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EXISTENCE OF MINIMAL AND MAXIMAL SOLUTIONS FOR A QUASILINEAR DIFFERENTIAL EQUATION WITH NONLOCAL BOUNDARY CONDITION ON THE HALF-LINE

MOHAMMED DERHAB[1] AND HAYAT MEKNI[2]

[1]Dynamic Systems and Applications Laboratory, Department of Mathematics,
Faculty of Sciences, University Abou-Bekr Belkaid Tlemcen
B.P.119, Tlemcen, 13000, Algeria
E-mail: derhab@yahoo.fr

[2]Dynamic Systems and Applications Laboratory, Department of Mathematics,
Faculty of Sciences, University Abou-Bekr Belkaid Tlemcen
B.P.119, Tlemcen, 13000, Algeria
E-mail: h.mekni1@yahoo.fr

ABSTRACT. This work is concerned with the construction of minimal and maximal solutions for a second order quasilinear differential equation on the half-line with nonlocal boundary condition and without condition at infinity, where the nonlinearity is a continuous function depending on the first derivative of the unknown function. We also give an example to illustrate our results.

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1. INTRODUCTION

The purpose of this work is the construction of minimal and maximal solutions for a class of second order quasilinear differential equation on the half-line with nonlocal boundary condition and without condition at infinity. More specifically, we consider the following nonlinear boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = f(x,u,u'),\ x \in I, \\ u(0) - au'(0) = L(u), \end{cases} \tag{1.1}$$

where $\varphi_p(y) = |y|^{p-2}\,y$, $y \in \mathbb{R}$, $p > 1$, $I = \left]0,+\infty\right[$, $f : \overline{I} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $L : C^1(\overline{I}) \to \mathbb{R}$ is an increasing continuous function and a is a positive real number.

Boundary value problems on infinite intervals arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena, such as the theory of drain flows, the theory of colloids, plasma physics, unsteady flow of gas through a semi-infinite porous media, semi-conductor devices,

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non-Newtonian fluid theory, nuclear physics and in determining the electrical potential in an isolated neutral atom (see[[1]- [6]], [9], [17], [20], [23], [24], [34], [35] and [42]).

Boundary value problems with nonlocal boundary conditions on semi-infinite intervals have been studied by several authors using the upper and lower solutions method, the monotone iterative method, the coincidence degree theory of Mawhin and fixed point theorems in cones (see [22], [[26]- [31]], [[47]- [48]]). We note also that boundary value problems on unbounded domains and without conditions at infinity have been studied by several authors (see [11], [[27]- [28]], [32] and [[40]- [41]]).

It is well known that the method of upper and lower solutions coupled with monotone iterative technique has been used to prove existence of minimal and maximal solutions of nonlinear boundary value problems on unbounded domains by various authors (see [15], [19], [[36]- [37]], [38] and [[40]- [41]]).

The purpose of this work is to show that it can be applied successfully to problems of type (1.1) and consequently our results improve and generalize those obtained in [36], [40] and [41, Chapter 7 Section 10]. To the best of our knowledge this is the first paper which uses the method of upper and lower solutions coupled with monotone iterative technique to prove the existence of minimal and maximal solutions for quasilinear boundary value problems of type (1.1).

The plan of this paper is organized as follows. In Section 2, we give some definitions. The main result is presented and proved in Section 3, followed by an example in Section 4 to illustrate the application of our result.

2. DEFINITIONS

In this section, we give some definitions.

First let$A$ be a subset of $\mathbb{R}$, we denote by $C^1_{loc}(A)$ the space of functions which are $C^1(K)$ for every bounded domain K in A.

Definition 2.1. *We say that u is a solution of (1.1) if*

(i) $u \in C^1_{loc}(\overline{I})$ *and* $\varphi_p(u') \in C^1_{loc}(I)$.

(ii) $$\begin{cases} -(\varphi_p(u'))' = f(x,u,u'),\ x \in (0,b)\,, \text{ for each } b > 0, \\ u(0) - au'(0) = L(u). \end{cases}$$

Definition 2.2. *We say that $\underline{u}$ is a lower solution of (1.1) if*

(i) $\underline{u} \in C^1_{loc}(\overline{I})$ *and* $\varphi_p(\underline{u}') \in C^1_{loc}(I)$.

