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POSITIVE SOLUTION FOR A SECOND ORDER BVP WITH SINGULAR SIGN-CHANGING NONLINEARITY

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ABSTRACT. We discuss existence of at least one positive solution to the singular second-order two point boundary value problem,

$$\begin{cases} -\left(pu'\right)'(t) = f(t, u(t)), \;\; t \in (0,1), \\ au(0) - b\lim_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim_{t\to 1} p(t)u'(t) = 0, \end{cases}$$

where $a, b, c, d \in [0, +\infty)$, $p : (0, 1) \to [0, +\infty)$ is a measurable function, and $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

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1. INTRODUCTION

The theory of boundary value problems associated with differential equations has become an important branch of applied mathematics. This importance is due essentially to the fact that many physical and biological phenomenas are modeled by such problems. Many hundreds of papers where several classes of boundary value problems have been investigated, have appeared during the last three decades. In particular, study of existence and multiplicity of positive solutions for boundary value problems associated with differential equations have received a great interest; see [1], [3], [4], [5], [7]–[11], [13]–[16] and references therein.

Motivated by the works in [2], [7], [12], [13] and [16], we will investigate in this work a second order boundary value problem with a Carathéodory sign-changing nonlinearity which may have singularities. So, this paper deals with existence of at least one positive solution to the second-order nonlinear boundary value problem (bvp

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for short),

$$\begin{cases} -\left(pu'\right)'(t) = f(t, u(t)), \text{ a.e. } t \in (0,1), \\ au(0) - b\lim_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim_{t\to 1} p(t)u'(t) = 0, \end{cases} \tag{1.1}$$

where $a, b, c, d \in \mathbb{R}^+ := [0, +\infty)$, $p : (0,1) \to \mathbb{R}^+$ is a measurable function, and $f : (0,1) \times (0,+\infty) \to \mathbb{R}$ is a Carathéodory function which may change its sign.

We recall that a function $h : (0,1) \times I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$, is said to to be a Carathéodory function if

- $h(\cdot, u)$ is a measurable function for all $u \in I$, and
- $h(t, \cdot)$ is continuous for a.e. $t \in (0,1)$.

Throughout, we assume that

$$\int_0^1 \frac{d\tau}{p(\tau)} < \infty, \tag{1.2}$$

$$\Delta = ad + ac\int_0^1 \frac{d\tau}{p(\tau)} + bc > 0. \tag{1.3}$$

In all this paper, G denotes the Green's function associated with the bvp

$$\begin{cases} -\left(pu'\right)'(t) = 0, \text{ a.e. } t \in (0,1), \\ au(0) - b\lim_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim_{t\to 1} p(t)u'(t) = 0. \end{cases}$$

We have

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \Phi_{ab}(s)\Psi_{cd}(t), & 0 \le s \le t \le 1, \\ \Phi_{ab}(t)\Psi_{cd}(s), & 0 \le t \le s \le 1, \end{cases}$$

where

$$\Phi_{ab}(x) = b + a\int_0^x \frac{d\tau}{p(\tau)} \text{ and } \Psi_{cd}(x) = d + c\int_x^1 \frac{d\tau}{p(\tau)},$$

are well defined on $[0,1]$.

Also, throughout, we let

$$L_G^1 = \left\{ q : (0,1) \to \mathbb{R} \ \text{ measurable}, \ \int_0^1 G(t,t)\left|q(t)\right| dt < +\infty \right\}$$

equipped with the norm $|\cdot|_G$, defined for $q \in L_G^1$ by

$$|q|_G = \int_0^1 G(t,t)\left|q(t)\right| dt$$

and

$$K_G = \left\{ q \in L_G^1 \ \text{ such that } q(t) \ge 0 \text{ for a.e. } t \in (0,1) \right\}.$$

A function $h : (0,1) \times I \longrightarrow \mathbb{R}$ where I is an interval of $\mathbb{R}$, is said to be an L_G^1-Carathéodory function, if

- h is a Carathéodory function, and

- for each compact $Q \subset (0,+\infty)$ there exists $\psi_Q \in K_G$ such that $|h(t,u)| \leq \psi_Q(t)$ for a.e. $t \in (0,1)$ and for all $u \in Q$.

Throughout this paper, we assume that the function f satisfies the following hypotheses:

$$f \text{ is an } L^1_G\text{-Carathéodory function on } (0,1) \times (0,+\infty), \tag{1.4}$$

and there exists a function $q \in K_G \cap L^1[0,1]$ such that

$$f(t,u) + q(t) \geq 0, \text{ for all } u > 0 \text{ and a.e. } t \in (0,1). \tag{1.5}$$

Note that under the above hypotheses, the nonlinearity f may be singular at $u = 0$, that is, there may exist $t_0 \in (0,1)$ such that $\lim_{u\to 0} f(t_0,u) = +\infty$. Note also, that the hypotheses assumed here on the weight p and the nonlinearity f are less restrictive than those imposed in [7], [12], [13] and [16].

The paper is organized as follows. Section 2 deals with preliminaries, and in Section 3, we present our main results and their proofs. We end the paper with Section 4, where we provide illustrative examples. The main tool in this paper is Krasnoselskii's theorem of expansion and compression of a cone in a Banach space.

2. PRELIMINARIES AND AUXILIARY LEMMAS

The main results of this paper are obatained by means of the following Krasnoselskii's fixed point theorem.

