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ON φ_h-PREINVEX FUNCTIONS

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ABSTRACT. In this paper, we define a new class of functions called φ_h-preinvex functions, which generalize preinvex, φ-convex and h-preinvex functions. Some examples are constructed which show that it is the most generalized class. Furthermore, several properties are discussed and some integral inequalities are established.

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1. INTRODUCTION

Convexity plays a central and fundamental role in engineering, mathematical finance, economics and optimization theory. Due to its applications and significant importance, it has been extended and generalized in several directions see [1-4]. The theory of convex functions are very much related with the theory of inequalities. For recent developments of classical inequalities for convex functions and its generalizations, see [5-9]. An important generalization of convexity is that of invexity introduced by Hanson [10].

Youness [11] introduced the concept of E-convex sets and E-convex functions while Iqbal et. al. [12, 13] generalized it for the Riemannian manifolds. The initial results of Youness [11] inspired a great deal of subsequent work, however Yang [14] showed that some of the results obtained by Youness [11] are incorrect. Fulga and Preda [15] extended the class of E-convex functions to E-preinvex functions and discussed some of their properties. Fulga and Preda [15] established that under the mild conditions, a local minimum is a global minimum.

Motivated by earlier research works, we introduce a new class of functions, called φ-preinvex and φ_h-preinvex functions, which are the most generalized classes. It generalize φ-convex function given by Cristescu [1] and h-preinvex functions given by Matloka [8]. Some examples are constructed to show that there exist functions

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which are φ-preinvex and φ_h-preinvex but not preinvex and h-preinvex functions. In addition, we establish some new integral inequalities.

2. PRELIMINARIES

Firstly, we recall some definitions and known results, which will be used throughout the paper. Let R^n denotes the n-dimensional Euclidean space.

Definition 2.1. [10] A set $S \subseteq R^n$ is said to be invex with respect to $\eta : S \times S \to R^n$ if for every $x, y \in S$ and $t \in [0, 1]$

$$y + t\eta(x, y) \in S$$

The definition says that there is a path starting from y which is contained in S. It is not necessary that x should be one of the end points of the path. However, if we require that x should be an end point of the path for every pair x, y, then $\eta(x, y) = x - y$, reducing to convexity. A significant generalization of convex functions called preinvex was given by Weir and Mond [15].

Definition 2.2. [16] Let $S \subseteq R^n$ be an invex set with respect to $\eta : S \times S \to R^n$. Then, the function $f : S \to R$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y).$$

Every convex function function is a preinvex function but the converse need not true. For example, the function $f(x) = - \mid x \mid$ is not a convex function but it is a preinvex function with respect to η, where

$$\eta(x, y) := \begin{cases} x - y, & if\ x \leq 0, y \leq 0\ and\ x \geq 0, y \geq 0, \\ \\ y - x, & otherwise. \end{cases}$$

In 2007, Varosanec [17], introduced the concept of h-convex functions which generalize convex, s-convex, Godunova-Levin functions and P-functions.

Definition 2.3. [17] Let I be a real interval and let $h : (0, 1) \to (0, \infty)$ be a given function. A nonnegative function $f : I \to R$ is said to be h-convex if

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$. If inequality is reversed, f is called h-concave.

Example 2.1. [17] Let h be a non-negative function such that $h(t) \geq (t)$ and let $h_k(x) = x^k,\ k < 0$. Then, $f : [a, b] \to R$ such that

$$f(x) = \begin{cases} 1, \ x \neq \frac{a+b}{2} \\ \\ 2^{1-k}, \ x = \frac{a+b}{2} \end{cases}$$

is not a convex function but it is h_k-convex.

Definition 2.4. [11] A function $f : R^n \to R$ is said to be E-convex on a set $S \subset R^n$ iff there is a map $E : R^n \to R^n$ such that S is an E-convex set and

$$f(tE(x) + (1-t)E(y)) \leq tf(E(x)) + (1-t)f(E(y)),$$

for each $x, y \in S$ and $t \in [0, 1]$.

