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POSITIVE PERIODIC SOLUTIONS OF SECOND-ORDER SYSTEMS WITH SINGULARITIES AND DEVIATING ARGUMENTS

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ABSTRACT. In this paper, we use Schauder's fixed point theorem to prove that the second-order systems with singularities and deviating arguments

$$\begin{cases} x''(t) + a_1(t)\, x(t) = f_1(t, y(t - \tau_1(t))) + e_1(t), \\ y''(t) + a_2(t)\, y(t) = f_2(t, x(t - \tau_2(t))) + e_2(t), \end{cases}$$

has a positive periodic solution, where $a_i, \tau_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}_+)$, $e_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and f_i $(i = 1, 2)$ are Carathéodory functions and have singularities at the origin. An example is also given to illustrate our work.

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1. Introduction

Due to their important in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence of positive periodic solutions for second-order differential equations with singularities; see [1]–[12] and references therein.

The study of singular problems began with the paper of Taliaferro. In 1979, Taliaferro [9] discussed a model equation with singularity

$$\begin{aligned} x''(t) + \frac{q(t)}{x^{\lambda}(t)} &= 0,\ 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned}$$

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and obtained the existence of solution for the problem. Here, $\lambda > 0$, $q \in C((0,1))$ with $q > 0$ on $(0,1)$ and $\int_0^1 t(1-t)\,q(t)\,dt < \infty$. We call this equation with a strong force condition if $\lambda \geq 1$ and with a weak force condition if $0 < \lambda < 1$.

In 1987, Lazer and Solimini [5] proved that a necessary and sufficient condition for the existence of a positive periodic solution of the problem

$$\begin{aligned} x''(t) &= \frac{1}{x^{\lambda}(t)} + c(t), \\ x(0) &= x(T), \end{aligned}$$

is that the mean value of c is negative, i.e.,

$$\bar{c} = \frac{1}{T}\int_0^T c(t)\,dt < 0,$$

for $\lambda \geq 1$. Moreover, if $0 < \lambda < 1$, they found examples of functions c with negative mean values and such that periodic solutions do not exist.

Recently, Ma, Chen and He [7] discussed the existence of positive periodic solutions of second-order differential equations with weak singularities

$$x''(t) + a(t)\,x(t) = f(t, x(t)) + e(t),$$

where $a \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}_+)$, $e \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, f is a Carathéodory function and is singular at $x = 0$. By employing Schauder's fixed point theorem, the authors obtained existence results for positive periodic solutions, which improve and generalize some results of Torres [11].

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of second-order systems with singularities and deviating arguments. Inspired and motivated by the works mentioned above and the references therein, we concentrate on the existence of positive periodic solutions of the second-order systems with singularities and deviating arguments

$$\begin{cases} x''(t) + a_1(t)\,x(t) = f_1(t, y(t - \tau_1(t))) + e_1(t), \\ y''(t) + a_2(t)\,y(t) = f_2(t, x(t - \tau_2(t))) + e_2(t), \end{cases} \tag{1.1}$$

where $a_i, \tau_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}_+)$, $e_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, and $f_i \in Car(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}_+, \mathbb{R})$ which means $f_i\,|_{[0,T]}\colon [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ are L^1-Carathéodory functions, and have singularities at the origin, i.e. $\lim_{x \to 0^+} \sup f_i(t, x) = +\infty$. By means of Schauder fixed point theorem, we obtain sufficient conditions of the existence of positive periodic solutions for (1.1). An example is also given to illustrate our results.

Throughout this paper, we use $\mathbb{R}$ and $\mathbb{R}_+$ to denote the real number set $(-\infty, +\infty)$ and the positive real number set $(0, +\infty)$, respectively. For a given function $\xi \in L^1[0,T]$, we denote the essential supremum and infimum of ξ if they exist by $\xi^*$and ξ_*, respectively.

2. Main results

It is the purpose of this section to study the existence of positive periodic solutions of (1.1) under the assumptions.

(A1) The linear equation $x''(t) + a_i(t)\, x(t) = 0$ is nonresonant and the corresponding Green's function

$$G_i(t,s) \geq 0, \quad (t,s) \in [0,T] \times [0,T], \quad (i = 1,2).$$

(A2) $f_i\,|_{[0,T]}\colon [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, $(i = 1,2)$ are L^1-Carathéodory functions.