Theorem 2.1 ([6])**.** *Let X be a real Banch space, K a cone of X and let Ω_1, Ω_2 be open bounded subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. If $T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ is a completely countinuous operator such that, either*

1. *$\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or*
2. *$\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Now, let us introduce some spaces and operators needed for proofs of the main results. Hereafter, in this paper, E is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ equipped with the norm

$$\|u\| = \sup\{|u(t)|,\ 0 \leq t \leq 1\},$$

K and P are the cones of E given by

$$K = \{u \in E : u(t) \geq 0 \text{ for all } t \in [0,1]\},$$

$$P = \{u \in E : u(t) \geq \rho(t)\|u\| \text{ for all } t \in [0,1]\}, \tag{2.1}$$

where for $t \in [0,1]$,

$$\rho(t) = \begin{cases} \dfrac{1}{\Delta} \min\left(c\Phi_{ab}(t), a\Psi_{cd}(t)\right), \text{ if } ac \neq 0, \\ \dfrac{\Psi_{cd}(t)}{\Psi_{cd}(0)}, \text{ if } a = 0, \\ \dfrac{\Phi_{ab}(t)}{\Phi_{ab}(1)}, \text{ if } c = 0. \end{cases}$$

Note that Hypothesis (1.3) implies that the situation $a = c = 0$ is not possible and also makes $\rho > 0$ on $(0,1)$.

In all this paper, we write $u > v$ for $u, v \in E$ if $u(t) > v(t)$ for all $t \in (0,1)$.

It is well known that the Green's function G satisfies

$$G(t,s) \leq G(s,s), \text{ for } 0 \leq t, s \leq 1, \tag{2.2}$$

Therefore, for all $u \in L^1_G$,

$$\left| \int_0^1 G(t,s)\, u(s)\, ds \right| \leq \int_0^1 G(s,s) \left| u(s) \right| ds < \infty.$$

Lemma 2.2. *Assume that* (1.2) *and* (1.3) *hold. Then the operator* $\pounds : L^1_G \to E$ *defined for* $u \in L^1_G$ *by*

$$\pounds u(t) = \int_0^1 G(t,s)\, u(s)\, ds$$

is linear, continuous, and maps K_G *into* P.

Proof. Let, for $n = 3, 4, \ldots$, and $u \in L^1_G$,

$$\pounds_n u(t) = \int_0^1 G(t,s)\, \gamma_n(s)\, u(s)\, ds,$$

where

$$\gamma_n(s) = \begin{cases} 0 \text{ if } s \in (0, 1/n) \cup (1 - 1/n, 1), \\ 1 \text{ if } s \in (1/n, 1 - 1/n). \end{cases} \tag{2.3}$$

Since $L^1_G \subset L^1_{loc}(0,1)$, we have that $\gamma_n u \in L^1[0,1]$. Hence, $\pounds_n u \in E$ for all $u \in L^1_G$ and $\pounds_n$ maps L^1_G into E.

Clearly, we have for all $u \in L^1_G$

$$\|\pounds_n u\| \leq \int_0^1 G(s,s)\, \gamma_n(s) \left| u(s) \right| ds = |u|_G,$$

and $\pounds_n : L^1_G \to E$ is a linear continuous operator.

Now, since for $u \in L^1_G$, $\pounds_n u \in E$ and

$$\sup_{t \in [0,1]} \left| \pounds u(t) - \pounds_n u(t) \right| \leq \int_0^1 G(s,s) \left| u(s) \right| (1 - \gamma_n(s))\, ds,$$

we obtain from the Lebesgue dominated convergence theorem that $\lim \|\pounds u - \pounds_n u\| = 0$. Therefore, $\pounds u \in E$ for all $u \in L^1_G$ and $\pounds : L^1_G \to E$ is well defined.

Also, we have for all $u \in L^1_G$

$$\|\pounds u\| = \sup_{t\in[0,1]} |\pounds u(t)| \leq \int_0^1 G(s,s)|u(s)|\,ds = |u|_G .$$

This shows that $\pounds : L^1_G \to E$ is a linear continuous operator.

At the end, the proof of $T(K_G) \subset P$ is similar to that of Lemma 2.8 in [2]. □

It is easy to prove the following Lemma.

Lemma 2.3. *Assume that* (1.2) *and* (1.3) *hold. Then for all* $u \in L^1_G$, $\pounds u$ *is the unique solution of*

$$\begin{cases} -(pv')'(t) = u(t), \ a.e.\ t \in (0,1), \\ av(0) - b\lim_{t\to 0} p(t)v'(t) = 0, \\ cv(1) + d\lim_{t\to 1} p(t)v'(t) = 0. \end{cases}$$

Following Lemma 2.3, $\pounds q$ is the unique solution of the bvp

$$\begin{cases} -(pu')'(t) = q(t), \ \text{a.e. } t \in (0,1), \\ au(0) - b\lim_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim_{t\to 1} p(t)u'(t) = 0. \end{cases} \tag{2.4}$$

Throughout this paper, we denote $\phi = \pounds q$, and we have that

$$\begin{aligned} \phi(t) = \textstyle\int_0^1 G(t,s)q(s)ds = \frac{1}{\Delta}\int_0^t \Phi_{ab}(s)\Psi_{cd}(t)q(s)ds \\ +\frac{1}{\Delta}\textstyle\int_t^1 \Phi_{ab}(t)\Psi_{cd}(s)q(s)ds. \end{aligned} \tag{2.5}$$