Cristescu et. al. [1] improved the definition given by Youness [11] as follows:

Definition 2.5. [1] A function $f : [a, b] \to R$ is said to be φ-convex on $[a, b]$ if there is a map $\varphi : [a, b] \to [a, b]$ satisfying the following inequality

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y))$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Fulga and Preda [15] extended the class of E-convex sets and E-convex functions as follows:

Definition 2.6. [15] Let $E : R^n \to R^n$. A set $S \subset R^n$ is said to be E-invex with respect to $\eta : R^n \times R^n \to R^n$ if

$$E(y) + t\eta(E(x), E(y)) \in S, \ \forall x, y \in S, t \in [0, 1]$$

Definition 2.7. [15] A function $f : R^n \to R$ is said to be E-preinvex on an E-invex set S with respect to η if $\forall x, y \in S$, $t \in [0, 1]$,

$$f(E(y) + t\eta(E(x), E(y))) \leq tf(E(x)) + (1-t)f(E(y))$$

Recently, Matloka [8] introduced the class of h-preinvex functions and established some new inequalities.

Definition 2.8. [8] Let $h : [0, 1] \to R$ be a nonnegative function such that $h \neq 0$. The function f on the invex set S is said to be h-preinvex with respect to η, if

$$f(y + t\eta(x, y)) \leq h(t)f(x) + h(1-t)f(y)$$

for every $x, y \in S$ and $t \in [0, 1]$ where $f(.) > 0$. It is to be noted that every convex function is a h-preinvex function with respect to $\eta(x, y) = x - y$ and $h(t) = t$ for any $t \in [0, 1]$.

3. MAIN RESULTS

Let A be a nonempty subset of R and $\eta : A \times A \to R$. We also consider $\varphi : [a, b] \to [a, b]$ be a given function, where $[a, b] \subset R$. For the sake of convenience we consider a set to be φ-invex if $n = 1$ in the Definition 2.6. Now we define a new improved class of functions as follows:

Definition 3.1. A function $f : [a, b] \to R$ is said to be φ-preinvex on $[a, b]$ with respect to η if for every points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds

$$f(\varphi(y) + t\eta(\varphi(x), \varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y))$$

Note that every preinvex function is φ-preinvex if $\varphi(x) = x$ for all $x \in [a, b]$ and every φ-convex function is φ-preinvex function with respect to $\eta(\varphi(x), \varphi(y)) = \varphi(x) - \varphi(y)$.

Example 3.1. The following example illustrates that φ-preinvex function need not be a preinvex function with respect to the same η. Let $X = [-1, 1]$, and let $f : X \to R$ be defined as follows:

$$f(x) = \begin{cases} 1; \ x > 0 \\ \\ -x; \ x \leq 0, \end{cases}$$

and

$$\eta(x, y) := \begin{cases} x - y, \ if \ x \leq 0, y \leq 0 \ and \ x \geq 0, y \geq 0, \\ \\ y - x, \ otherwise. \end{cases}$$

Then, from Definition 3.1, f is φ-preinvex function on X with respect to η, where $\varphi(x) = e^x$. While if we take $x = 1$, $y = 0$ and $t = \frac{1}{2}$, then
$f(y + t\eta(x, y)) = f(\frac{1}{2}) = 1 > \frac{1}{2}f(1) + \frac{1}{2}f(0) = \frac{1}{2}$. Thus, f is not preinvex function on X.

Definition 3.2. Let I be an interval in R and $h : (0, 1) \to (0, \infty)$ be a given function. A function $f : I \to [0, \infty)$ is said to be φ_h-preinvex if

$$f(\varphi(y) + t\eta(\varphi(x), \varphi(y))) \leq h(t)f(\varphi(x)) + h(1 - t)f(\varphi(y))$$

for all $x, y \in I$ and $t \in (0, 1)$.

Note that every h-preinvex function is φ_h-preinvex if $\varphi(x) = x$ for all $x \in [a, b]$.

Example 3.2. If the functions f and η are defined same as in Ex.3.1. and let $h(t) = t^k$, $k \leq 1$, $t \in (0, 1)$, then f is φ_h-preinvex for $\varphi(x) = e^x$ but not h-preinvex for $x = 1$, $y = 0$, $t = \frac{1}{2}$ and $k = \frac{1}{2}$.

