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SOME APPROXIMATION RESULTS FOR THE STANCU TYPE Q-BERNSTEIN-SCHURER-KANTOROVICH OPERATORS

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ABSTRACT. In this paper we introduce the Stancu type generalization of the q-Bernstein-Schurer-Kantorovich operators and examine their approximation properties. We investigate the convergence of our operators with the help of the Korovkin's approximation theorem and examine the convergence of these operators in the Lipschitz class of functions. Finally, we introduce the bivariate analogue of these operators and study some results for the bivariate case.

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1. INTRODUCTION AND PRELIMINARIES

The q-calculus has played an important role in the field of approximation theory since last three decades. Lupaş [9] was the first to apply the q-calculus to approximation theory who introduced the q-analogue of the well known Bernstein polynomials. Another q-analogue of the classical Bernstein polynomials was given by Phillips [19]. Ostrovska [18] obtained more results on the q-Bernstein polynomials. In the sequel many researchers have studied the q-analogues of many well known operators like Baskakov operators, Meyer-König-Zeller operators, Szász-Mirakyan operators, Bleimann-Butzer-Hahn operators (written succinctly as BBH). In [11] Muraru defined the q-Bernstein-Schurer operators and obtained the rate of convergence of these operators. Recently, approximation properties of q-analogues of various operators have been studied in [1], [2], [11]–[17] and [20].

Kantorovich introduced the following integral type generalization of the classical Bernstein operatos

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$$L_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1]$$

is the Bernstein basis function.

Below we give rudiments of the q-calculus.
For any fixed real number $q > 0$ and $k \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, the q-integer of k, denoted by $[k]_q$, is defined by

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

and the q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} [k]_q[k-1]_q ... [1]_q, & k \geq 1 \\ 1, & k = 0. \end{cases}$$

The q-concept can be extended to any real number k. For integers n and k such that $0 \leq k \leq n$, the q-analogue of the binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For the q-binomial coefficient the following relations hold:

$$\begin{aligned} \binom{n}{k}_q &= \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q \\ \binom{n}{k}_q &= q^{n-k}\binom{n-1}{k-1}_q + \binom{n-1}{k}_q. \end{aligned}$$

For $x \in [0,1]$ and $n \in \mathbb{N}^0$, the q-analogue of $(1+x)^n$, denoted by $(1+x)_q^n$, is defined by the polynomial

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)...(1+q^{n-1}x), & n = 1,2,3,... \\ 1, & n = 0. \end{cases}$$

For $0 < q < 1,\ a > 0$, the q-definite integral of a real valued function f is defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n,\ a\in R,$$

and over the interval $[a,b]$, $0<a<b$, it is defined by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

For details of these integrals one is referred to [5], [7], [8]. In some situations the above integrals are not appropriate to obtain the q-analogues of well known integrals. So we use another more general integrals, the Reimann type q-integrals, defined as follows:

$$\int_a^b f(x)d_q^Rx = (1-q)(b-a)\sum_{s=0}^{\infty} f(a+(b-a)q^s)q^s,$$

where a,b are such that $0\leq a<b$ and q is as above. The latter integrals were introduced by Gauchman [7] and Marinković et al. [10].
In this paper, let I denote the interval $[0,1+l]$ and the space $C[0,1+l]$ equipped with the norm

$$\|f\|_{C[0,1+l]} = \sup_{x\in[0,1+l]} |f(x)|.$$

2. CONSTRUCTION OF OPERATORS

In 2015, Agarwal et al. [3] introduced the following Kantorovich type generalization of the q-Bernstein-Schurer operators

$$K_{n,p}(f;q,x) = [n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}^q(x)q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q^Rt,\ x\in[0,1], \tag{2.1}$$

where $b_{n+p,k}^q(x) = \binom{n+p}{k}_q x^k(1-x)_q^{n+p-k}$ is the q-Bernstein basis function. They have investigated the approximation properties of these operators using the Korovkin's approximation theorem. Inspired by their work, we introduce the Stancu type generalization of the Bernstein-Schurer-Kantorovich operators based on q-integers as follows:

$$L_{n,l}^{\alpha,\beta}(f;q;x) = ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t)d_q^Rt,\ x\in[0,1], \tag{2.2}$$

where $b^k_{n,l}(q;x) = \binom{n+l}{k}_q x^k(1-x)^{n+l-k}_q$ is the q-Bernstein basis function and α, β are such that $0 \leq \alpha \leq \beta$.

$\alpha = 0,\ \beta = 0$ reduce the operators (2.2) to the operators (2.1). So the newly constructed operators are a generalization of the operators in (2.1). We shall investigate some approximation results for the operators in (2.2). To examine the approximation results, we need the following lemmas.

Lemma 2.1. *Let $L^{\alpha,\beta}_{n,l}(f;q;x)$ be given by (2.2). Then the followings hold:*

(i) $L^{\alpha,\beta}_{n,l}(1;q;x) = 1$,

(ii) $L^{\alpha,\beta}_{n,l}(t;q;x) = \dfrac{\alpha}{[n+l]_q} + \dfrac{1}{([n+1]_q+\beta)[2]_q} + \dfrac{2q[n+l]_q}{([n+1]_q+\beta)[2]_q}x,$

(iii) $$L^{\alpha,\beta}_{n,l}(t^2;q;x) = \frac{1}{([n+1]_q+\beta)^2[3]_q} + \frac{2\alpha}{([n+1]_q+\beta)^2[2]_q} + \frac{\alpha^2}{([n+1]_q+\beta)^2} + \frac{q[n+l]_q((3+4\alpha)+(5+4\alpha)q+4(1+\alpha)q^2)}{([n+1]_q+\beta)^2[2]_q[3]_q}x + \frac{q^2[n+l]_q[n+l-1]_q(1+q+4q^2)}{([n+1]_q+\beta)^2[2]_q[3]_q}x^2.$$