(A3) There exist $b_i, c_i \in L^1(0,T)$ with $b_i, c_i \geq 0$, $\alpha_i, \beta_i \in \mathbb{R}_+$, $m_i \leq 1 \leq M_i$, $(i = 1,2)$, such that

$$0 \leq f_i(t,x) \leq \frac{b_i(t)}{x^{\alpha_i}}, \quad x \in (M_i, \infty), \quad a.e.\ t \in [0,T],$$

and

$$0 \leq f_i(t,x) \leq \frac{c_i(t)}{x^{\beta_i}}, \quad x \in (0, m_i), \ a.e.\ t \in [0,T].$$

(A4) There exist $b_{1i}, b_{2i}, c_i \in L^1(0,T)$ with $b_{1i}, b_{2i}, c_i \geq 0$ and $\alpha_i, \beta_i, \mu_i, \nu_i \in (0,1)$, $(i = 1,2)$, such that

$$\begin{aligned} 0 &\leq \frac{b_{1i}(t)}{x^{\alpha_i}} \leq f_i(t,x) \leq \frac{b_{2i}(t)}{x^{\beta_i}}, \quad x \in [1,\infty), \ a.e.\ t \in [0,T], \\ 0 &\leq \frac{b_{1i}(t)}{x^{\mu_i}} \leq f_i(t,x) \leq \frac{c_i(t)}{x^{\nu_i}}, \quad x \in (0,1), \ a.e.\ t \in [0,T]. \end{aligned}$$

Now, we introduce the following notations.

$$\begin{aligned} \gamma_i(t) &= \int_0^T G_i(t,s)\, e_i(s)\, ds, \quad (i = 1,2), \\ B_i(t) &= \int_0^T G_i(t,s)\, b_i(s)\, ds, \quad (i = 1,2), \\ C_i(t) &= \int_0^T G_i(t,s)\, c_i(s)\, ds, \quad (i = 1,2), \\ B_{ii}(t) &= \int_0^T G_i(t,s)\, b_{ii}(s)\, ds, \quad (i = 1,2), \\ \rho_i^* &= C_i^* + B_{2i}^*, \ \sigma_i = \max\{\mu_i, \alpha_i\}, \ \delta_i = \max\{\nu_i, \beta_i\}. \end{aligned}$$

Theorem 2.1. *Let (A1), (A2) and (A3) hold. If $\gamma_{i*} > 0$, then (1.1) has a positive T-periodic solution.*

Proof. We denote the set of continuous T-periodic functions as C_T. A T-periodic solution of (1.1) is just a fixed point of the completely continuous map $A : C_T \times C_T \to C_T \times C_T$, $A(x, y) = (A_1 y, A_2 x)$, where

$$\begin{aligned}
(A_1 y)(t) &= \int_0^T G_1(t, s)\left(f_1(s, y(s - \tau_1(s))) + e_1(s)\right) ds \\
&= \int_0^T G_1(t, s) f_1(s, y(s - \tau_1(s))) ds + \gamma_1(t), \\
(A_2 x)(t) &= \int_0^T G_2(t, s)\left(f_2(s, x(s - \tau_2(s))) + e_2(s)\right) ds \\
&= \int_0^T G_2(t, s) f_2(s, x(s - \tau_2(s))) ds + \gamma_2(t),
\end{aligned}$$

By Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \{(x, y) \in C_T \times C_T : r_1 \le x(t) \le R_1,\ r_2 \le y(t) \le R_2,\ \forall t \in [0, T]\},$$

into itself, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$, are positive constants to be fixed properly. For given $(x, y) \in K$, let us denote

$$\begin{aligned}
I_1 &= \{t \in [0, T] \mid r_1 \le x(t) < m_1,\ r_2 \le y(t) < m_2\}, \\
I_2 &= \{t \in [0, T] \mid R_1 \ge x(t) > M_1,\ R_2 \ge y(t) > M_2\}, \\
I_3 &= [0, T] \setminus (I_1 \cup I_2).
\end{aligned}$$