Remark 3.1. Let $h : (0,1) \to (0,\infty)$ be a given function such that $h(t) \geq t$ for all $t \in (0,1)$. If f is φ-preinvex on I, then for every $x, y \in I$ and $t \in (0,1)$

$$\begin{aligned} f(\varphi(y) + t\eta(\varphi(x), \varphi(y))) &\leq tf(\varphi(x)) + (1-t)f(\varphi(y)) \\ &\leq h(t)f(\varphi(x)) + h(1-t)f(y) \end{aligned}$$

Proposition 3.1. *Let $h_1, h_2 : (0,1) \to (0,\infty)$ be the functions such that $h_2(t) \leq h_1(t)$ for all $t \in (0,1)$. If f is φ_{h2}-preinvex on I, then for $x, y \in I$, f is φ_{h1}-preinvex on I.*

Proof. Since f is φ_{h2}-preinvex on I , for $x, y \in I$ and $t \in (0,1)$, we have

$$\begin{aligned} f(\varphi(y) + t\eta(\varphi(x), \varphi(y)) &\leq h_2(t)f(\varphi(x)) + h_2(1-t)f(\varphi(y)) \\ &\leq h_1(t)f(\varphi(x)) + h_1(1-t)f(\varphi(y)) \end{aligned}$$

Proposition 3.2. *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f, g : I \to [0,\infty)$ are φ_h-preinvex functions on I and $\alpha > 0$, then for all $t \in (0,1)$, $f+g$ and αf are φ_h-preinvex on I.*

Proof. Using definition of φ_h-preinvexity, proof is obvious.

There are some properties of E-preinvex functions which remain valid for φ_h-preinvex functions.

Theorem 3.1. *Let $A \subseteq R$ be a φ-invex set and $f : A \to R$ be φ_h-preinvex function. If $g : I \to R$ be an increasing convex function such that range $(f) \subset I$, then the composite function $g \circ f$ is φ_h-preinvex on A.*

Proof. Since f is φ_h-preinvex function, we have

$$f(\varphi(y) + t\eta(\varphi(x), \varphi(y))) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y))$$

for all $x, y \in A$ and $\forall\ t \in [0,1]$. Since g is an increasing convex function, we get

$$\begin{aligned} g[f(\varphi(y) + t\eta(\varphi(x), \varphi(y)))] &\leq g[h(t)f(\varphi(x)) + h(1-t)f(\varphi(y))] \\ &\leq h(t)g[f(\varphi(x))] + h(1-t)g[f(\varphi(y))] \\ &= h(t)(g \circ f)(\varphi(x)) + h(1-t)(g \circ f)(\varphi(y)), \end{aligned}$$

which shows that $g \circ f$ is φ_h-preinvex on A.

Theorem 3.2. *Let $A \subseteq R$ be a φ-invex set and $\{f_j\}_{j\in J}$ be a family of real valued functions defined on A such that $\sup\limits_{j\in J} f_j(x)$ exists in R, for all $x \in A$. Let $f : A \to R$ be a real function defined by $f(x) = \sup\limits_{j\in J} f_j(x)$, $\forall\ x \in A$. If $f_j : A \to [0,\infty)$, for any $j \in J$, are φ_h-preinvex functions on A, then the function f is φ_h-preinvex on A.*

Proof. Let $f_j : A \to [0,\infty)$, for any $j \in J$, are φ_h-preinvex functions on A, then for every $x, y \in A$ and $\forall\ t \in [0,1]$,

$$f_j(\varphi(y) + t\eta(\varphi(x), \varphi(y))) \leq h(t)f_j(\varphi(x)) + h(1-t)f_j(\varphi(y)),$$

$$\begin{aligned}\sup_{j\in J} f_j(\varphi(y) + t\eta(\varphi(x), \varphi(y))) &\leq \sup_{j\in J}[h(t)f_j(\varphi(x)) + h(1-t)f_j(\varphi(y))] \\ &= h(t)\sup_{j\in J} f_j(\varphi(x)) + h(1-t)\sup_{j\in J} f_j(\varphi(y)) \\ &= h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)),\end{aligned}$$

or

$$f(\varphi(y) + t\eta(\varphi(x), \varphi(y))) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)).$$

Hence, the result.