(ii) $$\begin{cases} -(\varphi_p(\underline{u}'))' \leq f(x,\underline{u},\underline{u}'),\ x \in (0,b)\,, \text{ for each } b > 0, \\ \underline{u}(0) - a\underline{u}'(0) \leq L(\underline{u}). \end{cases}$$

Definition 2.3. *We say that $\overline{u}$ is an upper solution of (1.1) if*

(ii) $\overline{u} \in C^1_{loc}(\overline{I})$ *and* $\varphi_p(\overline{u}') \in C^1_{loc}(I)$.

$$\text{(ii)} \begin{cases} -(\varphi_p(\overline{u}'))' \geq f(x, \overline{u}, \overline{u}'), \ x \in (0, b)\,, \textit{ for each } b > 0, \\ \\ \overline{u}(0) - a\overline{u}'(0) \geq L(\overline{u}). \end{cases}$$

Definition 2.4. *Nagumo-Wintner conditions*

We say that the function $f : \overline{I} \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies Nagumo-Wintner conditions with respect to the pair of functions* $\underline{u}$ *and* $\overline{u}$, *if there exist* $C \geq 0$ *and a functions* $Q \in L^p(\overline{I})$ *and* $\Psi : [0, +\infty) \to (0, +\infty)$ *continuous such that*

$$|f(x, u, v)| \leq \Psi(|v|) \left(Q(x) + C\,|v|^{\frac{1}{p-1}}\right), \tag{2.1}$$

for all $(x, u, v) \in D$, *where*

$$D = \left\{(x, u, v) \in \overline{I} \times \mathbb{R}^2 : \underline{u}(x) \leq u(x) \leq \overline{u}(x)\right\},$$

and

$$\int_0^{+\infty} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds = +\infty. \tag{2.2}$$

3. MAIN RESULT

In this section, we state and prove our main result.

On the nonlinearity f, we shall impose the following condition

(H) There exists a positive real number M such that $u \mapsto f(x, u, v) + M\varphi_p(u)$ is increasing, for all $x \in \overline{I}$, and all $v \in \mathbb{R}$.

Also, we will assume the existence of an ordered pair of lower and upper solutions $\underline{u}$ and $\overline{u}$ satisfying

$$\underline{u}(x) \leq \overline{u}(x), \text{ for all } x \in \overline{I}.$$

The main result of this work is the following Theorem.

Theorem 3.1. *Let* $\underline{u}$ *and* $\overline{u}$ *be lower and upper solutions respectively for problem (1.1) such that* $\underline{u} \leq \overline{u}$ *in* $\overline{I}$. *Assume that the condition (*H*) is satisfied and the Nagumo-Wintner conditions relative to* $\underline{u}$ *and* $\overline{u}$ *holds. Then the problem (1.1) has a maximal solution* u^* *and a minimal solution* u_* *such that for every solution* u *of (1.1) with* $\underline{u} \leq u \leq \overline{u}$ *in* $\overline{I}$, *we have*

$$\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u} \textit{ on } \overline{I}.$$

Proof. The proof will be given in several steps.

Let us consider the following auxiliary boundary value problem

$$\begin{cases} -(\varphi_p(u_n'))' = f(x, u_n, u_n'), \ x \in I_n, \\ u_n(0) - a u_n'(0) = L(u_n), \\ u_n(n) = \overline{u}(n), \end{cases} \qquad (P_n)$$

where $I_n =]0, n[$ with $n \in \mathbb{N}^*$.

By using a proof similar to that of Theorem 4 in [12], we prove that the problem (P_n) has a maximal solution $\overline{u}_n$ and a minimal solution $\underline{u}_n$ such that

$$\underline{u} \leq \underline{u}_n \leq \overline{u}_n \leq \overline{u} \text{ on } [0, n].$$

Now we construct the sequences of functions $\left(\overline{U}_n\right)_{n\geq 1}$ and $(\underline{U}_n)_{n\geq 1}$ in the following way

$$\overline{U}_n(x) = \begin{cases} \overline{u}_n(x), \ x \in \overline{I}_n, \\ \overline{u}(x), \ x \in I \backslash \overline{I}_n, \end{cases}$$

and

$$\underline{U}_n(x) = \begin{cases} \underline{u}_n(x), \ x \in \overline{I}_n, \\ \underline{u}(x), \ x \in I \backslash \overline{I}_n. \end{cases}$$

Step 1: For all $n \in \mathbb{N}^*$, we have

$$\underline{u} \leq \underline{U}_1 \leq ... \leq \underline{U}_n \leq \underline{U}_{n+1} \leq \overline{U}_{n+1} \leq \overline{U}_n \leq ... \leq \overline{U}_1 \leq \overline{u} \text{ on } \overline{I}.$$

Proof: It is not difficult to see that for all $n \in \mathbb{N}^*$, we have $\underline{u} \leq \overline{U}_n \leq \overline{u}$ and $\underline{u} \leq \underline{U}_n \leq \overline{u}$ on $\overline{I}$.