Given $(x, y) \in K$, by the nonnegativity of G_i and f_i, we have

$$\begin{aligned}
(A_1 y)(t) &= \int_0^T G_1(t, s)\left(f_1(s, y(s - \tau_1(s))) + e_1(s)\right) ds \\
&= \int_0^T G_1(t, s) f_1(s, y(s - \tau_1(s))) ds + \gamma_1(t) \\
&= \int_{I_1} G_1(t, s) f_1(s, y(s - \tau_1(s))) ds + \int_{I_2} G_1(t, s) f_1(s, y(s - \tau_1(s))) ds \\
&\quad + \int_{I_3} G_1(t, s) f_1(s, y(s - \tau_1(s))) ds + \gamma_1(t) \\
&\ge \gamma_1(t) \ge \gamma_{1*} = r_1,
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s)\left(f_2(s,x(s-\tau_2(s)))+e_2(s)\right)ds \\
&= \int_0^T G_2(t,s)\, f_2(s,x(s-\tau_2(s)))\,ds+\gamma_2(t) \\
&= \int_{I_1} G_2(t,s)\, f_2(s,x(s-\tau_2(s)))\,ds+\int_{I_2} G_2(t,s)\, f_2(s,x(s-\tau_2(s)))\,ds \\
&\quad +\int_{I_3} G_2(t,s)\, f_2(s,x(s-\tau_2(s)))\,ds+\gamma_2(t) \\
&\geq \gamma_2(t)\geq \gamma_{2*}=r_2.
\end{aligned}$$

Let

$$\begin{aligned}
\Lambda_1 &= \sup\left\{\max_{t\in[0,T]}\int_0^T G_1(t,s)\, f_1(s,y(s-\tau_1(s)))\,ds : m_2\leq y(s-\tau_1(s))\leq M_2\right\}, \\
\Lambda_2 &= \sup\left\{\max_{t\in[0,T]}\int_0^T G_2(t,s)\, f_2(s,x(s-\tau_2(s)))\,ds : m_1\leq x(s-\tau_2(s))\leq M_1\right\}.
\end{aligned}$$

Then, it follows from (A2) that $\Lambda_1<+\infty$, $\Lambda_2<+\infty$, and consequently, for every $(x,y)\in K$,

$$\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s)\, f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t) \\
&= \int_{I_1} G_1(t,s)\, f_1(s,y(s-\tau_1(s)))\,ds+\int_{I_2} G_1(t,s)\, f_1(s,y(s-\tau_1(s)))\,ds \\
&\quad +\int_{I_3} G_1(t,s)\, f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t) \\
&\leq \int_{I_1} G_1(t,s)\,\frac{c_1(s)}{y(s-\tau_1(s))^{\beta_1}}ds+\int_{I_2} G_1(t,s)\,\frac{b_1(s)}{y(s-\tau_1(s))^{\alpha_1}}ds+\Lambda_1+\gamma_1^* \\
&\leq \int_0^T G_1(t,s)\,\frac{c_1(s)}{y(s-\tau_1(s))^{\beta_1}}ds+\int_{I_2} G_1(t,s)\, b_1(s)\,ds+\Lambda_1+\gamma_1^* \\
&\leq \int_0^T G_1(t,s)\,\frac{c_1(s)}{r_2^{\beta_1}}ds+\int_0^T G_1(t,s)\, b_1(s)\,ds+\Lambda_1+\gamma_1^* \\
&\leq \frac{C_1^*}{r_2^{\beta_1}}+(B_1^*+\Lambda_1+\gamma_1^*)<\frac{C_1^*}{r_2^{\beta_1}}+(B_1^*+\Lambda_1+\gamma_1^*)=R_1,
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&= \int_{I_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \int_{I_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds \\
&\quad + \int_{I_3} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&\le \int_{I_1} G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \int_{I_2} G_2(t,s) \frac{b_2(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \Lambda_2 + \gamma_2^* \\
&\le \int_0^T G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \int_{I_2} G_2(t,s) b_2(s)\, ds + \Lambda_2 + \gamma_2^* \\
&\le \int_0^T G_2(t,s) \frac{c_2(s)}{r_1^{\beta_2}} ds + \int_0^T G_2(t,s) b_2(s)\, ds + \Lambda_2 + \gamma_2^* \\
&\le \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*) < \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*) = R_2.
\end{aligned}$$

Therefore, $A(x,y) = (A_1y, A_2x) \in K$ if $r_1 = \gamma_{1*}$, $r_2 = \gamma_{2*}$, $R_1 = \frac{C_1^*}{r_2^{\beta_1}} + (B_1^* + \Lambda_1 + \gamma_1^*)$, $R_2 = \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*)$, and the proof is finished. □

Theorem 2.2. *Let (A1), (A2) and (A4) hold. If $\gamma_{i*} = 0$, then (1.1) has a positive T-periodic solution.*