Let $S \subset R^n$ be an E-invex set. It follows from Lemma 2.1 in [14], $E(S) \subseteq S$. Hence, for any $f : S \to R$, the restriction $\tilde{f} : E(S) \to R$ of $f : S \to R$ to $E(S)$ is defined by

$$\tilde{f}(\tilde{x}) = f(\tilde{x}), \text{ for all } \tilde{x} \in E(S).$$

Theorem 3.3. *Let $A \subset R$ be a φ-invex set and $f : A \to R$ be φ_h-preinvex function on A. Then, the restriction $\tilde{f} : C \to R$ of f to any nonempty invex subset C of $\varphi(A)$ is h-preinvex function on C.*

Proof. Let $x, y \in C \subset \varphi(A)$. Then there exist $x', y' \in A$ such that $x = \varphi(x')$ and $y = \varphi(y')$. Since C is an invex set, we have $\varphi(y') + t\eta(\varphi(x'), \varphi(y')) \in C, \ \forall\, t \in [0,1]$. Therefore, we have

$$\begin{aligned}\tilde{f}(y + t\eta(x,y)) &= f(\varphi(y') + t\eta(\varphi(x'), \varphi(y'))) \\ &\leq h(t)f(\varphi(x')) + h(1-t)f(\varphi(y')) \\ &= h(t)f(x) + h(1-t)f(y).\end{aligned}$$

Theorem 3.4. *Let $A \subset R$ be a φ-invex set, $f : A \to R$ be a real function and $\varphi(A)$ be an invex set. Then f is φ_h-preinvex on A if and only if its restriction $\tilde{f} = f\,|_{\varphi(A)}$ is h-preinvex function on $\varphi(A)$.*

Proof. The if condition is true due to the Theorem 3.3. Conversely, let $x, y \in A$, then $\varphi(x), \varphi(y) \in \varphi(A)$ and $\varphi(y) + t\eta(\varphi(x), \varphi(y)) \in \varphi(A) \subseteq A, \ \forall\, t \in [0,1]$. Since $\varphi(A) \subseteq A$, we have

$$\begin{aligned}f(\varphi(y) + t\eta(\varphi(x), \varphi(y)) &= \tilde{f}(\varphi(y) + t\eta(\varphi(x), \varphi(y)) \\ &\leq h(t)\tilde{f}(\varphi(x)) + h(1-t)\tilde{f}(\varphi(y)) \\ &= h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)).\end{aligned}$$

Definition 3.3. If $S \subseteq R^n \times R$ and $E : R^n \to R^n$, then the set S is said to be E-invex iff $(x, \alpha), (y, \beta) \in S$ imply

$$(E(y) + t\eta(E(x), E(y)), t\alpha + (1-t)\beta) \in S, \ t \in [0, 1].$$

Epigraph $epi(f)$ of f is given by:

$$epi(f) = \{(E(x), \alpha) : x \in R^n, \alpha \in R, f(E(x)) \leq \alpha\}.$$

The following theorem gives a sufficient condition for f to be a φ-preinvex function.

Theorem 3.5. *Let φ be a linear idempotent map. Assume that $A \subset R$ be a φ-invex set and $f : A \to R$ be a real valued function. If $epi(f)$ is φ-invex set, then f is a φ-preinvex function on A.*

Proof. Let $x, y \in A$ and $(\varphi(x), f(\varphi(x)))$, $(\varphi(y), f(\varphi(y))) \in epi(f)$. Since φ is an idempotent and $epi(f)$ is φ-invex set, we have

$$(\varphi(\varphi(y)) + t\eta(\varphi(\varphi(x)), \varphi(\varphi(y)), tf(\varphi(x)) + (1-t)f(\varphi(y))) \in epi(f),$$

or

$$(\varphi(y) + t\eta(\varphi(x), \varphi(y), tf(\varphi(x)) + (1-t)f(\varphi(y)) \in epi(f),$$

which implies,

$$f(\varphi(y) + t\eta(\varphi(x), \varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y))$$

Hence, f is φ-preinvex function on A.