Now, we are going to prove that

$$\forall n \in \mathbb{N}^*, \ \overline{U}_{n+1} \leq \overline{U}_n \text{ and } \underline{U}_n \leq \underline{U}_{n+1} \text{ on } \overline{I}.$$

Let $n \in \mathbb{N}^*$, we distinguish three cases

Case1 If $x \in I \backslash \overline{I}_{n+1}$, we have

$$\underline{u}(x) \leq \overline{u}(x) = \overline{U}_{n+1}(x) = \overline{U}_n(x) \leq \overline{u}(x).$$

Case2 If $x \in \overline{I}_{n+1} \backslash \overline{I}_n$, we have

$$\underline{u}(x) \leq \overline{u}_{n+1}(x) = \overline{U}_{n+1}(x) \leq \overline{u}(x) = \overline{U}_n(x).$$

Case3 If $x \in \overline{I}_n$, we have $\overline{U}_{n+1}(x) = \overline{u}_{n+1}(x)$ and since the function $\overline{u}_{n+1}$ satisfies the first equation of (P_{n+1}) and $\overline{I}_n \subset \overline{I}_{n+1}$, we obtain

$$-(\varphi_p(\overline{u}_{n+1}'))' = f(x, \overline{u}_{n+1}, \overline{u}_{n+1}'), \ x \in I_n. \qquad (3.1)$$

On the other hand, one has

$$\begin{cases} \overline{u}_{n+1}(0) - a\overline{u}'_{n+1}(0) = L(\overline{u}_{n+1}), \\ \overline{u}_{n+1}(n) \leq \overline{u}(n), \end{cases}$$

which means that $\overline{u}_{n+1}$ is a lower solution of (P_n) and consequently, we have $\overline{u}_{n+1} \leq \overline{u}_n$ on $\overline{I}_n$.

Thus, we obtain

$$\overline{U}_{n+1} \leq \overline{U}_n \text{ on } \overline{I}.$$

Similarly, we prove that

$$\underline{U}_n \leq \underline{U}_{n+1} \text{ on } \overline{I}.$$

Now we are going to prove that

$$\forall n \in \mathbb{N}^*, \, \underline{U}_n \leq \overline{U}_n \text{ on } \overline{I}.$$

Let $n \in \mathbb{N}^*$, we have

$$\begin{cases} \underline{U}_n(x) = \underline{u}(x) \leq \overline{u}(x) = \overline{U}_n(x), \text{ if } x \in I \backslash \overline{I}_n, \\ \underline{U}_n(x) = \underline{u}_n(x) \leq \overline{u}_n(x) = \overline{U}_n(x), \text{ if } x \in \overline{I}_n, \end{cases}$$

then, it follows that

$$\forall n \in \mathbb{N}^*, \, \underline{U}_n \leq \overline{U}_n \text{ on } \overline{I}.$$

The proof of **Step 1** is complete**.**

Now let $A = [0, b]$ with $b > 0$, then there exists a integer l such that $A \subset \overline{I}_l$.

For each $k \geq l + 1$, the restriction of $\overline{U}_k$ to $\overline{I}_l$ satisfies

$$\begin{cases} -(\varphi_p(\overline{U}_k)')' = f(x, \overline{U}_k, \overline{U}'_k), \, x \in I_l, \\ \overline{U}_k(0) - a\overline{U}'_k(0) = L(\overline{U}_k), \\ \overline{U}_k(l) \leq \overline{u}(l). \end{cases}$$

Step 2: There exists a constant K_l such that $\left\|\overline{U}'_k\right\|_0 := \max_{x \in \overline{I}_l} \left|\overline{U}'_k(x)\right| \leq K_l$, for all $k \geq l + 1$.