Lemma 2.4. *Assume that* (1.2) *and* (1.3) *hold, then* ϕ *satisfies the following upper bound:*

$$\phi(t) \leq \phi^*\rho(t) \ \ \textit{for all } t \in [0,1],$$

where

$$\phi^* = \begin{cases} \max\left(\frac{1}{c}\int_0^1 \Psi_{cd}(s)q(s)ds, \frac{1}{a}\int_0^1 \Phi_{ab}(s)q(s)ds\right) \ \textit{if } ac \neq 0, \\ \frac{\Phi_{ab}(1)}{\Delta}\int_0^1 \Psi_{cd}(s)q(s)ds \ \textit{if } c = 0, \\ \frac{\Psi_{cd}(0)}{\Delta}\int_0^1 \Phi_{ab}(s)q(s)ds \ \textit{if } a = 0. \end{cases}$$

Proof. The proof is based on the fact that the functions Φ_{ab} and Ψ_{cd} are, respectively, increasing and decreasing on $[0,1]$. We distinguish the following cases:

- $a = 0$: In this case we have $\rho(t) = \frac{\Psi_{cd}(t)}{\Psi_{cd}(0)}$ and $c\Phi_{ab}(t) = bc > 0$. Thus, we have from (2.5),

$$\begin{aligned}\phi(t) &\leq \frac{1}{\Delta}\Psi_{cd}(t)\int_0^t bq(s)ds + \frac{1}{\Delta}\Psi_{cd}(t)\int_t^1 bq(s)ds \\ &= \frac{1}{\Delta}\Psi_{cd}(t)\int_0^t \Phi_{ab}(s)q(s)ds + \frac{1}{\Delta}\Psi_{cd}(t)\int_t^1 \Phi_{ab}(s)q(s)ds \\ &= \frac{\Psi_{cd}(0)}{\Delta}\rho(t)\int_0^1 \Phi_{ab}(s)q(s)ds = \phi^*\rho(t).\end{aligned}$$

- $c = 0$: In this case we have $\rho(t) = \frac{\Phi_{ab}(t)}{\Phi_{ab}(1)}$ and $a\Psi_{cd}(t) = ad > 0$. Thus, we have from (2.5),

$$\begin{aligned}\phi(t) &\leq \frac{1}{\Delta}\Phi_{ab}(t)\int_0^t dq(s)ds + \frac{1}{\Delta}\Phi_{ab}(t)\int_t^1 dq(s)ds \\ &= \frac{1}{\Delta}\Phi_{ab}(t)\int_0^t \Psi_{cd}(s)q(s)ds + \frac{1}{\Delta}\Phi_{ab}(t)\int_t^1 \Psi_{cd}(s)q(s)ds \\ &= \frac{\Phi_{ab}(1)}{\Delta}\rho(t)\int_0^1 \Psi_{cd}(s)q(s)ds = \phi^*\rho(t).\end{aligned}$$

- $ac \neq 0$: In this case, and because the function $c\Phi_{ab} - a\Psi_{cd}$ is increasing on $[0,1]$, we distinguish the following three possibilities.

  i) $c\Phi_{ab}(t) \leq a\Psi_{cd}(t)$ for all $t \in [0,1]$. In this subcase $\rho(t) = \frac{c}{\Delta}\Phi_{ab}(t)$. Thus, we have from (2.5),

$$\begin{aligned}\phi(t) &\leq \frac{1}{\Delta}\Phi_{ab}(t)\int_0^t \Psi_{cd}(s)q(s)ds + \frac{1}{\Delta}\Phi_{ab}(t)\int_t^1 \Psi_{cd}(s)q(s)ds \\ &= \frac{1}{\Delta}\Phi_{ab}(t)\int_0^1 \Psi_{cd}(s)q(s)ds = \phi^*\rho(t).\end{aligned}$$

  ii) $a\Psi_{cd}(t) \leq c\Phi_{ab}(t)$ for all $t \in [0,1]$. In this subcase $\rho(t) = \frac{a}{\Delta}\Psi_{cd}(t)$. Thus, we have from (2.5),

$$\begin{aligned}\phi(t) &\leq \frac{1}{\Delta}\Psi_{cd}(t)\int_0^t \Phi_{ab}(s)q(s)ds + \frac{1}{\Delta}\Psi_{cd}(t)\int_t^1 \Phi_{ab}(s)q(s)ds \\ &= \frac{1}{\Delta}\Psi_{cd}(t)\int_0^1 \Phi_{ab}(s)q(s)ds = \phi^*\rho(t).\end{aligned}$$

  iii) There exists a unique $t^* \in (0,1)$ such that $c\Phi_{ab}(t) < a\Psi_{cd}(t)$ for all $t \in (0,t^*)$ and $c\Phi_{ab}(t) > a\Psi_{cd}(t)$ for all $t \in (t^*,1)$. In this subcase

$$\rho(t) = \begin{cases} \frac{c}{\Delta}\Phi_{ab}(t) & \text{if } t \in [0,t^*], \\ \frac{a}{\Delta}\Psi_{cd}(t) & \text{if } t \in [t^*,1]. \end{cases}$$

Thus, we have from (2.5), if $t \in [0, t^*]$,

$$\begin{aligned} \phi(t) &\leq \frac{1}{\Delta}\Phi_{ab}(t)\int_0^t \Psi_{cd}(s)q(s)ds + \frac{1}{\Delta}\Phi_{ab}(t)\int_t^1 \Psi_{cd}(s)q(s)ds \\ &= \frac{1}{c}\left(\frac{c}{\Delta}\Phi_{ab}(t)\right)\int_0^1 \Psi_{cd}(s)q(s)ds \leq \phi^*\rho(t), \end{aligned}$$

and if $t \in [t^*, 1]$,

$$\begin{aligned} \phi(t) &\leq \frac{1}{\Delta}\Psi_{cd}(t)\int_0^t \Phi_{ab}(s)q(s)ds + \frac{1}{\Delta}\Psi_{cd}(t)\int_t^1 \Phi_{ab}(s)q(s)ds \\ &= \frac{1}{a}\left(\frac{a}{\Delta}\Psi_{cd}(t)\right)\int_0^1 \Phi_{ab}(s)q(s)ds \leq \phi^*\rho(t). \end{aligned}$$

