z(v) = \int_v^u L_3 z(s)\Delta s \ge (u-v)L_3 z(u),$$

that is, $L_2 z(u) \ge (u-v)L_3 z(u)$. Hence,

$$z^{\Delta^2}(u) \ge \frac{(u-v)}{r(u)}L_3 z(u). \tag{3.2}$$

For $s > \sigma(t) > t > t_2$, it is easy to verify that

$$\int_{t_2}^{s}(s-\sigma(t))z^{\Delta^2}(t)\Delta t = z(s) - z(t_2) - (s-t_2)z^{\Delta}(t_2).$$

Therefore,

$$z(s) > \int_{t_2}^{s}(s-\sigma(t))z^{\Delta^2}(t)\Delta t \tag{3.3}$$

implies that

$$\begin{aligned} z(s) &> \int_{t_2}^{s}(s-\sigma(t))\frac{(t-v)}{r(t)}L_3 z(t)\Delta t \\ &\ge L_3 z(s)\int_{v}^{s}(s-\sigma(t))\frac{(t-v)}{r(t)}\Delta t \\ &= L_3 z(s)A[s,v], \text{ for } s > v \ge t_2 \end{aligned}$$

due to (3.2). Letting $s=\beta(\theta)$ and $v=\beta(s)$, we get

$$z(\beta(\theta)) \ge L_3 z(\beta(\theta))A[\beta(\theta),\beta(s)], \tag{3.4}$$

for $\beta(\theta) > \beta(s) \ge t_2$. Using (1.1), it is easy to verify that

$$\begin{aligned} 0 &= L_4 z(t) + q(t)f(y(\beta(t))) + f(a)L_4 z(\alpha(t)) + f(a)q(\alpha(t))f(y(\beta(\alpha(t)))) \\ &\ge L_4 z(t) + f(a)L_4 z(\alpha(t)) + Q(t)[f(y(\beta(t))) + f(a)f(y(\alpha(\beta(t))))] \\ &\ge L_4 z(t) + f(a)L_4 z(\alpha(t)) + \lambda Q(t)f(z(\beta(t))) \end{aligned}$$

due to (H_1), (H_3), (H_4), and (H_5), where we have used the fact that $z(t) \leq y(t) + ay(\alpha(t))$. Using (3.4), the last inequality becomes

$$\begin{aligned} 0 &\geq L_4z(\theta) + f(a)L_4z(\alpha(\theta)) + \lambda Q(\theta)f(L_3z(\beta(\theta))A[\beta(\theta), \beta(s)]) \\ &\geq L_4z(\theta) + f(a)L_4z(\alpha(\theta)) + \lambda Q(\theta)f(L_3z(\beta(\theta)))f(A[\beta(\theta), \beta(s)]). \end{aligned}$$

Integrating the above inequality from $\alpha(s)$ to s, we obtain

$$\begin{aligned} \lambda \int_{\alpha(s)}^{s} Q(\theta)f(L_3z(\beta(\theta)))f[A(\beta(\theta), \beta(s))]\Delta\theta &\leq L_3z(\alpha(s)) + f(a)L_3z(\alpha(\alpha(s))) \\ &\leq (1 + f(a))L_3z(\alpha^2(s)), \end{aligned}$$

where we have used the fact that $\alpha^2(s)) \leq \alpha(s)$. As a result,

$$\lambda f(L_3z(\beta(s))) \int_{\alpha(s)}^{s} Q(\theta)f[A(\beta(\theta), \beta(s))]\Delta\theta \leq (1 + f(a))L_3z(\alpha^2(s)),$$

that is,

$$\int_{\alpha(s)}^{s} Q(\theta)f[A(\beta(\theta), \beta(s))]\Delta\theta \leq \frac{(1 + f(a))L_3z(\alpha^2(s))}{\lambda f(L_3z(\alpha^2(s)))} \leq \frac{(1 + f(a))}{\lambda M_1},$$

a contradiction to our hypothesis (H_7) due to (H_6).