Proof: We put by definition

$$S = 2\max\{\|\overline{u}\|_0, \|\underline{u}\|_0\},$$

and

$$\eta_l = \max\{|\underline{u}(l) - \overline{u}(0)|, |\overline{u}(l) - \underline{u}(0)|\},$$

Take $K_l \geq \max\{\|\overline{u}'\|_0, \|\underline{u}'\|_0, \eta_l\}$ such that

$$\int_{\varphi_p(\eta_l)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds > \|Q\|_p S^{\frac{p-1}{p}} + CS^{\frac{p-1}{p}}. \tag{3.2}$$

Since $\underline{u}(x) \leq \overline{U}_k(x) \leq \overline{u}(x)$, for all x in $\overline{I}_l$, then we have

$$\underline{u}(l) - \overline{u}(0) \leq \overline{U}_k(l) - \overline{U}_k(0) \leq \overline{u}(l) - \underline{u}(0).$$

By the mean value theorem, there exists $x_1 \in (0, l)$ such that

$$\overline{U}_k(l) - \overline{U}_k(0) = l\overline{U}'_k(x_1).$$

Then

$$\left|\overline{U}'_k(x_1)\right| \leq \eta_l.$$

We put by definition

$$L = \left|\overline{U}'_k(x_1)\right|.$$

Suppose on the contrary that there exists $\widetilde{x} \in [0, l]$ such that $\left|\overline{U}'_k(\widetilde{x})\right| > K_l$, then by the continuity of $\overline{U}'_k$, we can choose $x_2 \in [0, l]$ verifying one of the following situations:

(i) $\overline{U}'_k(x_1) = L, \overline{U}'_k(x_2) = K_l$ and $L \leq \overline{U}'_k(x) \leq K_l$, for all $x \in (x_1, x_2)$.

(ii) $\overline{U}'_k(x_1) = L, \overline{U}'_k(x_2) = K_l$ and $L \leq \overline{U}'_k(x) \leq K_l$, for all $x \in (x_2, x_1)$.

(iii) $\overline{U}'_k(x_1) = -L, \overline{U}'_k(x_2) = -K_l$ and $-K_l \leq \overline{U}'_k(x) \leq -L$, for all $x \in (x_2, x_1)$.

(iv) $\overline{U}'_k(x_1) = -L, \overline{U}'_k(x_2) = -K_l$ and $-K_l \leq \overline{U}'_k(x) \leq -L$, for all $x \in (x_2, x_1)$.

Assume that the case (i) holds. The others can be handled in similar way.

Since $\overline{U}_k$ is a solution of the problem (P_l) and by Nagumo-Wintner conditions, then for each $x \in (x_1, x_2)$, we have

$$\begin{aligned}
(\varphi_p(\overline{U}'_k))'(x) &\leq \left|f(x, \overline{U}_k(x), \overline{U}'_k(x))\right| \\
&\leq \Psi(\left|\overline{U}'_k(x)\right|)(Q(x) + C\left|\overline{U}'_k(x)\right|^{\frac{1}{p-1}}).
\end{aligned} \tag{3.3}$$

Since $L \leq \eta_l$ and φ_p is increasing, one has

$$\int_{\varphi_p(\eta_l)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds \leq \int_{\varphi_p(L)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds. \tag{3.4}$$

Now if we put $s = \varphi_p(\overline{U}'_k(x))$, we obtain

$$\int_{\varphi_p(L)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds = \int_{x_1}^{x_2} \frac{\left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}}}{\Psi(\overline{U}'_k(x))} (\varphi_p(\overline{U}'_k))'(x) dx, \tag{3.5}$$

then by (3.3),(3.4) and (3.5), it follows that

$$\begin{aligned}
\int_{\varphi_p(L)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds &= \int_{x_1}^{x_2} \frac{\left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}}}{\Psi(\overline{U}'_k(x))} (\varphi_p(\overline{U}'_k))'(x) dx \\
&\leq \int_{x_1}^{x_2} \left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}} (Q(x) + C \left|\overline{U}'_k(x)\right|^{\frac{1}{p-1}}) dx.
\end{aligned}$$

This means that

$$\int_{\varphi_p(L)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds \leq \int_{x_1}^{x_2} \left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}} (Q(x) + C \left|\overline{U}'_k(x)\right|^{\frac{1}{p-1}}) dx. \tag{3.6}$$