Proof. Before proving the above lemma, we shall first prove the followings:

$$\sum_{k=0}^{n+l} b^k_{n,l}(q;x)q^k = 1-(1-q)[n+l]_q x \tag{2.3}$$

and

$$\sum_{k=0}^{n+l} b^k_{n,l}(q;x)q^{2k} = 1-(1-q^2)[n+l]_q x + q(1-q)^2[n+l][n+l-1]_q x^2, \tag{2.4}$$

where $b^k_{n,l}(q;x) = \binom{n+l}{k}_q x^k(1-x)^{n+l-k}_q$. In fact we have

$$\begin{aligned}
\sum_{k=0}^{n+l} b^k_{n,l}(q;x)q^k &= \sum_{k=0}^{n+l} b^k_{n,l}(q;x)\big(1-[k]_q+q[k]_q\big) \\
&= \sum_{k=0}^{n+l} b^k_{n,l}(q;x) - (1-q)[n+l]_q \sum_{k=0}^{n+l} b^k_{n,l}(q;x)\frac{[k]_q}{[n+l]_q} \\
&= 1-(1-q)[n+l]_q \sum_{k=0}^{n+l-1} \binom{n+l-1}{k}_q x^{k+1}(1-x)^{n+l-k-1}_q \\
&= 1-(1-q)[n+l]_q x
\end{aligned}$$

and

$$
\begin{aligned}
\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{2k} &= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)\Big(1-[k]_q+q^2[k]_q+q(1-q)^2[k]_q[k-1]_q\Big) \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)-(1-q^2)[n+l]_q\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{[k]_q}{[n+l]_q} \\
&\quad +q(1-q)^2[n+l]_q[n+l-1]_q\times\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{[k]_q}{[n+l]_q}\frac{[k-1]_q}{[n-1]_q} \\
&= 1-(1-q^2)[n+l]_q\sum_{k=0}^{n+l}\binom{n+l-1}{k}_q x^k x(1-x)_q^{n+l-k-1} \\
&\quad +q(1-q)^2[n+l]_q[n+l-1]_q \\
&\quad \times\sum_{k=0}^{n+l-2}\binom{n+l-2}{k}_q x^k x^2(1-x)_q^{n+l-k-2}.
\end{aligned}
$$

Now we prove the lemma.

$$
\begin{aligned}
L_{n,l}^{\alpha,\beta}(1;q;x) &= ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} 1 d_q^R t \\
&= ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}(1-q)\frac{([k+1]_q-[k]_q)}{([n+1]_q+\beta)}\sum_{s=0}^{\infty} q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)=1,
\end{aligned}
$$

which proves (i).

$$
\begin{aligned}
L_{n,l}^{\alpha,\beta}(t;q;x) &= ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t d_q^R t \\
&= ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}(1-q)\frac{([k+1]_q-[k]_q)}{([n+1]_q+\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{s=0}^{\infty} f\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{[k+1]_q-[k]_q}{[n+1]_q+\beta}q^s\right)q^s \\
=\ & \sum_{k=0}^{n+l} b_{n,l}^k(q;x)(1-q)\sum_{s=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{q^kq^s}{[n+1]_q+\beta}\right)q^s \\
=\ & \sum_{k=0}^{n+l} b_{n,l}^k(q;x)\left(\frac{[k]_q+\alpha}{([n+1]_q+\beta)}+\frac{q^k}{([n+1]_q+\beta)}\frac{1}{[2]_q}\right) \\
=\ & \frac{[n+l]_q}{[n+1]_q+\beta}\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{[k]_q+\alpha}{[n+l]_q} \\
&+\frac{1}{[2]_q([n+1]_q+\beta)}\left(1-(1-q)[n+l]_qx\right) \\
=\ & \frac{[n+l]_q}{[n+1]_q+\beta}\left[\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{[k]_q}{[n+l]_q}+\sum_{k=0}^{n+l} b_{n,p}^k(q;x)\frac{\alpha}{[n+l]_q}\right] \\
&+\frac{1-(1-q)[n+l]_qx}{[2]_q([n+1]_q+\beta)} \\
=\ & \frac{[n+l]_q}{[n+1]_q+\beta}\left[\sum_{k=0}^{n+l}\binom{n+l}{k}_q x^k(1-x)^{n+l-k}\frac{[k]_q}{[n+l]_q}+\frac{\alpha}{[n+l]_q}\right] \\
&+\frac{1-(1-q)[n+l]_qx}{[2]_q([n+1]_q+\beta)} \\
=\ & \frac{[n+l]_q}{[n+1]_q+\beta}\left[\sum_{k=1}^{n+l-1}\binom{n+l-1}{k-1}_q x^k(1-x)^{n+l-k-1}+\frac{\alpha}{[n+l]_q}\right] \\
&+\frac{1-(1-q)[n+l]_qx}{[2]_q([n+1]_q+\beta)} \\
=\ & \frac{[n+l]_q}{[n+1]_q+\beta}x+\frac{\alpha}{[n+l]_q}+\frac{1}{[2]_q([n+1]_q+\beta)}-\frac{(1-q)[n+l]_qx}{[2]_q([n+1]_q+\beta)} \\
=\ & \left[\frac{[n+l]_q}{[n+1]_q+\beta}-\frac{(1-q)[n+l]_q}{[2]_q([n+1]_q+\beta)}\right]x+\frac{\alpha}{[n+l]_q}+\frac{1}{[2]_q([n+1]_q+\beta)} \\
=\ & \frac{\alpha}{[n+l]_q}+\frac{1}{[2]_q([n+1]_q+\beta)}+\frac{2q[n+l]_q}{[2]_q([n+1]_q+\beta)}x,
\end{aligned}
$$

which proves (ii).

$$
\begin{aligned}
L_{n,l}^{\alpha,\beta}(t^2;q;x) &= ([n+1]_q+\beta)\sum_{k=0}^{n+l} b_{n,l}^k(q;x)q^{-k}\frac{[k+1]_q-[k]_q}{([n+1]_q+\beta)}\sum_{s=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}\right.\\
&\left.+\frac{[k+1]_q-[k]_q}{[n+1]_q+\beta}q^s\right)^2 q^s\\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)(1-q)\sum_{s=0}^{\infty}\left(\frac{([k]_q+\alpha)^2}{([n+1]_q+\beta)^2}+\frac{q^{2k}q^{2s}}{([n+1]_q+\beta)^2}\right.\\
&\left.+\frac{2q^kq^s([k]_q+\alpha)}{([n+1]_q+\beta)^2}\right)q^s\\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)\left(\frac{([k]_q+\alpha)^2}{([n+1]_q+\beta)^2}+\frac{2q^k([k]_q+\alpha)}{([n+1]_q+\beta)^2(1+q)}\right.\\
&\left.+\frac{q^{2k}}{([n+1]_q+\beta)^2(1+q+q^2)}\right)\\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{[k]_q^2}{([n+1]_q+\beta)^2}+\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{2\alpha[k]_q}{([n+1]_q+\beta)^2}\\
&+\sum_{k=0}^{n+l} b_{n,p}^k(q;x)\frac{\alpha^2}{([n+1]_q+\beta)^2}\\
&+\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{2q^k[k]_q}{([n+1]_q+\beta)^2[2]_q}+\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{2\alpha q^k}{([n+1]_q+\beta)^2[2]_q}\\
&+\sum_{k=0}^{n+l} b_{n,l}^k(q;x)\frac{q^{2k}}{([n+1]_q+\beta)^2[3]_q}\\
&= \frac{[n+l]_q}{[n+1]_q+\beta}x\left(\frac{1}{[n+1]_q+\beta}+q\frac{[n+l-1]_q}{[n+1]_q+\beta}x\right)\\
&+\frac{2\alpha[n+l]_q}{([n+1]_q+\beta)^2}x+\frac{\alpha^2}{([n+1]_q+\beta)^2}+\frac{2q[n+l]_q}{[2]_q([n+1]_q+\beta)^2}x-\frac{2q(1-q)}{[2]_q}\\
&\times\frac{[n+l]_q[n+l-1]_q}{([n+1]_q+\beta)^2}x^2+\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}\Big(1-(1-q)[n+l]_qx\Big)
\end{aligned}
$$