Therefore, for all $t \in [0, 1]$,

$$\phi(t) \leq \phi^*\rho(t).$$

The proof is complete. □

Let r, R be two real numbers such that $R > r > \phi^*$. We have from the definition of the cone P and Lemma 2.4 that, for all $v \in P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$,

$$v(t) - \phi(t) \geq (\|v\| - \phi^*)\rho(t) \geq (r - \phi^*)\rho(t) > 0.$$

Therefore, for all $v \in P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$ the expression

$$F_{r,R,\phi}v(t) = f(t, v(t) - \phi(t)) + q(t) \tag{2.6}$$

is defined for all $t \in (0, 1)$.

Lemma 2.5. *Assume that* (1.4) *and* (1.5) *hold and*

$$f(t,\cdot) \text{ is continuous at } u = 0 \text{ a.e. } t \in (0,1) \text{ and } f(\cdot, 0) \in L^1_G. \tag{2.7}$$

Then for all $r, R \in \mathbb{R}$ *with* $R > r > \phi^*$ *and all* $v \in P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$, $F_{r,R,\phi}v \in K_G$ *and the operator* F_ϕ *defined by expression* (2.6) *maps* $P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$ *into* K_G. *Moreover if* (1.2), (1.3) *hold, then the operator*

$$T_{r,R,\phi} = \pounds F_{r,R,\phi} : P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right) \to P$$

is compact and for any fixed point v *of* $T_{r,R,\phi}$, *satisfying* $v > \phi$, $u = v - \phi$ *is a positive solution of bvp* (1.1).

Proof. Fix $R, r \in \mathbb{R}$ with $R > r > \phi^*$ and set $\Omega = P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$. Note that Hypotheses (1.4) and (2.7) imply that the nonlinearity f is an L^1_G-Carathéodory function, and there exists $\psi_R \in K_G$ such that

$$|f(t,z)| \leq \psi_R(t) \text{ for all } z \in [0, R] \text{ and a.e. } t \in (0,1). \tag{2.8}$$

Since, for all $v \in \Omega$,

$$0 \leq (r - \phi^*)\rho(t) \leq v(t) - \phi(t) \leq R,$$

we have that

$$0 \leq F_{r,R,\phi}v(t) = f(t, v(t) - \phi(t)) + q(t) \leq \psi_R(t) + q(t) \text{ a.e. } t \in (0,1),$$

and this shows that $F_{r,R,\phi}v \in K_G$.

Now, if Hypotheses (1.2) and (1.3) hold, then Lemma 2.2 guarantees that for $v \in \Omega$, $\pounds F_{r,R,\phi}v \in P$ and the operator $T_{r,R,\phi} = \pounds F_{r,R,\phi} : \Omega \to P$ is well defined.

Consider the linear continuous operator $\pounds_1 : L^1[0,1] \to C^1[0,1]$ defined for $u \in L^1[0,1]$ by

$$\pounds_1 u(t) = \int_0^1 G(t,s)\, u(s)\, ds,$$

and let, for $n = 3, 4, \ldots$ and $u \in \Omega$, $F_n u$ be defined by

$$F_n u(t) = \gamma_n(t)\, F_{r,R,\phi}u(t),$$

where γ_n is the function defined by (2.3). Since $L^1_G \subset L^1_{loc}(0,1)$, the mapping $F_n : \Omega \to K_G \cap L^1[0,1]$ is well defined. Furthermore, since for all $u, v \in \Omega$

$$\begin{aligned} \int_0^1 |F_n u(t) - F_n v(t)|\, dt &= \int_{\frac{1}{n}}^{1-\frac{1}{n}} |F_{r,R,\phi}u(t) - F_{r,R,\phi}v(t)|\, dt \\ &= \int_{\frac{1}{n}}^{1-\frac{1}{n}} |f(t, u(t) - \phi(t)) - f(t, v(t) - \phi(t))|\, dt, \end{aligned}$$

we obtain from (2.8) and the Lebesgue dominated convergence theorem that F_n is a continuous and bounded mapping.

Now, let $T_n = i \circ \pounds_1 \circ F_n$ where i is the compact embedding of $C^1[0,1]$ into E. Clearly, T_n is a compact mapping.

We have from (2.2), (1.4) and (2.7) that, for all $u \in \Omega$,

$$\begin{aligned} |T_{r,R,\phi}u(t) - T_n u(t)| &\leq \int_0^1 G(s,s)(1 - \gamma_n(s))\, |f(s, u(s) - \phi(s)) + q(s)|\, ds \\ &\leq \int_0^1 G(s,s)(1 - \gamma_n(s))(\psi_R(s) + q(s))\, ds. \end{aligned}$$

Thus, we obtain by means of the Lebesgue dominated convergence theorem that $\|Tu - T_n u\|_{C_b(\Omega,E)} \to 0$ as $n \to \infty$ and this shows that $T \in C_b(\Omega, E)$ and T is compact. Here $C_b(\Omega, E)$ is the Banach space of all continuous bounded maps from Ω into E equipped with the sup-norm.