By (3.4), (3.6) and using Hölder inequality, one has

$$\begin{aligned}
\int_{\varphi_p(\eta_l)}^{\varphi_p(K_l)} \frac{s^{\frac{1}{p}}}{\Psi(|s|^{\frac{1}{p-1}})} ds &\leq \int_{x_1}^{x_2} \left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}} Q(x) dx + C \int_{x_1}^{x_2} \left(\varphi_p(\overline{U}'_k(x))\right)^{\frac{1}{p}} \left|\overline{U}'_k(x)\right|^{\frac{1}{p-1}} dx \\
&\leq \int_{x_1}^{x_2} \left(\overline{U}'_k(x)\right)^{\frac{p-1}{p}} Q(x) dx + C \int_{x_1}^{x_2} \left(\overline{U}'_k(x)\right)^{\frac{p-1}{p}} dx \\
&\leq \|Q\|_p \left(\int_{x_1}^{x_2} \left(\left(\overline{U}'_k(x)\right)^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\quad + C \left(\int_{x_1}^{x_2} \left(\left(\overline{U}'_k(x)\right)^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&= \|Q\|_p \left(\int_{x_1}^{x2} \overline{U}'_k(x) dx \right)^{\frac{p-1}{p}} + C \left(\int_{x_1}^{x_2} \overline{U}'_k(x) dx \right)^{\frac{p-1}{p}} \\
&= \|Q\|_p \left(\overline{U}_k(x_2) - \overline{U}_k(x_1)\right)^{\frac{p-1}{p}} + C \left(\overline{U}_k(x_2) - \overline{U}_k(x_1)\right)^{\frac{p-1}{p}} \\
&\leq (\|Q\|_p + C) S^{\frac{p-1}{p}},
\end{aligned}$$

which is a contradiction with (3.2) and consequently, we have

$$\left\|\overline{U}'_k\right\|_0 \leq K_l, \text{ for all } k \geq l+1.$$

The proof of **Step 2** is complete**.**

Step 3: The sequence $(\overline{U}_k)_{k\geq l+1}$ converges to a maximal solution of (1.1).

Proof: By **Step 2**, we have $\left\|\overline{U}'_k\right\|_0 \leq K_l$ for all $k \geq l+1$, then the sequence$(\overline{U}_k)_{k\geq l+1}$ is uniformly bounded in $C^1([0,l])$.

Now we are going to prove that the sequence $(\overline{U}'_k)_{k\geq l+1}$ is equicontinuous on $[0,l]$. For this let $\varepsilon > 0$ and $t, s \in [0,l]$ such that $t < s$, then for each $k \geq l+1$, one has

$$\begin{aligned}
\left|\varphi_p(\overline{U}'_{k+1}(s)) - \varphi_p(\overline{U}'_{k+1}(t))\right| &= \left|\int_t^s f(x, \overline{U}_{k+1}(x), \overline{U}'_{k+1}(x))dx\right| \\
&\leq \int_t^s \left|f(x, \overline{U}_{k+1}(x), \overline{U}'_{k+1}(x))\right| dx \\
&\leq M(f)\left|s-t\right|,
\end{aligned}$$

where

$$M(f) = \max\{\left|f(x,y,z)\right|, x \in \overline{I}_l,\ \underline{u} \leq y \leq \overline{u} \text{ and } \ |z| \leq K_l\}.$$

If we choose $|s-t| \leq \dfrac{\varepsilon}{M(f)+1}$, we obtain

$$\left|\varphi_p(\overline{U}'_{k+1}(s)) - \varphi_p(\overline{U}'_{k+1}(t))\right| < \varepsilon,$$

which means that the sequence $\left(\varphi_p(\overline{U}'_k)\right)_{k\geq l+1}$ is equicontinuous on $[0,l]$.

Now, since the mapping φ_p^{-1} is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, we deduce from

$$\left|\overline{U}'_{k+1}(s) - \overline{U}'_{k+1}(t)\right| = \left|\varphi_p^{-1}\left(\varphi_p(\overline{U}'_{k+1}(s))\right) - \varphi_p^{-1}\left(\varphi_p(\overline{U}'_{k+1}(t))\right)\right|,$$

that the sequence $(\overline{U}'_k)_{k\geq l+1}$ is equicontinuous on $[0,l]$. Hence by the Arzela-Ascoli theorem, there exists a subsequence $(\overline{U}_{k_j})_{k_j\in\mathbb{N}}$ of $(\overline{U}_k)_{k\geq l+1}$ which converges in $C^1([0,l])$.

Let

$$u = \lim_{k_j\to+\infty} \overline{U}_{k_j},$$

then

$$u' = \lim_{k_j\to+\infty} \overline{U}'_{k_j}.$$

But by **Step 1**, the sequence $(\overline{U}_k)_{k\geq l+1}$ is decreasing and bounded from below, then the pointwise limit of this sequence exists and it is denoted by u^*. Hence we have $u = u^*$ and moreover, the whole sequence converges in $C^1(\overline{I}_l)$ to u^*.