$$+\frac{1}{[3]_q([n+1]_q+\beta)^2}\left(1-(1-q^2)[n+l]_q x+q(1-q)^2[n+l]_q[n+l-1]_q x^2\right)$$

$$= \frac{1}{[3]_q([n+1]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}$$

$$+\left[\frac{[n+l]_q}{([n+1]_q+\beta)^2}+\frac{2\alpha[n+l]_q}{([n+1]_q+\beta)^2}+\frac{2q[n+l]_q}{[2]_q([n+1]_q+\beta)^2}\right.$$

$$\left.-\frac{2\alpha(1-q)[n+l]_q}{[2]_q([n+1]_q+\beta)^2}-\frac{(1-q^2)[n+l]_q}{[3]_q([n+1]_q+\beta)^2}\right]x+\left[\frac{q[n+l]_q[n+l-1]_q}{([n+1]_q+\beta)^2}\right.$$

$$\left.-\frac{2q(1-q)[n+l]_q[n+l-1]_q}{[2]_q([n+1]_q+\beta)^2}+\frac{q(1-q)^2[n+l]_q[n+l-1]_q}{[3]_q([n+1]_q+\beta)^2}\right]x^2$$

$$= \frac{1}{[3]_q([n+1]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}$$

$$+\frac{[n+l]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}$$

$$\times\left[[2]_q[3]_q+2\alpha[2]_q[3]_q+2q[3]_q-2\alpha(1-q)[3]_q-(1-q^2)[2]_q\right]x$$

$$+\frac{q[n+l]_q[n+l-1]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}\times\left[[2]_q[3]_q-2(1-q)[3]_q+(1-q)^2[2]_q\right]x^2$$

$$= \frac{1}{[3]_q([n+1]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}$$

$$+\frac{[n+l]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}$$

$$\times q\Big((1+q)(1+2q)+(1+q+q^2)(4\alpha+2)\Big)x$$

$$+\frac{q[n+l]_q[n+l-1]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}\times\Big((1+q+q^2)(3q-1)+(1+q^2-2q)(1+q)\Big)x^2$$

$$= \frac{1}{[3]_q([n+1]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}$$

$$+\frac{q[n+l]_q\big((3+4\alpha)+(5+4\alpha)q+4(1+\alpha)q^2\big)}{([n+1]_q+\beta)^2[2]_q[3]_q}x$$

$$+\frac{q^2[n+l]_q[n+l-1]_q(1+q+4q^2)}{([n+1]_q+\beta)^2[2]_q[3]_q}x^2.$$

which proves (iii).

Hence the lemma.

Remark 2.1. From the Lemma 2.1, we have

(i) $L_{n,l}^{\alpha,\beta}((t-x);q;x)=\left(\frac{2q[n+l]_q}{[2]_q([n+1]_q+\beta}-1\right)x+\frac{1}{[2]_q([n+1]_q+\beta)}+\frac{\alpha}{[n+l]_q},$

(ii) $L_{n,l}^{\alpha,\beta}((t-x)^2;q;x)=\frac{\alpha^2}{([n+1]_q+\beta)^2}+\frac{2\alpha}{([n+1]_q+\beta)^2[2]_q}+\frac{1}{([n+1]_q+\beta)^2[3]_q}$
$+\left(\frac{q[n+l]_q\big((3+4\alpha)+(5+4\alpha)q+4(1+\alpha)q^2\big)}{([n+1]_q+\beta)^2[2]_q[3]_q}-\frac{2}{([n+1]_q+\beta)^2[2]_q}-\frac{2\alpha}{[n+l]_q}\right)x$
$+\left(\frac{q^2[n+l]_q[n+l-1]_q(1_q+4q^2)}{[n+1]_q+\beta)^2[2]_q[3]_q}-\frac{4q[n+l]_q}{[n+1]_q+\beta)[2]_q}+1\right)x^2.$

Lemma 2.2. *For any $f\in C(I)$, we have $\|L_{n,l}^{\alpha,\beta}(f;q;.)\|_{C[0,1]}\leq\|f\|_{C[0,1+l]}$.*

3. DIRECT THEOREMS

In this section, we prove some direct theorems for the operators $L_{n,l}^{\alpha,\beta}(f;q;x)$.

Theorem 3.1. *Let $f\in C(I)$ and $0<q_n<1$. Then the sequence of the operators $L_{n,l}^{\alpha,\beta}(f;q_n;.)$ converges uniformly to f on the compact interval $[0,1]$ if and only if $\lim\limits_{n\to\infty}q_n=1$.*

Proof. (Forward) Suppose that $\lim\limits_{n\to\infty}q_n=1$. Then we shall show that $L_{n,l}^{\alpha,\beta}(f;q_n;.)$ converges to f uniformly on $[0,1]$. Note that for $0<q_n<1$ and $q_n\to\infty$ for $n\to\infty$, we get $[n+1]_{q_n}\to\infty$ as $n\to\infty$. Now it is easily seen that

$$\frac{[n+l]_{q_n}}{[n+1]_{q_n}+\beta}=1+q_n^n\frac{([l]_{q_n}-1)}{[n+1]_{q_n}}-\frac{\beta}{[n+1]_{q_n}+\beta}.$$

So when $n\to\infty$,

$$\frac{[n+l]_{q_n}}{[n+1]_{q_n}+\beta}\to 1 \text{ and } \frac{[n+l]_{q_n}}{([n+1]_{q_n}+\beta)^2}\to 0.$$

Using this and the Lemma 2.1, we find that $L_{n,l}^{\alpha,\beta}(1;q_n;x)\to 1$, $L_{n,l}^{\alpha,\beta}(t;q_n;x)\to x$ and $L_{n,l}^{\alpha,\beta}(t^2;q_n;x)\to x^2$ uniformly on the compact set $[0,1]$ as $n\to\infty$. Therefore, by the Korovkin's theorem it implies that the sequence $L_{n,l}^{\alpha,\beta}(f;q_n;.)$ converges uniformly to f on $[0,1]$.