Finally, if v is a fixed point of $T_{r,R,\phi}$ with $v > \phi$, then $u = v - \phi$ is positive and satisfies $u + \phi = \pounds F_{r,R,\phi}(u + \phi)$. That is,

$$\begin{cases} -(pu')'(t) - (p\phi')'(t) = f(t, u(t)) + q(t), \text{ a.e. } t \in (0,1), \\ a(u+\phi)(0) - b\lim_{t\to 0} p(t)(u+\phi)'(t) = 0, \\ c(u+\phi)(1) + d\lim_{t\to 1} p(t)(u+\phi)'(t) = 0. \end{cases}$$

Taking into consideration that $\phi = \pounds q$ is the unique solution of (2.4), we obtain that $u = v - \phi$ is a positive solution of bvp (1.1). □

Lemma 2.6. *Assume that* (1.4) *and* (1.5) *hold and there exist functions* $m_1, m_2 \in K_G$*, a continuous decreasing function* $g : (0, +\infty) \to (0, +\infty)$*, a continuous increasing function* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $r_0 > \phi^*$ *such that*

$$\begin{cases} |f(t,u)| \leq m_1(t)\, g(u) + m_2(t)\, h(u) \text{ and all } u > 0 \text{ and a.e. } t \in (0,1), \text{ and} \\ \int_0^1 G(s,s)\, m_1(s)\, g(\rho(s)(r_0 - \phi^*))\, ds < \infty. \end{cases}$$

Then for all $R > r_0$ *and all* $v \in P \cap \left(\overline{B}(0,R) \backslash B(0,r_0)\right)$*,* $F_\phi v \in K_G$ *and the operator* $F_{r_0,R,\phi}$ *defined by expression* (2.6) *maps* $P \cap \left(\overline{B}(0,R) \backslash B(0,r_0)\right)$ *into* K_G*.*

Moreover if (1.2)*,* (1.3) *hold, then the operator*

$$T_{r_0,R,\phi} = \pounds F_{r_0,R,\phi} : P \cap \left(\overline{B}(0,R) \backslash B(0,r_0)\right) \to P$$

is compact and for any fixed point v *of* $T_{r_0,R,\phi}$*, satisfying* $v > \phi$*,* $u = v - \phi$ *is a positive solution of bvp* (1.1)*.*

Proof. Fix $R > r_0$ and set $\Omega = P \cap \left(\overline{B}(0,R) \backslash B(0,r)\right)$. Since, for all $v \in \Omega$,

$$0 \leq (r_0 - \phi^*)\, \rho(t) \leq v(t) - \phi(t) \leq R,$$

we have that

$$\begin{aligned} 0 &\leq F_{r_0,R,\phi} v(t) = f(t, v(t) - \phi(t)) + q(t) \\ &\leq m_1(t)\, g((r_0 - \phi^*)\, \rho(t)) + m_2(t)\, h(R) + q(t) \text{ a.e. } t \in (0,1), \end{aligned}$$

from which it follows that

$$\begin{aligned} &\int_0^1 G(s,s)\, F_{r_0,R,\phi} v(s)\, ds \leq \int_0^1 G(s,s)\, m_1(s)\, g(\rho(s)(r_0 - \phi^*))\, ds \\ &+ \int_0^1 G(s,s)\,(h(R)\, m_2(s) + q(s)) < \infty. \end{aligned}$$

All the above estimates show that $F_{r_0,R,\phi} v \in K_G$.

As in the proof Lemma 2.5, if Hypotheses (1.2) and (1.3) hold, then the operator $T_{r_0,R,\phi} = \pounds F_{r_0,R,\phi} : \Omega \to P$ is well defined

Now, let F_n be as defined in proof of Lemma 2.5. We have, for all $v \in \Omega$,

$$v(s) - \phi(s) \geq (r_0 - \phi^*)\, \rho(s) > 0 \text{ for all } s \in [0,1],$$

and this, together with Hypothesis (1.4), leads to

$$\lim_{n \to \infty} \gamma_n(s)\, f(s, v(s) - \phi(s)) = f(s, v(s) - \phi(s)) \text{ a.e. } s \in (0,1).$$

Also, we have for all $v \in \Omega$,

$$\begin{aligned}\|T_{r_0,R,\phi}v - T_n v\| &= \sup_{t\in[0,1]} |T_{r_0,R,\phi}v(t) - T_n v(t)| \\ &\leq \int_0^1 G(s,s)(1-\gamma_n(s))|f(s,v(s)-\phi(s)) + q(s)|\,ds \\ &\leq \int_0^1 G(s,s)(1-\gamma_n(s))(m_1(s)\,g((r_0-\phi^*)\rho(s)) \\ &\quad + h(R)\,m_2(s) + q(s))\,ds.\end{aligned}$$

Thus, we obtain by means of the Lebesgue dominated convergence theorem that $\|Tu - T_n u\|_{C_b(\Omega,E)} \to 0$ as $n \to \infty$. As in the proof of Lemma 2.5, if v is a fixed point of $T_{r_0,R,\phi}$, with $v > \phi$, then $u = v - \phi$ is a positive solution of bvp (1.1). This ends the proof. □

3. MAIN RESULTS

3.1. The regular case.

Theorem 3.1. *Suppose that Hypotheses* (1.2), (1.3), (1.4), (1.5) *and* (2.7) *hold and*

(a): *there exist a function $\alpha \in K_G$ and $R_1 > \max(\phi^*, \|\pounds\alpha\|)$ such that*

$$f(t,u) + q(t) \leq \alpha(t)$$

for a.e. $t \in (0,1)$ and all $u \in [0,R_1]$,

(b): *there exist $\sigma \in \left(0,\frac{1}{2}\right)$, a function $\beta \in K_G$ and a constant R_2, with $R_2 \neq R_1$, such that*

$$\phi^* < R_2 \leq \max_{t\in[0,1]} \int_\sigma^{1-\sigma} G(t,s)\,\beta(s)\,ds,$$

$$f(t,u) + q(t) \geq \beta(t),$$

for a.e. $t \in [\sigma, 1-\sigma]$ and all $u \in [\rho_\sigma(R_2-\phi^*), R_2]$, where $\rho_\sigma = \min_{s\in[\sigma,1-\sigma]}\rho(s)$.