Proof. We follow the same strategy and notations as in the proof of Theorem 2.1. Define a closed convex set

$$K = \{(x,y) \in C_T^2 : r_1 \le x(t) \le R_1,\ r_2 \le y(t) \le R_2,\ \forall t \in [0,T],\ R_1 > 1,\ R_2 > 1\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that $A(x,y) = (A_1y, A_2x)$ maps the closed convex set K into itself, where R_1, R_2 and r_1, r_2 are positive constants to be fixed properly and they should satisfy $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ and $R_1 > 1$, $R_2 > 1$. For given $(x,y) \in K$, let us denote

$$\begin{aligned}
J_1 &= \{t \in [0,T] \mid r_1 \le x(t) < 1,\ r_2 \le y(t) < 1\}, \\
J_2 &= \{t \in [0,T] \mid R_1 \ge x(t) \ge 1,\ R_2 \ge y(t) \ge 1\}.
\end{aligned}$$

Then for given $(x,y)\in K$, by the nonnegative sign of G_i and f_i, it follows that

$$\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t)\\
&= \int_{J_1} G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds\\
&\quad +\int_{J_2} G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t)\\
&\le \int_{J_1} G_1(t,s)\,\frac{c_1(s)}{y(s-\tau_1(s))^{\nu_1}}ds+\int_{J_2} G_1(t,s)\,\frac{b_{21}(s)}{y(s-\tau_1(s))^{\beta_1}}ds+\gamma_1^*\\
&\le \int_0^T G_1(t,s)\,\frac{c_1(s)}{r_2^{\nu_1}}ds+\int_{J_2} G_1(t,s)\,b_{21}(s)\,ds+\gamma_1^*\\
&\le \int_0^T G_1(t,s)\,\frac{c_1(s)}{r_2^{\nu_1}}ds+\int_0^T G_1(t,s)\,b_{21}(s)\,ds+\gamma_1^*\le \frac{C_1^*}{r_2^{\nu_1}}+(B_{21}^*+\gamma_1^*),
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s)\,f_2(s,x(s-\tau_2(s)))\,ds+\gamma_2(t)\\
&= \int_{J_1} G_2(t,s)\,f_2(s,x(s-\tau_2(s)))\,ds\\
&\quad +\int_{J_2} G_2(t,s)\,f_2(s,x(s-\tau_2(s)))\,ds+\gamma_2(t)\\
&\le \int_{J_1} G_2(t,s)\,\frac{c_2(s)}{x(s-\tau_2(s))^{\nu_2}}ds+\int_{J_2} G_2(t,s)\,\frac{b_{22}(s)}{x(s-\tau_2(s))^{\beta_2}}ds+\gamma_2^*\\
&\le \int_0^T G_2(t,s)\,\frac{c_2(s)}{r_1^{\nu_2}}ds+\int_{J_2} G_2(t,s)\,b_{22}(s)\,ds+\gamma_2^*\\
&\le \int_0^T G_2(t,s)\,\frac{c_2(s)}{r_1^{\nu_2}}ds+\int_0^T G_2(t,s)\,b_{22}(s)\,ds+\gamma_2^*\le \frac{C_2^*}{r_1^{\nu_2}}+(B_{22}^*+\gamma_2^*).
\end{aligned}$$

On the other hand, for every $(x,y)\in K$,

$$\begin{aligned}
(A_1 y)(t) &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&= \int_{J_1} G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds \\
&\quad + \int_{J_2} G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\mu_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\alpha_1}} ds + \gamma_{1*} \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{R_2^{\mu_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{R_2^{\alpha_1}} ds \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds \\
&\geq \int_0^T G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds \geq \frac{B_{11*}}{R_2^{\sigma_1}},
\end{aligned}$$

and

$$\begin{aligned}
(A_2 x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \gamma_{2*} \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\alpha_2}} ds \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds \\
&\geq \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds \geq \frac{B_{12*}}{R_1^{\sigma_2}}.
\end{aligned}$$

Thus $A(x,y) = (A_1 y, A_2 x) \in K$ if r_1, r_2 and R_1, R_2 are chosen so that

$$\begin{aligned}
r_1 &\leq \frac{B_{11*}}{R_2^{\sigma_1}}, \quad \frac{C_1^*}{r_2^{\nu_1}} + (B_{21}^* + \gamma_1^*) \leq R_1, \\
r_2 &\leq \frac{B_{12*}}{R_1^{\sigma_2}}, \quad \frac{C_2^*}{r_1^{\nu_2}} + (B_{22}^* + \gamma_2^*) \leq R_2.
\end{aligned}$$