We shall prove the converse by the method of contradiction. Suppose that the sequence (q_n) does not converge to 1. Then there must exist a subsequence (q_{n_i}) of the sequence (q_n) such that $q_{n_i}\in(0,1)$, $q_{n_i}\to\delta\in[0,1)$ as $i\to\infty$. Then

$$\frac{1}{[n_i+l]_{q_{n_i}}}=\frac{1-q_{n_i}}{1-(q_{n_i})^{n_i+l}}\to 1-\delta$$

as $i \to \infty$ because $(q_{n_i})^{n_i} \to 0$ as $i \to \infty$. Now if we choose $n = n_i, q = q_{n_i}$ in $L_{n,l}^{\alpha,\beta}(t; q; x)$ from the Lemma 2.1, then we get

$$L_{n,l}^{\alpha,\beta}(t; q; x) = \frac{2\delta}{(1+\delta)(1+\beta(1-\delta))}x + \frac{1-\delta}{1+\delta}\frac{1}{(1+\beta(1-\delta))} + \alpha(1-\delta),$$

which is different from x when $i \to \infty$, which contradicts our supposition. Therefore, $\lim_{n\to\infty} q_n = 1$. Hence the theorem is proved.

We define the following:

Let $f \in C(I)$, $\delta > 0$ and $W^2 = \{h : h', h'' \in C(I)\}$, then the Peetre's K-functional is defined by

$$K_2(f, \delta) = \inf_{h\in W^2}\{\|f - h\| + \delta\|h''\|\}.$$

By DeVore and Lorentz theorem (see [6]), there exists a constant $C > 0$ such that

$$K_2(f, \delta) \le C\omega_2(f, \sqrt{\delta}), \tag{3.1}$$

where $\omega_2(f, \sqrt{\delta})$, the second order modulus of continuity of $f \in C(I)$, is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0<p<\delta^{\frac{1}{2}}} \sup_{x\in I} |f(x + 2p) - 2f(x + p) + f(x)|.$$

Also by $\omega(f, \delta)$, we denote the first order modulus of continuity of $f \in C(I)$ defined as

$$\omega(f, \delta) = \sup_{0<p<\delta} \sup_{x\in I} |f(x + p) - f(x)|.$$

Next we prove the following theorem.

Theorem 3.2. *Let $L_{n,l}^{\alpha,\beta}(f; q; x)$ be the sequence of positive linear operators defined by (2.2) and $f \in C(I)$. Let (q_n) be the sequence with $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then there exists a constant $C > 0$, independent of n and x, such that*

$$\begin{aligned}\left|L_{n,l}^{\alpha,\beta}(f; q; x) - f(x)\right| \le{}& C\omega_2\left(f, \sqrt{\phi_{n,l}^{\alpha,\beta}(q_n; x)}\right)\\ &+ \omega\left(f, \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)}x - x\right),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\phi_{n,l}^{\alpha,\beta}(q_n; x) ={}& L_{n,l}^{\alpha,\beta}((t - x)^2; q_n; x)\\ &+ \left(\frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)}x - x\right)^2\end{aligned}$$

and $x \in [0, 1]$.

Proof. Let us define the associated operators

$$\bar{L}_{n,l}^{\alpha,\beta}(f;q_n;x)=L_{n,l}^{\alpha,\beta}(f;q_n;x)+f(x)$$
$$-f\left(\frac{\alpha}{[n+l]_{q_n}}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta)}+\frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}x\right). \quad (3.3)$$

In the light of the Lemma 2.1, it is easily seen that $\bar{L}_{n,l}^{\alpha,\beta}(1;q_n;x)=1$ and $\bar{L}_{n,l}^{\alpha,\beta}(t;q_n;x)=x$. Now from the Taylor's formula, for $g\in W^2$, we can write

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t (t-u)g''(u)du.$$

Operating by the $\bar{L}_{n,l}^{\alpha,\beta}$ on both sides of the above equation, we get

$$\bar{L}_{n,l}^{\alpha,\beta}(g;q_n;x)-g(x)=g'(x)\bar{L}_{n,l}^{\alpha,\beta}((t-x);q_n;x)+\bar{L}_{n,l}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du\right)$$

$$=\bar{L}_{n,l}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du,q_n;x\right)$$

$$=L_{n,l}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du,q_n;x\right)$$

$$-\int_x^{\frac{\alpha}{[n+l]_{q_n}}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta)}+\frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}x}\left(\frac{\alpha}{[n+l]_{q_n}}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta)}\right.$$
$$\left.+\frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}x-u\right)g''(u)du.$$

Therefore, we have

$$|\bar{L}_{n,l}^{\alpha,\beta}(g;q_n;x)-g(x)| \le L_{n,l}^{\alpha,\beta}((t-x)^2;q_n;x)\|g''\|_{C[0,1+l]}$$
$$+\left(\frac{\alpha}{[n+l]_{q_n}}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta)}\right.$$
$$\left.+\frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}x-x\right)^2\|g''\|_{C[0,1+l]}$$
$$=\phi_{n,l}^{\alpha,\beta}(q_n;x)\|g''\|_{C[0,1+l]}.$$

In view of (3.2), we obtain

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq |\bar{L}_{n,l}^{\alpha,\beta}(f-g;q_n;x)-g(x)|+|\bar{L}_{n,l}^{\alpha,\beta}(g;q_n;x)-g(x)| \\ &+\left|f\left(\frac{\alpha}{[n+l]_{q_n}}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta)}\right.\right. \\ &\left.\left.+\frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}x\right)-f(x)\right| \end{aligned}$$

and we have

$$\|\bar{L}_{n,l}^{\alpha,\beta}(f;q_n;x)\| \leq 3\|f\|_{C[0,1+l]},$$

by the Lemma 2.2, so we have

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq 4\|f-g\|_{C[0,1+l]}+\phi_{n,l}^{\alpha,\beta}(q_n;x)\|g''\|_{C[0,1+l]} \\ &+\omega\left(f,\left|\frac{1}{([n+1]_{q_n}+\beta)[2]_{q_n}}+\frac{[n+l]_{q_n}}{([n+1]_{q_n}+\beta)}\frac{2q_n}{[2]_{q_n}}x-x\right|\right). \end{aligned}$$

On taking the infimum of the right hand side running over all $g \in W^2$ and using the definition of the Peetre's K-functional, we get

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq 4K_2(f,\phi_{n,l}^{\alpha,\beta}(q_n;x)) \\ &+\omega\left(f,\frac{1}{([n+1]_{q_n}+\beta)[2]_{q_n}}+\frac{[n+l]_{q_n}}{([n+1]_{q_n}+\beta)}\frac{2q_n}{[2]_{q_n}}x-x\right|\right). \end{aligned}$$

In view of (3.1), we obtain

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq C\omega_2\left(f,\sqrt{\phi_{n,l}^{\alpha,\beta}(q_n;x)}\right) \\ &+\omega\left(f,\left|\frac{1}{([n+1]_{q_n}+\beta)[2]_{q_n}}+\frac{[n+l]_{q_n}}{([n+1]_{q_n}+\beta)}\frac{2q_n}{[2]_{q_n}}x-x\right|\right) \end{aligned}$$

and this completes the proof of the theorem.