Case (b) For $v > \sigma(t) > t > u \geq t_2$, it is easy to verify that

$$\begin{aligned} -z(v) &= -z(u) - (v - u)z^{\Delta}(v) + \int_{u}^{v} (\sigma(t) - u)z^{\Delta^2}(t)\Delta t \\ &\leq \int_{u}^{v} (\sigma(t) - u)z^{\Delta^2}(t)\Delta t. \end{aligned}$$

Following to *Case (a)* we find that $-L_2z(v) \geq (u - v)L_3z(u)$, that is, $-z^{\Delta^2}(v) \geq \frac{(u-v)}{r(v)}L_3z(u)$. Consequently,

$$\begin{aligned} z(v) &\geq \int_{u}^{v} (\sigma(t) - u)\frac{(u - t)}{r(t)}L_3z(u)\Delta t \\ &= L_3z(u)C[v, u], \text{ for } v \geq s > \sigma(t) > t > u \geq t_2. \end{aligned}$$

Letting v and u by $\beta(v)$ and $\beta(\theta)$ respectively in the last inequality, we get

$$z(\beta(v)) > L_3z(\beta(\theta))C[\beta(v), \beta(\theta)], \text{ for } \beta(v) \geq s > \sigma(t) > t > \beta(\theta) \geq t_2.$$

Proceeding as in *Case (a)*, we obtain

$$L_4z(v) + f(a)L_4z(\alpha(v)) + \lambda Q(v)[f(z(\beta(v))) \leq 0$$

and then integrating it from $\alpha(\theta)$ to θ, we get a contradiction to (H_8).

If $y(t) < 0$ for sufficiently large t on $[t_0, \infty)_{\mathbb{T}}$, then $-y(t)$ is also a solution of (1.1) due to Remark 1.1. Hence the details are omitted. This completes the proof of the theorem. □

Theorem 3.2. *Let* $0 \leq p(t) \leq a < 1$ *and* $\beta(t) \leq \alpha(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume that* (H_0)*–*(H_2), (H_6), (H_9) *and* (H_{10}) *hold. Then every solution of* (1.1) *oscillates.*

Proof. Proceeding as in the proof of Theorem 3.1, we get contradictions to (H_9) and (H_{10}). In this case, we may note that for each of *Cases (a)* and *(b)*

$$\begin{aligned}(1-p(t))z(t) &\leq z(t)-p(t)z(\alpha(t))\\ &= y(t)+p(t)y(\alpha(t))-p(t)y(\alpha(t))-p(t)p(\alpha(t))y(\alpha(\alpha(t)))\\ &= y(t)-p(t)p(\alpha(t))y(\alpha(\alpha(t)))<y(t).\end{aligned}$$

Hence, the theorem is proved. □

Theorem 3.3. *Let* $-1 \leq -b \leq p(t) \leq 0, b > 0$ *and* $\beta(t) \leq \alpha(t)$, *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If* $(H_0)-(H_2),(H_4),(H_6)$ *and* $(H_{11})-(H_{14})$ *hold, then every solution of* (1.1) *oscillates.*

Proof. Suppose on the contrary that $y(t)$ is a nonoscillatory solution of (1.1) on $[t_1,\infty)_{\mathbb{T}}$. The case $y(t) < 0$ can similarly be dealt with. In what follows, we apply Lemma 2.1, for $t \in [t_2,\infty)_{\mathbb{T}}$ with (3.1). Because $z(t)$ is monotonic, then we consider the cases when $z(t) > 0$ and $z(t) < 0$. Suppose there exists a $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $z(t) > 0$, for $t \geq t_3$. Then $z(t) \leq y(t)$, for $t \in [t_3,\infty]_{\mathbb{T}}$ and

$$L_4z(t)+q(t)f(z(\beta(t))) \leq 0. \tag{3.5}$$

Upon applying Lemma 2.1 to (3.5) and then proceeding as in the proof of Theorem 3.2, we get contradictions to (H_{11}) and (H_{12}).

Next, we suppose that $z(t) < 0$, for $t \in [t_3,\infty)_{\mathbb{T}}$. Clearly, $z(t) \geq -by(\alpha(t))$, for $t \geq t_3$ implies that there exists a $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $y(t) \geq (-\frac{1}{b})z(\alpha^{-1}(t))$, for $t \in [t_4,\infty)_{\mathbb{T}}$ due to (H_4). By Lemma 2.1, any one of *Cases (b)–(e)* holds on $[t_4,\infty)_{\mathbb{T}}$.