Let $x \in (0, l)$, we have

$$-\varphi_p(\overline{U}'_{k+1}(x)) = -\varphi_p(\overline{U}'_{k+1}(0)) + \int_0^x f(s, \overline{U}_{k+1}(s), \overline{U}'_{k+1}(s))ds$$

Now, as $k \to +\infty$, we obtain

$$f(s, \overline{U}_{k+1}, \overline{U}'_{k+1}) \underset{k\to+\infty}{\longrightarrow} f(s, u^*, u^{*\prime}).$$

Also, we have

$$\exists K > 0,\ \forall k \geq l+1, \forall s \in [0, l], \left| f(s, \overline{U}_{k+1}, \overline{U}'_{k+1}) \right| \leq K.$$

Hence, the dominated convergence theorem of Lebesgue implies that

$$-\varphi_p(u^{*\prime}(x)) = -\varphi_p(u^{*\prime}(0)) + \int_0^x f(s, u^*(s), u^{*\prime}(s))ds.$$

Thus, we obtain

$$\forall x \in (0, l),\ -(\varphi_p(u^{*\prime}))' = f(x, u^*, u^{*\prime}).$$

Also, we have

$$u^*(0) - au^{*\prime}(0) = L(u^*).$$

Consequently u^* is a solution of (P_l) and since A is an arbitrary bounded domain, then it is a solution of (1.1).

Now, we are going to prove that u^* is a maximal solution; i.e., if u is another solution of (1.1) such that $\underline{u} \leq u \leq \overline{u}$ on $\overline{I}$, then $u \leq u^*$ on $\overline{I}$.

Since u is a lower solution of (1.1), then by **Step 1,** we have

$$\forall n \in \mathbb{N},\ u \leq \overline{U}_n.$$

Letting $n \longrightarrow +\infty$, we obtain

$$u \leq \lim_{n\to+\infty} \overline{U}_n = u^*,$$

which means that u^* is a maximal solution of problem (1.1).

The proof of **Step 3** is complete**.**

Similarly, we can prove that the sequence $(\underline{U}_n)_{n\in\mathbb{N}}$ converges to a minimal solution of (1.1).

The proof of our result is complete. $\square$

4. APPLICATION

In this section, we apply the previous result to the following problem

$$\begin{cases} -(\varphi_p(u'))' = \dfrac{|u'(x)|^{\frac{p-1}{p}}}{(x^2+1)^{\frac{1}{p}}} \left(g(x+u(x)) - g(x)\,|u'(x)|^{\frac{1}{p-1}} \right), \ x \in I, \\ u(0) - au'(0) = \displaystyle\int_0^1 e^{-x} u(x)dx, \end{cases} \tag{4.1}$$

where $I =]0,+\infty[$, $g : \overline{I} \to \overline{I}$ is a continuous decreasing function and a is a positive real number such that $a \leq 1 + 3e^{-1}$.

We put by definition $\underline{u}(x) = 0$ and $\overline{u}(x) = x+1$, for all $x \in \overline{I}$.

First it is easy to check that $\underline{u}$ is a lower solution of (4.1).

Now $\overline{u}$ is an upper solution of (4.1) if we have

$$\begin{cases} -(\varphi_p(\overline{u}'))' \geq \dfrac{|\overline{u}'(x)|^{\frac{p-1}{p}}}{(x^2+1)^{\frac{1}{p}}} \left(g(x+\overline{u}(x)) - g(x)\,|\overline{u}'(x)|^{\frac{1}{p-1}} \right), \ x \in I, \\ \overline{u}(0) - a\overline{u}'(0) \geq \displaystyle\int_0^1 e^{-x} \overline{u}(x)dx. \end{cases}$$

That is

$$\begin{cases} 0 \geq g(2x+1) - g\left(x\right), \ \text{for all } x \in I, \\ 1 - a \geq 2 - 3e^{-1}. \end{cases}$$

Since g is a decreasing function and $a \leq 1 + 3e^{-1}$, we obtain $\overline{u}$ is an upper solution of (4.1).

On the other hand, it is not difficult to prove that the function $\widetilde{f}$ defined by

$$\widetilde{f}(x,u,v) = \frac{|v|^{\frac{p-1}{p}}}{(x^2+1)^{\frac{1}{p}}} \left(g(x+u(x)) - g(x)\,|v|^{\frac{1}{p-1}} \right)$$

satisfies the hypothesis of Theorem 3.1 and consequently, it follows that the problem (4.1) has a minimal solution and a maximal solution.

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