Note that $B_{1i*} > 0$, $C_i^* > 0$ and taking $r_1 = r_2 = r$, $R_1 = R_2 = R$, and $R = \frac{1}{r}$, it is sufficient to find $R > 1$ such that

$$B_{1i*} R^{1-\sigma_i} \geq 1, \quad C_i^* R^{\nu_i} + (B_{2i}^* + \gamma_i^*) \leq R,$$

and these inequalities hold for R big enough because $\sigma_i < 1$ and $\nu_i < 1$. □

Remark 2.3. *It is worth remarking that Theorem 2.2 is also valid for the special case that* $e_i(t) \equiv 0$, $(i = 1, 2)$, *which implies that* $\gamma_{i*} = 0$.

Theorem 2.4. *Let (A1), (A2) and (A4) hold. Assume that*

$$\rho_1^* > \max\left\{ (\delta_1 \sigma_2 B_{12*})^{\delta_1}, (\delta_1 \sigma_2 B_{12*})^{\frac{1}{\sigma_1}} \right\}, \tag{2.1}$$

$$\rho_2^* > \max\left\{ (\delta_2 \sigma_1 B_{11*})^{\delta_2}, (\delta_2 \sigma_1 B_{11*})^{\frac{1}{\sigma_2}} \right\}. \tag{2.2}$$

If $\gamma_i^* \leq 0$ *and*

$$\gamma_{1*} \geq \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}} \delta_2 \sigma_1 \right]^{\frac{1}{1-\delta_2\sigma_1}} \left(1 - \frac{1}{\delta_2 \sigma_1} \right), \tag{2.3}$$

$$\gamma_{2*} \geq \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}} \delta_1 \sigma_2 \right]^{\frac{1}{1-\delta_1\sigma_2}} \left(1 - \frac{1}{\delta_1 \sigma_2} \right), \tag{2.4}$$

then (1.1) has a positive T-periodic solution.

Proof. Define a closed convex set

$$K = \{(x, y) \in C_T \times C_T : r_1 \leq x(t) \leq R_1,\ r_2 \leq y(t) \leq R_2,$$

$$[1pt] \qquad \forall t \in [0, T],\ 0 < r_1 < 1 < R_1,\ 0 < r_2 < 1 < R_2\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R_1, R_2 and r_1, r_2 are positive constants to be fixed properly and they should satisfy $R_1 > 1 > r_1 > 0$, $R_2 > 1 > r_2 > 0$.

Recall that $\delta_i = \max\{\nu_i, \beta_i\}$ and $r_1 < 1$, $r_2 < 1$; therefore for given $(x, y) \in K$,

$$\begin{aligned}
(A_1 y)(t) &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&= \int_{J_1} G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds \\
&\quad + \int_{J_2} G_1(t,s) f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&\leq \int_{J_1} G_1(t,s) \frac{c_1(s)}{y(s-\tau_1(s))^{\nu_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{21}(s)}{y(s-\tau_1(s))^{\beta_1}} ds + \gamma_1^* \\
&\leq \int_{J_1} G_1(t,s) \frac{c_1(s)}{r_2^{\nu_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{21}(s)}{r_2^{\beta_1}} ds \\
&\leq \int_0^T G_1(t,s) \frac{c_1(s)}{r_2^{\delta_1}} ds + \int_0^T G_1(t,s) \frac{b_{21}(s)}{r_2^{\delta_1}} ds \leq \frac{\rho_1^*}{r_2^{\delta_1}},
\end{aligned}$$

and

$$\begin{aligned}
(A_2 x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&\leq \int_{J_1} G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\nu_2}} ds \\
&\quad + \int_{J_2} G_2(t,s) \frac{b_{22}(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \gamma_2^* \\
&\leq \int_{J_1} G_2(t,s) \frac{c_2(s)}{r_1^{\nu_2}} ds + \int_{J_2} G_2(t,s) \frac{b_{22}(s)}{r_1^{\beta_2}} ds \\
&\leq \int_0^T G_2(t,s) \frac{c_2(s)}{r_1^{\delta_2}} ds + \int_0^T G_2(t,s) \frac{b_{22}(s)}{r_1^{\delta_2}} ds \leq \frac{\rho_2^*}{r_1^{\delta_2}},
\end{aligned}$$

where J_i, δ_i, ρ_i^*, $(i = 1, 2)$, are defined previously in this Section.