In each of *Cases (c)* and *(d)*, $\lim_{t\to\infty} z(t) = -\infty$. However, $z(t) < 0$ for $t \geq t_4$ implies that $y(t) < y(\tau(t))$ and hence

$$y(t) < y(\tau(t)) < y(\tau^2(t)) < \cdots < y(\tau^n(t)) < \cdots,$$

that is, $y(t)$ is bounded due to (H_{13}) and so also $z(t)$, a contradiction.

For *Case (e)*, $L_2z(t)$ is nondecreasing on $[t_3,\infty)_{\mathbb{T}}$. Therefore, there exist a constant $C > 0$ and $t_4 > t_3$ such that $tz^{\Delta^2}(t) \geq \frac{Ct}{r(t)}$, for $t \geq t_4$ and applying integration by parts formula we obtain

$$tz^{\Delta}(t) \geq t_4z^{\Delta}(t_4)+z(\sigma(t))-z(\sigma(t_4))+\int_{t_4}^{t}\frac{Cs}{r(s)}\Delta s,$$

that is, $tz^{\Delta}(t) > 0$ for large t due to bounded $z(t)$, a contradiction.

As in *Case (b)* of Theorem 3.1, we have $-z^{\Delta^2}(v) \geq \frac{(u-v)}{r(v)}L_3z(u)$ which on integration from u to v, it follows that

$$z^{\Delta}(u) \geq L_3z(u)\int_u^v \frac{(u-t)}{r(t)}\Delta t,$$

that is,

$$-z(u) \geq -z(\sigma(u)) + (\sigma(u) - u)L_3 z(u) \int_u^v \frac{(u-t)}{r(t)} \Delta t$$
$$= L_3 z(u) B[v, u] \geq L_3 z(v) B[v, u].$$

Therefore,

$$-z(\alpha^{-1}(\beta(u))) \geq L_3 z(\alpha^{-1}(\beta(v))) B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))]. \tag{3.6}$$

Since, (1.1) can be viewed as

$$L_4 z(u) + q(u) f(-\frac{1}{b}) f(z(\alpha^{-1}(\beta(u)))) \leq 0, \tag{3.7}$$

then using (3.6) and (H_1), (3.7) yields

$$L_4 z(u) + q(u) f(\frac{1}{b}) f(L_3 z(\alpha^{-1}(\beta(v)))) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))]) \leq 0.$$

Integrating the last inequality from $\alpha^{-1}(\beta(v))$ to $\alpha^{-1}(v)$, it follows that

$$f\left(\frac{1}{b}\right) f(L_3 z(\alpha^{-1}(\beta(v)))) \int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))]) \Delta u$$
$$\leq L_3 z(\alpha^{-1}(\beta(v)))).$$

Consequently,

$$\int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))]) \Delta u \leq \frac{1}{M_1 f(\frac{1}{b})}$$

due to (H_6), a contradiction to our hypothesis (H_{14}). This completes the proof of the theorem. □

Theorem 3.4. *Let* $-\infty < -b \leq p(t) \leq -1, b > 0$ *and* $\beta(t) \leq \alpha(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $b > 0$. *If* (H_0)*–*(H_2), (H_4), (H_6), (H_{11}), (H_{12}), *and* (H_{14})*–*(H_{16}) *hold, then* (1.1) *is oscillatory.*

Proof. The proof of the theorem follows from the proof of Theorem 3.3. We consider *Cases (c)* and *(d)* of Lemma 2.1 only when $z(t) < 0$, for $t \in [t_3, \infty)_{\mathbb{T}}$, that is, there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $y(t) \geq (-\frac{1}{b}) z(\alpha^{-1}(t))$, for $t \in [t_4, \infty)_{\mathbb{T}}$ due to (H_4) and hence we have obtained (3.7). In *Case (c)*, $z(t)$ is nonincreasing. So, we can find $t_5 > t_4$ and $L > 0$ such that $z(t) \leq -L$, for $t \geq t_5$. Using (H_1) and therefore, (3.7) yields

$$L_4 z(t) + f(\frac{1}{b}) f(L) q(t) \leq 0, \quad t \geq t_5.$$

Integrating the above inequality from t_5 to ∞, we obtain a contradiction to (H_{15}).