Theorem 3.6. *Let $(f_j)_{j \in J}$ be a family of real-valued functions defined on φ-invex set $A \subseteq R$ which are bounded from above. If the epigraphs are φ-invex sets, then the function f defined by $f(x) = \sup_{j \in J} f_j(x)$, $\forall x \in A$ is φ-preinvex function on A.*

Proof. Since epigraphs,

$$epi(f_j) = \{(\varphi(x), \alpha) : x \in A, \alpha \in R, f_j(\varphi(x)) \leq \alpha\},$$

are φ-invex sets in $A \times R$, their intersection

$$\bigcap_{j \in J} epi(f_j) = \{(\varphi(x), \alpha) : x \in A, \alpha \in R, f_j(\varphi(x)) \leq \alpha \ ; j \in J\}$$

$$= \{(\varphi(x), \alpha) : x \in A, \alpha \in R, f(\varphi(x)) \leq \alpha\},$$

where $f(\varphi(x)) = \sup_{j \in J} f_j(\varphi(x))$, is also a φ-invex set. We can see that this intersection is an epigraph; hence by Theorem 3.5, f is φ-preinvex function on A.

4. SOME INTEGRAL INEQUALITIES

Theorem 4.1. *Let $f : I \to [0,\infty)$ be Lebesgue integrable and φ_h-preinvex function for continuous function $\varphi : [a,b] \to [a,b]$. For a given function $h : (0,1) \to (0,\infty)$, the following inequality holds:*

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))dx \le [f(\varphi(a))+f(\varphi(b)]\int_0^1 h(t)dt$$

Proof. Using φ_h-preinvexity of f, for every t $\in$ (0,1), we have

$$f(\varphi(b)+t\eta(\varphi(a),\varphi(b))) \le h(t)f(\varphi(a))+h(1-t)f(\varphi(b))$$

Integrating both sides of the above inequality over (0,1),we obtain

$$\int_0^1 f(\varphi(b)+t\eta(\varphi(a),\varphi(b)))dt \le f(\varphi(a))\int_0^1 h(t)dt + f(\varphi(b))\int_0^1 h(1-t)dt.$$

On substituting $x = \varphi(b)+t\eta(\varphi(a),\varphi(b))$ in the first integral, we get

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_{\varphi(b)}^{\varphi(b)+\eta(\varphi(a),\varphi(b))} f(x)dx \le f(\varphi(a))\int_0^1 h(t)dt + f(\varphi(b))\int_0^1 h(1-t)dt,$$

now putting $1-t=p$ in the second integral, we get

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_{\varphi(b)}^{\varphi(b)+\eta(\varphi(a),\varphi(b))} f(x)dx \le f(\varphi(a))\int_0^1 h(t)dt + f(\varphi(b))\int_0^1 h(t)dt.$$

Replacing x with $x+\varphi(b)$,

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))dx \le [f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)dt.$$

Which completes the proof.

Remark 4.1. If $\eta(\varphi(a),\varphi(b)) = \varphi(a)-\varphi(b)$ then the above inequality becomes

$$\frac{1}{\varphi(a)-\varphi(b)}\int_{\varphi(b)}^{\varphi(a)} f(x)dx \le [f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)dt,$$

which is the inequality for φ_h-convex functions proved by Sarikaya in [18].

Corollary 4.1. *Under the assumptions of Theorem 4.1 with $h(t) = t^s$ ($s \in (0,1)$, $t \in (0,1)$), we have*

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))dx \le \frac{f(\varphi(a))+f(\varphi(b))}{s+1}$$

Remark 4.2. If $h(t)=t$, $t \in (0,1)$, and φ is an identity function with $\eta(a,b) = a-b$, then the above inequality becomes

$$\frac{1}{a-b}\int_b^a f(x)dx \le \frac{f(a)+f(b)}{2}$$

Corollary 4.2. *Under the assumptions of Theorem 4.1 with* $h(t) = \frac{1}{t}$, $t \in (0,1)$, *we get*

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))dx \le \infty$$

Remark 4.3. If φ is an identity function with $\eta(a,b) = a-b$, then the above inequality reduces to inequality for Godunova-Levin functions given by Dragomir [3].