Next, we shall obtain an estimate of the rate of convergence for the operators defined in (2.2) using the Lipschitz-type maximal functions defined as follows:

For $x \in [0,1]$ and $\xi \in (0,1]$, the Lipschitz-type maximal function is defined as

$$\tilde{\omega}_\xi(f,x)=\sup_{t\neq x,t\in[0,1+l]}\frac{|f(t)-f(x)|}{|t-x|^\xi}. \tag{3.4}$$

We prove the following theorem.

Theorem 3.3. *Let $f \in C(I), 0 < \xi \leq 1$ and $q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$. Then for every $x \in [0,1]$, we have*

$$|L_{n,l}^{\alpha,\beta}(f;q_n;x) - f(x)| \leq \tilde{\omega}_{\xi}(f,x)(\gamma_{n,l}(q_n;x))^{\frac{\xi}{2}},$$

where $\gamma_{n,l}(q_n;x) = L_{n,l}^{\alpha,\beta}((t-x)^2;q_n;x)$.

Proof. In the light of the Lemma (2.1), we have

$$|L_{n,l}^{\alpha,\beta}(f;q_n;x) - f(x)| \leq L_{n,l}^{\alpha,\beta}(|f(t)-f(x)|;q_n;x) \leq \tilde{\omega}_{\xi}(f,x)L_{n,l}^{\alpha,\beta}(|t-x|^{\xi};q_n;x) \quad (3.5)$$

and in view of (3.4), we have

$$|f(t) - f(x)| \leq \tilde{\omega}_{\xi}(f,x)|t-x|^{\xi}. \quad (3.6)$$

Using the Hölder's inequality, we obtain

$$|L_{n,l}^{\alpha,\beta}(f;q_n;x) - f(x)| \leq \tilde{\omega}_{\xi}(f,x)L_{n,l}^{\alpha,\beta}(|t-x|^2;q_n;x)^{\frac{\xi}{2}} = \tilde{\omega}_{\xi}(f,x)(\gamma_{n,l}(q_n;x))^{\frac{\xi}{2}}$$

and hence the theorem.

To prove the next theorem we consider the following Lipschitz-type space of functions:

$$\tilde{Lip}_M(s) = \left\{ f \in C(I) : |f(t) - f(x)| \leq M\frac{|t-x|^s}{(t+x)^{\frac{s}{2}}} \right\},$$

where M is a positive constant and $0 < s \leq 1$.
We prove the following theorem.

Theorem 3.4. *Let $f \in \tilde{Lip}_M(s), s \in (0,1]$ and $q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$. Then for each $x \in (0,1]$, we have*

$$|L_{n,l}^{\alpha,\beta}(f;q_n;x) - f(x)| \leq M\left(\frac{\gamma_{n,l}(q_n;x)}{x}\right)^{\frac{s}{2}}$$

where $\gamma_{n,l}(q_n;x) = L_{n,l}^{\alpha,\beta}((t-x)^2;q_n;x)$.

Proof. Firstly we will prove the result for $s = 1$. In fact we have, for $f \in \tilde{Lip}_M(1)$,

$$
\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq ([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |f(t)-f(x)|d_{q_n}^{R}t \\
&\leq M([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} \frac{|t-x|}{\sqrt{t+x}}d_{q_n}^{R}t.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and the fact $\frac{1}{\sqrt{t+x}} \leq \frac{1}{\sqrt{x}}$, we obtain

$$
\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq \frac{M}{\sqrt{x}}([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |t-x|d_{q_n}^{R}t \\
&= \frac{M}{\sqrt{x}}L_{n,l}^{\alpha,\beta}(|t-x|;q_n;x) \\
&\leq M\left(\frac{\gamma_{n,l}(q_n;x)}{x}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus the result is established for $s=1$. Next we prove the result for $0<s<1$. On using the Hölder's inequality twice, we get

$$
|L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| \leq ([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |f(t)-f(x)|d_{q_n}^{R}t
$$

$$
\leq \left\{\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)\left(([n+1]_{q_n}+\beta)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |f(t)-f(x)|d_{q_n}^{R}t\right)^{\frac{1}{s}}\right\}^{s}
$$

$$
\leq \left\{([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |f(t)-f(x)|^{\frac{1}{s}}d_{q_n}^{R}t\right\}^{s}.
$$

As $f \in \tilde{Lip}_M(s)$, we obtain

$$
\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f;q_n;x)-f(x)| &\leq M\left\{([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} \frac{|t-x|}{\sqrt{t+x}} d_{q_n}^R t\right\}^s \\
&\leq \frac{M}{x^{\frac{s}{2}}}\left\{([n+1]_{q_n}+\beta)\sum_{k=0}^{n+l} b_{n,l}^{k}(q_n;x)q_n^{-k}\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}} |t-x| d_{q_n}^R t\right\}^s \\
&= \frac{M}{x^{\frac{s}{2}}}(L_{n,l}^{\alpha,\beta}(|t-x|;q_n;x))^s \\
&\leq M\left(\frac{\gamma_{n,l}(q_n;x)}{x}\right)^{\frac{s}{2}}.
\end{aligned}
$$

This completes the proof of the theorem.

4. BIVARIATE OPERATORS

In what follows we shall construct the bivariate extension of the operators defined by (2.2).

Let $I_1 = [0, 1 + l_1]$ and $I_2 = [0, 1 + l_2]$. We consider $C(I_1 \times I_2)$, the space of all real valued continuous functions defined on $I_1 \times I_2$ equipped with the following norm

$$\|f\|_{C(I_1\times I_2)} = \sup_{(x,y)\in I_1\times I_2} |f(x,y)|.$$

We define the bivariate generalization of the operators in (2.2) as follows

$$
\begin{aligned}
L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(f(t,s);q_1,q_2;x,y) &= ([n_1+1]_{q_1}+\beta_1)([n_2+1]_{q_2}+\beta_2)\sum_{k_1=0}^{n_1+l_1}\sum_{k_2=0}^{n_2+l_2} q_1^{-k_1}q_2^{-k_2} \\
&\times b_{n_1,n_2;l_1,l_2}^{k_1,k_2}(q_1,q_2;x,y)\int_{\frac{[k_1]_{q_1}+\alpha_1}{[n_1+1]_{q_1}+\beta_1}}^{\frac{[k_1+1]_{q_1}+\alpha_1}{[n_1+1]_{q_1}+\beta}}\int_{\frac{[k_2]_{q_2}+\alpha_2}{[n+1]_{q_2}+\beta_2}}^{\frac{[k+1]_{q_2}+\alpha_2}{[n+1]_{q_2}+\beta_2}} f(t,s)d_{q_1}^R t d_{q_2}^R t, \quad (4.1)
\end{aligned}
$$

where

$$b_{n_1,n_2;l_1,l_2}^{k_1,k_2}(q_1,q_2;x,y) = \binom{n_1+l_1}{k_1}_{q_1}\binom{n_2+l_2}{k_2}_{q_2} x^{k_1}y^{k_2}(1-x)_{q_1}^{n_1+l_1-k_1}(1-x)_{q_2}^{n_2+l_2-k_2},$$

$f \in C(I_1 \times I_2)$, $0 < q_1, q_2 < 1$, $(x, y) \in [0,1] \times [0,1] = R^2$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are such that $0 \le \alpha_1 \le \beta_1;\ 0 \le \alpha_2 \le \beta_2$.
Now we prove a lemma concerning the above operators.