Assume that *Case (d)* of Lemma 2.1 holds. Proceeding as in *Case (a)* of Theorem 3.1, we obtain

$$z^{\Delta^2}(u) \le \frac{(u-v)}{r(u)} L_3 z(v), \tag{3.8}$$

for $u > v > t_4$. For $s > \sigma(t) > t > t_4$, it is easy to verify that

$$z(s) = z(t_4) + (s - t_4) z^{\Delta}(t_2) + \int_{t_4}^{s} (s - \sigma(t)) z^{\Delta^2}(t) \Delta t.$$

Therefore, for $s > v \ge t_4$

$$\begin{aligned} z(s) &\le \int_{t_4}^{s} (s - \sigma(t)) z^{\Delta^2}(t) \Delta t \\ &\le \int_{t_4}^{s} (s - \sigma(t)) \frac{(t-v)}{r(t)} L_3 z(v) \Delta t \\ &\le L_3 z(v) \int_{v}^{s} (s - \sigma(t)) \frac{(t-v)}{r(t)} \Delta t = A[s, v] L_3 z(v) \end{aligned}$$

due to (3.8). Consequently,

$$z(\alpha^{-1}(\beta(\theta))) \le L_3 z(\alpha^{-1}(\beta(s))) A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]. \tag{3.9}$$

Using (3.9) in (3.7), it follows that

$$L_4 z(\theta) + q(\theta) f\left(-\frac{1}{b}\right) f(L_3 z(\alpha^{-1}(\beta(s)))) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]) \le 0$$

due to (H_1). Integrating the last inequality from $\beta(s)$ to $\alpha^{-1}(\beta(s))$, we obtain that

$$\begin{aligned} f\left(\frac{1}{b}\right) f(-L_3 z(\alpha^{-1}(\beta(s)))) \int_{\beta(s)}^{\alpha^{-1}(\beta(s))} & q(\theta) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]) \Delta\theta \\ & \le -L_3 z(\alpha^{-1}(\beta(s))), \end{aligned}$$

that is,

$$\int_{\beta(s)}^{\alpha^{-1}(\beta(s))} q(\theta) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]) \Delta\theta \le \frac{-L_3 z(\alpha^{-1}(\beta(s)))}{f(\frac{1}{b}) f(-L_3 z(\alpha^{-1}(\beta(s))))} \le \frac{1}{M_1 f(\frac{1}{b})},$$

a contradiction to (H_{16}). This completes the proof of the theorem. □

Theorem 3.5. *Let* $0 \le p(t) \le a < \infty$ *and* $\beta(t) \le \alpha^2(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* (H_0)–(H_6) *and* (H_{17}) *hold, then* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we consider *Cases (a)* and *(b)* of Lemma 2.1. For both the cases,

$$L_4 z(t) + f(a) L_4 z(\alpha(t)) + \lambda Q(t) f(z(\beta(t))) \le 0$$

holds true. To the last inequality, we apply Lemma 2.2 and therefore,

$$L_4 z(t) + f(a) L_4 z(\alpha(t)) + \lambda Q(t) f(R_T(\beta(t))) f(L_3 z(\beta(t))) \le 0 \tag{3.10}$$

due to (H_1). Integrating (3.10) from $\alpha(s)$ to s and using the same type of reasoning as in Theorem 3.1, we get a contradiction to (H_{17}). Hence the theorem is proved. □

Theorem 3.6. *Let $-1 \le -b \le p(t) \le 0, b > 0$ and $\beta(t) \le \alpha(t)$, for $t \in [t_0, \infty)_{\mathbb{T}}$. If (H_0)–(H_2), (H_4), (H_6), (H_{13}), (H_{18}) and (H_{19}) hold, then every solution of* (1.1) *oscillates.*

Proof. On the contrary, we proceed as in Theorem 3.3 to obtain (3.5), for $t \ge t_3$. The rest of this case follows from the proof of Theorem 3.5.