Theorem 4.2. *Let* $f : I \to [0,\infty)$ *be Lebesgue integrable and* φ_h*-preinvex function for continuous function* $\varphi : [a,b] \to [a,b]$. *For a given function* $h : (0,1) \to (0,\infty)$, *the following inequality holds:*

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))f(\varphi(a)-x)dx$$

$$\le [f^2(\varphi(a))+f^2(\varphi(b))]\int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt$$

Proof. Since f is φ_h-preinvex function , we have

$$f(\varphi(b)+t\eta(\varphi(a),\varphi(b))) \le h(t)f(\varphi(a)) + h(1-t)f(\varphi(b))$$

and

$$f(\varphi(a)-t\eta(\varphi(a),\varphi(b))) \le h(1-t)f(\varphi(a)) + h(t)f(\varphi(b)).$$

On multiplying the above two equations, we get

$$\begin{aligned} & f(\varphi(b)+t\eta(\varphi(a),\varphi(b)))f(\varphi(a)-t\eta(\varphi(a),\varphi(b))) \\ \le\ & h(t)h(1-t)[f^2(\varphi(a))+f^2(\varphi(b))] + (h^2(t)+h^2(1-t))f(\varphi(a))f(\varphi(b)). \end{aligned}$$

Integrating the above inequality with respect to t over $(0,1)$, we get

$$\begin{aligned} & \int_0^1 f(\varphi(b)+t\eta(\varphi(a),\varphi(b)))f(\varphi(a)-t\eta(\varphi(a),\varphi(b)))dt \\ & \quad \le [f^2(\varphi(a))+f^2(\varphi(b))]\int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt, \end{aligned}$$

on substituting $x = \varphi(b)+t\eta(\varphi(a),\varphi(b))$, we get

$$\begin{aligned} & \frac{1}{\eta(\varphi(a),\varphi(b))}\int_{\varphi(b)}^{\varphi(b)+\eta(\varphi(a),\varphi(b))} f(x)f(\varphi(a)+\varphi(b)-x)dx \\ & \quad \le [f^2(\varphi(a))+f^2(\varphi(b))]\int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt, \end{aligned}$$

on changing the limits of the first integral, we get

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))f(\varphi(a)-x)dx$$
$$\leq [f^2(\varphi(a))+f^2(\varphi(b))]\int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt.$$

Hence the result.

Theorem 4.3. *Let $f,\ g: I \to [0,\infty)$ be Lebesgue integrable and φ_h-preinvex function for continuous function $\varphi : [a,b] \to [a,b]$. For a given function $h : (0,1) \to (0,\infty)$, the following inequality holds:*

$$\frac{1}{\eta(\varphi(a),\varphi(b))}\int_0^{\eta(\varphi(a),\varphi(b))} f(x+\varphi(b))g(x+\varphi(b))dx$$
$$\leq M(a,b)\int_0^1 h^2(t)dt + N(a,b)\int_0^1 h(t)h(1-t)dt,$$

where

$$M(a,b) = f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b))$$
$$N(a,b) = f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a)).$$

Proof. Since $f, g : I \to R$ are φ_h-preinvex functions, we have

$$f(\varphi(b)+t\eta(\varphi(a),\varphi(b))) \leq h(t)f(\varphi(a)) + h(1-t)f(\varphi(b)),$$

$$g(\varphi(b)) + t\eta(\varphi(a),\varphi(b)) \leq h(t)g(\varphi(a)) + h(1-t)g(\varphi(b)).$$

On multiplying the above two equations, we get

$$\begin{aligned} &f(\varphi(b)+t\eta(\varphi(a),\varphi(b)))g(\varphi(b)+t\eta(\varphi(a),\varphi(b)))\\ &\quad \leq \ h^2(t)f(\varphi(a))g(\varphi(a)) + h^2(1-t)f(\varphi(b))g(\varphi(b))\\ &\quad + \ h(t)h(1-t)[f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))]. \end{aligned}$$

Integrating both sides of the above inequality over (0,1), we obtain

$$\begin{aligned} &\int_0^1 f(\varphi(b)+t\eta(\varphi(a),\varphi(b)))g(\varphi(b)+t\eta(\varphi(a),\varphi(b)))dt\\ &\quad \leq \ [f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b))]\int_0^1 h^2(t)dt\\ &\quad + \ f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))\int_0^1 h(t)h(1-t)dt. \end{aligned}$$

On substituting $x = \varphi(b)+t\eta(\varphi(a),\varphi(b))$ in the first integral and solving, we get the required inequality.

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