Then, bvp (1.1) *has at least one positive solution.*

Proof. Let $T_{R_1,R_2,\phi}$ be the operator defined in Lemma 2.5, where R_1 and R_2 are those in Theorem 3.1. We have for all $v \in P \cap \partial B(0,R_1)$ and $t \in [0,1]$

$$0 \leq (R_1 - \phi^*)\,\rho(t) \leq v(t) - \phi(t) \leq R_1.$$

Therefore, the following estimates hold, for all $u \in P \cap \partial B(0,R_1)$ and all $t \in [0,1]$,

$$\begin{aligned}T_{R_1,R_2,\phi}v(t) &= \int_0^1 G(t,s)(f(s,v(s)-\phi(s)) + q(s))\,ds \\ &\leq \int_0^1 G(t,s)\,\alpha(s)\,ds \leq \max_{t\in[0,1]}\int_0^1 G(t,s)\,\alpha(s)\,ds \leq R_1 = \|v\|.\end{aligned}$$

Passing to the supremum in the above estimates, we get

$$\|T_{R_1,R_2,\phi}v\| \leq \|v\| \text{ for all } v \in P \cap \partial B\,(0, R_1)\,.$$

Now, we have, for all $v \in P \cap \partial B\,(0, R_2)$ and $t \in [\sigma, 1-\sigma]$,

$$v(t) \geq v(t) - \phi(s) \geq (R_2 - \phi^*)\,\rho(s) = (R_2 - \phi^*)\,\rho_\sigma > 0. \tag{3.1}$$

Assumption **(b)** and (3.1) lead to the following estimates

$$\begin{aligned} \|T_{R_1,R_2,\phi}u\| &\geq \max_{t\in[0,1]} \int_\sigma^{1-\sigma} G(t,s)\,(f(s, v(s) - \phi(s)) + q(s))\,ds \\ &\geq \max_{t\in[0,1]} \int_\sigma^{1-\sigma} G(t,s)\beta\,(s)\;ds \geq R_2 = \|u\|. \end{aligned}$$

Therefore,

$$\|T_{R_1,R_2,\phi}v\| \geq \|v\| \text{ for all } v \in P \cap \partial B\,(0, R_2)\,.$$

Thus, it follows from Theorem A that $T_{R_1,R_2,\phi}$ has a fixed point v such that

$$\min(R_1, R_2) \leq \|v\| \leq \max(R_1, R_2).$$

Moreover, since $v \in P$, we have for all $t \in [0, 1]$,

$$v(t) \geq p(t)\|v\| \geq p(t)\min(R_1, R_2) > \phi^* p(t) \geq \phi(t).$$

So, we deduce from Lemma 2.5 that $u = v - \phi$ is a positive solution to bvp (1.1). □

In what follows, we consider the particular case where the nonlinearity f is continuous and we suppose that,

$$\begin{gathered} \text{there exists } M > 0 \text{ such that} \\ f(t,u) + M \geq 0 \text{ for all } t \in [0,1] \text{ and } u \geq 0, \end{gathered} \tag{3.2}$$

$$\text{there exists } \ r > \phi^*, \text{ such that } \frac{S_f(r) + M}{r}\left(\max_{t\in[0,1]} \int_0^1 G(t,s)ds\right) \leq 1 \tag{3.3}$$

where

$$S_f(r) = \max\{f(t,u),\, t \in [0,1],\ u \in [0,r]\},$$

and

$$\begin{gathered} \text{there exists } \sigma \in \left(0, \tfrac{1}{2}\right) \text{ such that} \\ f_\infty\,(\sigma) = \lim x \to +\infty \inf\left(\min\nolimits_{t\in[\sigma,1-\sigma]} \frac{f(t,x)}{x}\right) > (\rho_\sigma G_\sigma)^{-1}, \end{gathered} \tag{3.4}$$

where for $\sigma \in \left(0, \frac{1}{2}\right)$

$$G_\sigma = \max_{t\in[0,1]} \int_\sigma^{1-\sigma} G\,(t,s)\,ds.$$

Corollary 3.2. *Suppose that f is continuous, and Hypotheses* (1.2), (1.3), (3.2), (3.3) *and* (3.4) *hold. Then bvp* (1.1) *has at least one positive solution.*

Proof. We have to show the conditions in Theorem 3.1 are satisfied. Clearly, if the nonlinearity f is continuous, then Hypotheses (1.4) and (2.7) are satisfied.

Now, let us prove that if Hypothesis (3.3) hold then Condition (**a**) in Theorem 3.1 is satisfied. Take $R_1 = r$ and $\alpha(t) = S_f(r) + M$. We have then

$$f(t,u) + M \leq \alpha(t)$$

for all $t \in [0,1]$ and $u \in [0, R_1]$.