When $z(t) < 0$, for $t \ge t_3$, we consider *Case (b)* of Lemma 2.1 only. Using (H_4) in (3.7), it follows that

$$\begin{aligned} f(-\frac{1}{b})q(t)f(z(\beta(\alpha^{-1}(t)))) &\le -L_3^{\Delta}z(t) \\ &= \frac{-L_3z(\sigma(t)) + L_3z(t)}{\sigma(t)-t} \\ &\le \frac{L_3z(t)}{(\sigma(t)-t)} = \frac{L_2^{\Delta}z(t)}{(\sigma(t)-t)} \\ &\le \frac{-L_2z(t)}{(\sigma(t)-t)^2}, \end{aligned}$$

for $t \ge t_4 > t_3$. Consequently,

$$f\left(-\frac{1}{b}\right)\frac{(\sigma(t)-t)^2q(t)}{r(t)}f(z(\beta(\alpha^{-1}(t)))) \le -z^{\Delta^2}(t) \le \frac{z^{\Delta}(t)}{(\sigma(t)-t)}$$

implies that

$$z^{\Delta}(t) + f\left(\frac{1}{b}\right)\frac{(\sigma(t)-t)^3q(t)}{r(t)}f(z(\beta(\alpha^{-1}(t)))) \ge 0,$$

and because of (H_6), the above inequality reduces to

$$z^{\Delta}(t) + M_1 f\left(\frac{1}{b}\right)\frac{(\sigma(t)-t)^3q(t)}{r(t)}z(\beta(\alpha^{-1}(t))) \ge 0 \qquad (3.11)$$

which in turn concludes that (3.11) can not have an eventually negative solution (because of Lemma 2.3) due to (H_{19}), a contradiction. The rest of the proof follows from the proof of Theorem 3.3. This completes the proof of the theorem. □

Theorem 3.7. *Let $-\infty \le -b \le p(t) \le -1, b > 0$ and $\beta(t) \le \alpha(t)$, for $t \in [t_0, \infty)_{\mathbb{T}}$. If (H_0)–(H_2), (H_4), (H_6), (H_{15}), (H_{18}) and (H_{19}) hold, then every bounded solution of* (1.1) *oscillates.*

Proof. If possible, let $y(t)$ be a bounded nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Clearly, $z(t)$ is bounded. The rest of the proof follows from the proof of Theorems 3.4 and 3.6 and hence the details are omitted. Thus the proof of the theorem is complete. □

4. DISCUSSION and EXAMPLES

Often, it is more challenging to study an all solution oscillatory problem (linear/nonlinear) than a problem (linear/nonlinear) dealing with asymptotic solutions. The later problem may get usual procedure to study than the former one. Eventhough, (1.1) is highly nonlinear, still all our results are hold true for linear, sublinear and as well as superlinear.

This work deserves a different approach to that of [13] as long as oscillation results are concerned. However, existence of nonoscillation results we take into account. It would be interesting to work out the results of this work for (1.2) and (1.3) respectively. In the following examples, we illustrate our main result:

Example 4.1. Let $\mathbb{T} = \mathbb{Z}$. Consider

$$\triangle^2 \left(\frac{n}{2} \triangle^2 \left(y(n) + \frac{1}{3}(1 + (-1)^n) y(n-2) \right) \right) + 8(n+1) y^3(n-5) = 0, \qquad (4.1)$$

where $0 \leq p(n) = \frac{1}{3}(1 + (-1)^n) \leq \frac{2}{3}, r(n) = \frac{n}{2}$ and $G(u) = u^3$. Clearly, all the conditions of Theorem 3.2 are satisfied. Hence (4.1) is oscillatory. Indeed, $y(n) = (-1)^n$ is one of the oscillatory solutions of (4.1).

Example 4.2. On $\mathbb{T} = \mathbb{R}$, consider

$$(y(t) + 2y(t - \pi))'''' + y(t - 4\pi) = 0, \qquad (4.2)$$

where $r(t) = 1, p(t) = 2, \alpha(t) = t - \pi, \beta(t) = t - 3\pi, q(t) = 1$. Clearly, all the conditions of Theorem 3.2 are satisfied for (4.2) when $\mathbb{T} = \mathbb{R}$. Hence, (4.2) is oscillatory. Indeed, $y(t) = sint$ is an oscillatory solution of (4.2).

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