Lemma 4.1. *Let* $(t, s) \in I_1 \times I_2$, $(i, j) \in N^0 \times N^0$ *with* $i + j \le 2$, *and* $t^i s^j$ *by* $e_{ij}(t, s)$ *be the two dimensional test functions. Then the following equalities hold for the operators* (4.1)*:*

(i) $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(e_{00}; q_1, q_2; x, y) = 1,$

(ii) $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(e_{10}; q_1, q_2; x, y) = \dfrac{\alpha_1}{[n_1+l_1]_{q_1}} + \dfrac{1}{([n_1+1]_{q_1}+\beta_1)[2]_{q_1}} + \dfrac{2q_1[n_1+l_1]_{q_1}}{([n_1+1]_{q_1}+\beta_1)[2]_{q_1}}x,$

(iii) $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(e_{01}; q_1, q_2; x, y) = \dfrac{\alpha_2}{[n_2+l_2]_{q_2}} + \dfrac{1}{([n_2+1]_{q_2}+\beta_2)[2]_{q_2}} + \dfrac{2q_2[n_2+l_2]_{q_2}}{([n_2+1]_{q_2}+\beta_2)[2]_{q_2}}y,$

(iv) $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(e_{20}; q_1, q_2; x, y) = \dfrac{1}{([n_1+1]_{q_1}+\beta_1)^2[3]_{q_1}} + \dfrac{2\alpha_1}{([n_1+1]_{q_1}+\beta_1)^2[2]_{q_1}} + \dfrac{\alpha_1^2}{([n_1+1]_{q_1}+\beta_1)^2} + \dfrac{q_1[n_1+l_1]_{q_1}((3+4\alpha_1)+(5+4\alpha_1)q_1+4(1+\alpha_1){q_1}^2)}{([n_1+1]_{q_1}+\beta_1)^2[2]_{q_1}[3]_{q_1}}x + \dfrac{{q_1}^2[n_1+l_1]_{q_1}[n_1+l_1-1]_{q_1}(1+q_1+4{q_1}^2)}{([n_1+1]_{q_1}+\beta_1)^2[2]_{q_1}[3]_{q_1}}x^2,$

(v) $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(e_{02}; q_1, q_2; x, y) = \dfrac{1}{([n_2+1]_{q_2}+\beta_2)^2[3]_{q_2}} + \dfrac{2\alpha_2}{([n_2+1]_{q_2}+\beta_2)^2[2]_{q_2}} + \dfrac{\alpha_2^2}{([n_2+1]_{q_2}+\beta_2)^2} + \dfrac{q_2[n_2+l_2]_{q_2}((3+4\alpha_2)+(5+4\alpha_2)q_2+4(1+\alpha_2){q_2}^2)}{([n_2+1]_{q_2}+\beta_1)^2[2]_{q_2}[3]_{q_2}}y + \dfrac{{q_2}^2[n_2+l_2]_{q_2}[n_2+l_2-1]_{q_2}(1+q_2+4{q_2}^2)}{([n_2+1]_{q_2}+\beta_2)^2[2]_{q_2}[3]_{q_2}}y^2.$

Proof. In the light of the Lemma 2.1 and noting that

$$L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(t^i s^j; q_1, q_2; x, y) = L^{\alpha_1,\beta_1}_{n_1,l_1}(t^i; q_1, x) \times L^{\alpha_2,\beta_2}_{n_2,l_2}(s^j; q_2, y),\ \ for\ 0 \le i, j \le 2.$$

the proof is plain and straightforward. So we omit the details.
To prove the next theorem we first define the followings:
Let $f \in C(I_1 \times I_2)$ and $\delta_1, \delta_2 > 0$. Then the first order complete modulus of continuity for the bivariate case, denoted by $\omega(f; \delta_1, \delta_2)$ is defined as follows:

$$\omega(f, \delta_1, \delta_2) = sup\big\{|f(t, s) - f(x, y)| : |t - x| \le \delta_1, |s - y| \le \delta_2\big\}.$$

Two of the important properties of $\omega(f, \delta_1, \delta_2)$ are the following:

(i) $\omega(f, \delta_1, \delta_2) \to 0$ as $\delta_1 \to 0$ and $\delta_2 \to 0$,

(ii) $|f(t,s)-f(x,y)| \leq \omega(f,\delta_1,\delta_2)\left(1+\dfrac{|t-x|}{\delta_1}\right)\left(1+\dfrac{|s-y|}{\delta_2}\right)$.

Next we state and prove a theorem regarding the rate of convergence of the bivariate operators. To do it we consider a sequence (q_{n_i}) with $q_{n_i} \in (0,1)$ such that $q_{n_i} \to 1$ and $q_{n_i}^{n_i} \to a_i, (0 \leq a_i < 1)$ as $n_i \to \infty$ for $i = 1,2$. Also we denote $L_{n_1,l_1}^{\alpha_1,\beta_1}(t-x)^2; q_{n_1}, x)$ and $L_{n_2,l_2}^{\alpha_2,\beta_2}(s-y)^2; q_{n_2}, y)$ by $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ respectively. We have the following theorem.