In this case, we have

$$\begin{aligned} \|\pounds\alpha\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\,\alpha(s)\,ds \\ &= (S_f(R_1) + M) \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds < R_1. \end{aligned}$$

It remains to show that if Hypothesis (3.4) holds, then Condition (**b**) of Theorem 3.1 is satisfied. Let $\epsilon > 0$ be small enough such that $f_\infty(\sigma) - \epsilon > (\rho_\sigma G_\sigma)^{-1}$. Hypothesis (3.4) implies that there exists $R_\infty > 0$ such that

$$f(t,x) + M \geq (f_\infty(\sigma) - \epsilon)\,x + M,$$

for all $t \in [\sigma, 1-\sigma]$ and $x \geq R_\infty$.

Let

$$\begin{aligned} R_2 &> \max\left(\phi^* + \frac{R_\infty}{\rho_\sigma}, \frac{\rho_\sigma G_\sigma\,(f_\infty(\sigma) - \epsilon)\,\phi^* - MG_\sigma}{(f_\infty(\sigma) - \epsilon)\,\rho_\sigma G_\sigma - 1}\right) \text{ and} \\ \beta(t) &= (f(\sigma) - \epsilon)\,\rho_\sigma\,(R_{2-}\phi^*) + M. \end{aligned}$$

We have from the choice of R_2,

$$\begin{aligned} \max_{t\in[\sigma,1-\sigma]} \int_\sigma^{1-\sigma} G(t,s)\,\beta(s)\,ds &= G_\sigma\,((f(\sigma) - \epsilon)\,\rho_\sigma\,(R_2 - \phi^*) + M) \\ &> (R_2 + G_\sigma\,(f(\sigma) - \epsilon)\,\rho_\sigma\phi^* - G_\sigma M) \\ &\quad - G_\sigma\,(f(\sigma) - \epsilon)\,\rho_\sigma\phi^* + G_\sigma M \\ &= R_2. \end{aligned}$$

This ends the proof. □

3.2. The singular case.

Theorem 3.3. *Suppose that Hypotheses* (1.2)–(1.5) *hold and*

(c): *there exist functions* $m_1, m_2 \in K_G$*, a continuous decreasing function* $g : (0,+\infty) \to (0,+\infty)$*, a continuous increasing function* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $R_1 > \phi^*$ *such that*

$$\begin{cases} |f(t,u)| \leq m_1(t)\,g(u) + m_2(t)\,h(u) \text{ and all } u > 0 \text{ and a.e. } t \in (0,1) \text{ and} \\ \int_0^1 G(s,s)\,(m_1(s)\,g(\rho(s)\,(R_1 - \phi^*)) + m_2(s)\,h(R_1) + q(s))\,ds \leq R_1, \end{cases}$$

(d): *there exist* $\sigma \in \left(0, \frac{1}{2}\right)$, *a function* $\beta \in K_G$ *and a constant* R_2, *with* $R_2 \neq R_1$, *such that*

$$\phi^* < R_2 \leq \max_{t\in[0,1]} \int_{\sigma}^{1-\sigma} G(t,s)\,\beta(s)\,ds,$$
$$f(t,u) + q(t) \geq \beta(t),$$

for a.e. $t \in [\sigma, 1-\sigma]$ *and all* $u \in [\rho_\sigma (R_2 - \phi^*), R_2]$ *where* $\rho_\sigma = \min_{s\in[\sigma,1-\sigma]} \rho(s)$.

Then, bvp (1.1) *has at least one positive solution.*

Proof. Let $T_{R_1,R_2,\phi}$ be the operator defined in Lemma 2.6 where R_1 and R_2 are those of Theorem 3.3. We have, for all $v \in P \cap \partial B(0, R_1)$ and $t \in [0, 1]$,

$$0 \leq (R_1 - \phi^*)\,\rho(t) \leq v(t) - \phi(t) \leq R_1.$$

Taking into account Assumption **(c)**, the following estimates hold, for all $u \in P \cap \partial B(0, R_1)$ and all $t \in [0, 1]$,

$$\begin{aligned} T_{R_1,R_2,\phi}v(t) &= \int_0^1 G(t,s)\left(f(s, v(s) - \phi(s)) + q(s)\right)ds \\ &\leq \int_0^1 G(s,s)\left(m_1(s)\,g(\rho(s)(R_1 - \phi^*)) + m_2(s)\,h(R_1) + q(s)\right)ds \\ &\leq R_1 = \|v\|. \end{aligned}$$

Passing to the supremum in the above estimates, we get

$$\|T_{R_1,R_2,\phi}v\| \leq \|v\| \text{ for all } u \in P \cap \partial B(0, R_1).$$

Now, we have, for all $v \in P\cap\, \partial B(0, R_2)$ and $t \in [\sigma, 1-\sigma]$,

$$v(t) \geq v(t) - \phi(s) \geq (R_2 - \phi^*)\,\rho(s) = (R_2 - \phi^*)\,\rho_\sigma > 0.$$

This together with Assumption **(d)** leads to the following estimates,

$$\begin{aligned} \|T_{R_1,R_2,\phi}v\| &\geq \max_{t\in[0,1]} \int_{\sigma}^{1-\sigma} G(t,s)\left(f(s, v(s) - \phi(s)) + q(s)\right)ds \\ &\geq \max_{t\in[0,1]} \int_{\sigma}^{1-\sigma} G(t,s)\beta(s)\;ds \geq R_2 = \|v\|. \end{aligned}$$

Therefore,

$$\|T_{R_1,R_2,\phi}v\| \geq \|v\| \text{ for all } v \in P \cap \partial B(0, R_2).$$