Theorem 4.2. *Let $f \in C(I_1 \times I_2)$. Then for all $(x,y) \in R^2$, we have*

$$|L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)| \leq 4\omega(f,(\delta_{n_1}(x))^{\frac{1}{2}}(\delta_{n_2}(y))^{\frac{1}{2}}).$$

Proof. Using the fact that the operators $L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(f;q_{n_1},q_{n_2};x,y)$ are linear and positive together with the property (ii) of the modulus of continuity, we obtain

$$\begin{aligned}
&|L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)| \\
&\qquad \leq |L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2};x,y) - f(x,y)| \\
&\leq \omega(f;(\delta_{n_1}(x))^{\frac{1}{2}},(\delta_{n_2}(y))^{\frac{1}{2}}))\left(L_{n_1,l_1}^{\alpha_1,\beta_1}(1;q_{n_1},x) + \frac{1}{(\delta_{n_1}(x))^{\frac{1}{2}}} L_{n_1,l_1}^{\alpha_1,\beta_1}(|t-x|;q_{n_1},x)\right) \\
&\qquad \times \left(L_{n_2,l_2}^{\alpha_2,\beta_2}(1;q_{n_2},y) + \frac{1}{(\delta_{n_2}(y))^{\frac{1}{2}}} L_{n_2,l_2}^{\alpha_2,\beta_2}(|s-y|;q_{n_2},y)\right).
\end{aligned}$$

On applying the Cauchy-Schwartz inequality,

$$\begin{aligned}
|L_{n_1,n_2;l_1,l_2}^{\alpha_1,\alpha_2;\beta_1,\beta_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)| &\leq \omega(f;(\delta_{n_1}(x))^{\frac{1}{2}},(\delta_{n_2}(y))^{\frac{1}{2}})) \\
&\times \left(1 + \frac{1}{(\delta_{n_1}(x))^{\frac{1}{2}}} (L_{n_1,l_1}^{\alpha_1,\beta_1}((t-x)^2;q_{n_1},x))^{\frac{1}{2}}\right) \\
&\times \left(1 + \frac{1}{(\delta_{n_2}(y))^{\frac{1}{2}}} (L_{n_2,l_2}^{\alpha_2,\beta_2}((s-y)^2;q_{n_2},y))^{\frac{1}{2}}\right)
\end{aligned}$$

and we obtain the required result.

In this section we prove some theorems regarding the degree of approximation for the bivariate operators through the Lipschitz class. The Lipschitz class for the bivariate case, denoted by $Lip_M(\alpha_1,\alpha_2)$ is defined as under:

Let $0 < \alpha_1, \alpha_2 \leq 1$. A function f is said to be in $Lip_M(\alpha_1,\alpha_2)$ if it satisfies the following inequality:

$$|f(x,y)-f(x',y')|\leq M|x-x'^{\alpha_1}|y-y'^{\alpha_2}$$

for all $(x,y),(x',y')\in I_1\times I_2$. We have the following theorem.

Theorem 4.3. *Let $f\in Lip_M(\alpha_1,\alpha_2)$. Then*

$$|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)|\leq M\sqrt{(\delta_{n_1}(x))^{\alpha_1}}\sqrt{(\delta_{n_2}(y))^{\alpha_2}}$$

holds for all $(x,y)\in R^2$.

Proof. Using the hypothesis, we can write

$$\begin{aligned}|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)| \leq& L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2};x,y)\\ \leq& ML^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(|t-x|^{\alpha_1}|s-y|^{\alpha_2};q_{n_1},q_{n_2};x,y)\\ =& ML^{\alpha_1;\beta_1}_{n_1;l_1}(|t-x|^{\alpha_1};q_{n_1};x)L^{\alpha_2;\beta_2}_{n_2;l_2}(|s-y|^{\alpha_2};q_{n_2};y).\end{aligned}$$

Applying the Hölder's inequality, we get

$$\begin{aligned}|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)| \leq& ML^{\alpha_1;\beta_1}_{n_1;l_1}((t-x)^2;q_{n_1};x)^{\frac{\alpha_1}{2}}L^{\alpha_1;\beta_1}_{n_1;l_1}(1;q_{n_1};x)^{\frac{2-\alpha_1}{2}}\\ & L^{\alpha_2;\beta_2}_{n_2;l_2}((s-y)^2;q_{n_2};y)^{\frac{\alpha_2}{2}}L^{\alpha_2;\beta_2}_{n_2;l_2}(1;q_{n_2};y)^{\frac{2-\alpha_2}{2}}\\ =& M(\delta_{n_1}(x))^{\frac{\alpha_1}{2}}(\delta_{n_2}(y))^{\frac{\alpha_2}{2}}\\ =& M\sqrt{(\delta_{n_1}(x))^{\alpha_1}}\sqrt{(\delta_{n_2}(y))^{\alpha_2}}\end{aligned}$$

and this completes the proof of the theorem.

In the ensuing, we will use the following notations:

$$C^1(I_1\times I_2)=\left\{f\in C(I_1\times I_2):f'_x,f'_y\in C(I_1\times I_2)\right\}.$$

We prove the following theorem.

Theorem 4.4. *Let $f\in C^1(I_1\times I_2)$ and $(x,y)\in R^2$. Then*

$$\begin{aligned}&|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)|\\ &\qquad\leq \|f'_x\|_{C(I_1\times I_2)}\sqrt{\delta_{n_1}(x)}+\|f'_y\|_{C(I_1\times I_2)}\sqrt{\delta_{n_2}(y)}.\end{aligned}$$

Proof. For a fixed $(x,y)\in J^2$, let us write

$$f(t,s)-f(x,y)=\int\limits_x^t f'_u(u,s)d_qu+\int\limits_y^s f'_v(x,v)d_qv.$$

Operating by $L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}$ on both sides of the above equation, we get

$$\begin{aligned}|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)| \leq & L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}\left(|\int\limits_t^x |f'_u(u,s)|d_qu|;q_{n_1},q_{n_2};x,y\right)\\ &+L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}\left(|\int\limits_y^s |f'_v(x,v)|d_qv|;q_{n_1},q_{n_2};x,y\right).\end{aligned}$$

Since

$$|\int\limits_t^x |f'_u(u,s)|d_qu|\leq \|f'_x\|_{C(I_1\times I_2)}|t-x|$$

and

$$|\int\limits_y^s |f'_v(x,v)|d_qv|\leq \|f'_y\|_{C(I_1\times I_2)}|s-y|,$$

we obtain

$$\begin{aligned}&|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)|\\ &\qquad\leq \|f'_x\|_{C(I_1\times I_2)}L^{\alpha_1,\beta_1}_{n_1,l_1}\left(|t-x|;q_{n_1};x\right)+\|f'_y\|_{C(I_1\times I_2)}L^{\alpha_2,\beta_2}_{n_2,l_2}\left(|s-y|;q_{n_2};y\right).\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y)-f(x,y)|\\ \leq\ & \|f'_x\|_{C(I_1\times I_2)}\left(L^{\alpha_1,\beta_1}_{n_1,l_1}\left((t-x)^2;q_{n_1};x\right)\right)^{\frac{1}{2}}\left(L^{\alpha_1,\beta_1}_{n_1,l_1}\left(1;q_{n_1};x\right)\right)^{\frac{1}{2}}\\ &+\|f'_y\|_{C(I_1\times I_2)}\left(L^{\alpha_1,\beta_1}_{n_1,l_1}\left((s-y)^2;q_{n_2};y\right)\right)^{\frac{1}{2}}\left(L^{\alpha_1,\beta_1}_{n_1,l_1}\left(1;q_{n_2};y\right)\right)^{\frac{1}{2}}\\ =\ & \|f'_x\|_{C(I_1\times I_2)}\sqrt{\delta_{n_1}(x)}+\|f'_y\|_{C(I_1\times I_2)}\sqrt{\delta_{n_2}(y)}.\end{aligned}$$

Hence the theorem.

We define the following:
If $f \in C(I_1 \times I_2)$ and $\delta > 0$, then the partial moduli of continuity of f with respect to s and t, is defined by

$$\tilde{\omega_1}(f;\delta) = sup\big\{|f(x_1,t) - f(x_2,t)| : t \in I_2 \text{ and } |x_1 - x_2| \leq \delta\big\}$$

and

$$\tilde{\omega_2}(f;\delta) = sup\big\{|f(s,y_1) - f(s,y_2)| : s \in I_1 \text{ and } |y_1 - y_2| \leq \delta\big\}.$$

Overtly they satisfy the properties of the usual modulus of continuity.

Next we have the following theorem.

Theorem 4.5. *Let* $f \in C(I_1 \times I_2)$ *and* $(x,y) \in R^2$. *Then*

$$|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)| \leq 2\tilde{\omega_1}(f;\sqrt{\delta_{n_1}(x)}) + 2\tilde{\omega_1}(f;\sqrt{\delta_{n_1}(y)})$$

holds.

Proof. Making use of the definition of partial moduli of continuity, we obtain

$$\begin{aligned}
|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)| &\leq L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(|f(t,s) - f(x,y)|;q_{n_1},q_{n_2};x,y) \\
&\leq L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(|f(t,s) - f(t,y)|;q_{n_1},q_{n_2};x,y) \\
&\quad + L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}(|f(t,y) - f(x,y)|;q_{n_1},q_{n_2};x,y) \\
&\leq \tilde{\omega_1}(f;\delta_{n_1}(x))\big(L^{\alpha_1,\beta_1}_{n_1,l_1}(1;q_{n_1};x) \\
&\quad + \frac{1}{\sqrt{\delta_{n_1}(x)}} L^{\alpha_1,\beta_1}_{n_1,l_1}(|t-x|;q_{n_1};x)\big) \\
&+ \tilde{\omega_2}(f;\delta_{n_2}(y))\big(L^{\alpha_2,\beta_2}_{n_2,l_2}(1;q_{n_2};y) \\
&\quad + \frac{1}{\sqrt{\delta_{n_2}(y)}} L^{\alpha_2,\beta_2}_{n_2,l_2}(|s-y|;q_{n_2};y)\big).
\end{aligned}$$

Applying the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|L^{\alpha_1,\alpha_2;\beta_1,\beta_2}_{n_1,n_2;l_1,l_2}&(f;q_{n_1},q_{n_2};x,y)-f(x,y)| \\
&\leq \tilde{\omega_1}(f;\delta_{n_1}(x))\big(1+\frac{1}{\sqrt{\delta_{n_1}(x)}}L^{\alpha_1,\beta_1}_{n_1,l_1,}((t-x)^2;q_{n_1};x)\big) \\
&\quad+\tilde{\omega_2}(f;\delta_{n_2}(y))\big(1+\frac{1}{\sqrt{\delta_{n_2}(y)}}L^{\alpha_2,\beta_2}_{n_2,l_2}((s-y)^2;q_{n_2};y)\big) \\
&=2\tilde{\omega_1}(f;\sqrt{\delta_{n_1}(x)})+2\tilde{\omega_2}(f;\sqrt{\delta_{n_1}(y)})
\end{aligned}
$$

and the proof is completed.

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REFERENCES

[1] P.N. Agarwal, V. Gupta, A.S. Kumar, On q-analogue of Bernstein-Schurer-Stancu operators, *Appl. Math. Comput.* 219 **14** (2013), 7754-7764.

[2] P.N. Agarwal, A.S. Kumar, T.A.K. Sinha, Stancu type generalization of modified Schurer operators based on q-integers, *Appl. Math. Comput.*, **226** (2014), 765-776.

[3] P.N. Agarwal, Zoltán Finta, A Satish Kumar, Bernstein-Schurer-Kantorovich operators based on q-integers, *Appl. Math. Comput.*, **256** (2015), 222-231.

[4] A. Aral, Generalization of Szász-Mirakyan operators based on q-integers, *Math. Comput. Modelling*, **49 (9-10)** (2008), 1052-1062.

[5] A. Aral, V. Gupta, R.P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, 2013.

[6] R.A. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[7] H. Gauchman, Integral inequalities in q-calculs, *Comput. Math. Appl.*, **47** (2004), 281-300.

[8] F.H. Jackson, On a q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193-203.

[9] A. Lupaş, A q-analogue of the Bernstein operator, *Rocky Mountain J. Math.* 36 **5** (2006), 1615-1629.

[10] S. Marinković, P. Rajković, M. Stanković, The inequalities for some types of q-integrals, *Comput. Math. Appl.*, **56** (2008), 2490-2498.

[11] C.V. Muraru, Note on q-Bernstein-Schurer operators, *Studia Univ. Babeş-Bolyai, Mathematica*, 56 **2** (2011), 489-495.

[12] M. Mursaleen, Asif Khan, H.M. Srivastava, K.S. Nisar, Operators constructed by means of q-Lagrange polynomials and A-statistical approximation, *Appl. Math. Comput.*, **219** (2013), 6911-6918.

[13] M. Mursaleen, Asif Khan, Generalized q-Bernstein-Schurer operators and some approximation theorems, *J. Funct. Spaces*, (2013), Article ID 719834, 7 pages.

[14] M. Mursaleen, Asif Khan, Statistical approximation properties of modified q-Stancu-Beta operators, *Bull. Malays. Math. Sci. Soc.*(2), 36 **3** (2013), 683-690.

[15] M. Mursaleen, Faisal Khan, Asif Khan, Approximation properties for modified q-Bernstein-Kantorovich operators, *Numer. Funct. Anal. Optim.*, 36 **9** (2015), 1178-1197.

[16] M. Mursaleen, Faisal Khan, Asif Khan, Approximation properties for King's type modified q-Bernstein-Kantorovich operators, *Math. Meth. Appl. Sci.*, 38 **2015**, 5242-5252.

[17] M. Örkcü, O. Doğru, Statistical approximaion of a kind of Kantorovich type q-Szász-Mirakjan operators, *Nonlinear Anal.*, 75 **5** (2012), 2874-2882.

[18] S. Ostrovska, q-Bernstein polynomials and their iterates, *J. Approx. Theory*, 123 **2** (2003), 232-255.

[19] G.M. Phillips, Bernstein polynomials based on the q-integers, *Ann. Numer. Math.*, **4** (1997), 511-518.

[20] M. Ren, X.M. Zeng, King type modification of q-Bernstein-Schurer operators, *Czechoslovak Math. J.*, 63 **138** (2013), 805-817.