Thus, it follows from Theorem A that $T_{R_1,R_2,\phi}$ admits a fixed point v such that

$$\min(R_1, R_2) \leq \|v\| \leq \max(R_1, R_2).$$

Moreover, since $v \in P$, we have for all $t \in (0, 1)$

$$v(t) \geq p(t)\|v\| \geq p(t)\min(R_1, R_2) > \phi^* p(t) \geq \phi(t).$$

So, we deduce from Lemma 2.6 that $u = v - \phi$ is a positive solution to bvp (1.1). □

4. EXAMPLES

4.1. Example 1.

Consider the bvp (1.1) with $p = 1,\ a = c = 1,\ b = d = 0$ and

$$f(t,u) = -\frac{1}{\sqrt{t}} + \frac{1}{t(1-t)}\frac{u}{1+u} + \frac{Au^2}{A+u}.$$

Note that Hypothesis (1.5) is satisfied for $q(t) = \frac{1}{\sqrt{t}}$, and by simple computations, we obtain $\phi(t) = \frac{4}{3}t\left(1-\sqrt{t}\right),\ \max_{t\in[0,1]}\int_0^1 G(t,s)\,ds = \frac{1}{8}$ and $\phi^* = \frac{4}{3}$. We have

$$f(t,u) + q(t) \leq \tfrac{2}{3}\tfrac{1}{t(1-t)} + 4 = \alpha(t) \text{ for all } u \in [0,2], \text{ and}$$
$$\max_{t\in[0,1]}\int_0^1 G(t,s)\,\alpha(s)\,ds = 2\ln 2 + \tfrac{1}{2} < 2.$$

Thus Condition **(a)** of Theorem 3.1 is satisfied for $R_1 = 2$.

Also, we have

$$f(t,u) + q(t) \geq \frac{B^2}{2} = \beta(t) \text{ for } u \geq B.$$

Choosing $\sigma = \frac{1}{3}$, we obtain after simple computations $\max_{t\in[0,1]}\int_\sigma^{1-\sigma} G(t,s)\,\beta(s)\,ds = \frac{B^2}{18}$.

Taking $R_2 = 3B + \frac{2}{3}$, one can see that Condition **(b)** of Theorem 3.1 is satisfied for all B satisfying

$$\frac{B^2}{18} - 3B - \frac{2}{3} > 0,$$

and in this case, bvp (1.1) admits a positive solution.

4.2. Example 2.

Consider the bvp (1.1) with $p = 1,\ a = c = 1,\ b = d = 0$ and

$$f(t,u) = (-t)\frac{u}{1+u} + \frac{Au}{A^2+u^2} + \frac{Bu^2}{B+u},$$

where A, B are positive real numbers.

Note that Hypothesis (3.2) is satisfied for $M = 1$ and in this case we have $\phi(t) = \frac{1}{2}t(1-t) = \int_0^1 G(t,s)\,ds,\ \max_{t\in[0,1]}\int_0^1 G(t,s)\,ds = \frac{1}{8}$ and $\phi^* = \frac{1}{2}$. Taking $\sigma = \frac{1}{3}$, straightforward computations lead to $\rho_{\frac{1}{3}} = \frac{1}{3},\ G_{\frac{1}{3}} = \frac{1}{9}$. Note also that

$$\lim_{u\to+\infty}\frac{f(t,u)}{u} = B \text{ uniformly for } t \in [0,1].$$

By simple computations, we get $S_f(r) = \frac{1}{2} + \frac{Br^2}{B+r}$, and Hypothesis (3.3) holds for $r = 1$. Thus, we deduce from Corollary 3.2 that bvp (1.1) admits a positive solution for all $A > 0$ and $B > 27$.

4.3. Example 3. Consider the bvp (1.1) with $p=1,\ a=c=1,\, b=d=0$ and

$$f\left(t,u\right)=\left(-2t\right)+\frac{1}{u}+\frac{Bu^{2}}{B+u}.$$

Note that Hypothesis (1.5) is satisfied for $q\left(t\right)=2t$, and by simple computations, we obtain $\phi\left(t\right)=\frac{1}{6}t\left(1-t^{2}\right),\ \max_{t\in[0,1]}\int_{0}^{1}G\left(t,s\right)ds=\frac{1}{8}$ and $\phi^{*}=\frac{2}{3}$. The first inequality in Condition **(a)** of Theorem 3.3 is satisfied for $m_1=m_2=1,\ g(u)=\frac{1}{u}$ and $h(u)=\frac{Bu^2}{B+u}$. Thus, we have for all $R>\phi^{*}$,

$$\int_{0}^{1}G\left(s,s\right)\left(m_{1}\left(s\right)g\left(\rho\left(s\right)\left(R-\phi^{*}\right)\right)+m_{2}\left(s\right)h\left(R\right)+q\left(s\right)\right)ds$$
$$=\tfrac{3}{4}\left(R-\tfrac{2}{3}\right)^{-1}+\tfrac{1}{6}\tfrac{BR^{2}}{B+R}+\tfrac{1}{6},$$

and the second inequality is satisfied for $R_1=2$. Thus Condition **(a)** of Theorem 3.3 is satisfied for $R_1=2$.

Also, in this example we have $f\left(t,u\right)+q\left(t\right)\geq\frac{B^{2}}{2}=\beta\left(t\right)$ for $u\geq B$. So, bvp (1.1) admits a positive solution whenever $\frac{B^2}{18}-3B-\frac{2}{3}>0$.

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