On the other hand, since $\sigma_i = \max\{\mu_i, \alpha_i\}$, and $R_1 > 1$, $R_2 > 1$, for every $(x, y) \in K$,

$$\begin{aligned}
(A_1 y)(t) &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&= \int_{J_1} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds \\
&\quad + \int_{J_2} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\mu_1}} ds \\
&\quad + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\alpha_1}} ds + \gamma_{1*} \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \gamma_{1*} \\
&\geq \int_0^T G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_0^T G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \gamma_{1*} \\
&\geq \frac{B_{11*}}{R_2^{\sigma_1}} + \gamma_{1*},
\end{aligned}$$

and

$$\begin{aligned}
(A_2 x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \gamma_{2*} \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \gamma_{2*} \\
&\geq \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \gamma_{2*} \\
&\geq \frac{B_{12*}}{R_1^{\sigma_2}} + \gamma_{2*},
\end{aligned}$$

where B_{1i*} and σ_i, $(i = 1, 2)$, are defined previously in this Section.

In this case, to prove that $A(K) \subset K$, it is sufficient to find

$r_1 < R_1$, $r_2 < R_2$ with $R_1 > 1 > r_1 > 0$, $R_2 > 1 > r_2 > 0$ such that

$$\frac{B_{11*}}{R_2^{\sigma_1}} + \gamma_{1*} \geq r_1, \quad \frac{\rho_1^*}{r_2^{\delta_1}} \leq R_1, \tag{2.5}$$

$$\frac{B_{12*}}{R_1^{\sigma_2}} + \gamma_{2*} \geq r_2, \quad \frac{\rho_2^*}{r_1^{\delta_2}} \leq R_2. \tag{2.6}$$

If we fix $R_1 = \frac{\rho_1^*}{r_2^{\delta_1}}$, $R_2 = \frac{\rho_2^*}{r_1^{\delta_2}}$, then to get the first inequality of (2.5), we need to find r_1 such that

$$\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}} r_1^{\sigma_1\delta_2} + \gamma_{1*} \geq r_1,$$

or equivalently,

$$\gamma_{1*} \geq f(r_1) = r_1 - \frac{B_{11*}}{(\rho_2^*)^{\sigma_1}} r_1^{\sigma_1\delta_2}.$$

The function $f(r_1)$ possesses a minimum in $r_{10} = \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}}\sigma_1\delta_2\right]^{\frac{1}{1-\sigma_1\delta_2}}$. Taking $r_1 = r_{10}$, (2.1) implies that $r_1 < 1$. Then the first inequality in (2.5) holds if $\gamma_{1*} \geq f(r_{10})$, which is just condition (2.3).

The first inequality of (2.6) holds when

$$\gamma_{2*} \geq g(r_2) = r_2 - \frac{B_{12*}}{(\rho_1^*)^{\sigma_2}} r_2^{\sigma_2\delta_1}.$$

Since $g(r_2)$ get the minimum at $r_{20} = \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}}\sigma_2\delta_1\right]^{\frac{1}{1-\sigma_2\delta_1}}$; taking $r_2 = r_{20}$, (2.2) implies that $r_2 < 1$. Then the first inequality in (2.6) holds if $\gamma_{2*} \geq g(r_{20})$, which is just condition (2.4).

Now we choose $r_1 = r_{10}$, $r_2 = r_{20}$. The second inequality in (2.5) holds directly by the choice of R_1, and it would remain to prove that $R_1 = \frac{\rho_1^*}{r_{20}^{\delta_1}} > 1$. To the end, it follows from (2.1) that

$$R_1 = \frac{\rho_1^*}{r_{20}^{\delta_1}} > \frac{(\delta_1\sigma_2 B_{12*})^{\delta_1} (\rho_1^*)^{\frac{\delta_1\sigma_2}{1-\delta_1\sigma_2}}}{(\delta_1\sigma_2 B_{12*})^{\frac{\delta_1}{1-\delta_1\sigma_2}}} > \frac{(\delta_1\sigma_2 B_{12*})^{\delta_1} (\delta_1\sigma_2 B_{12*})^{\frac{\delta_1^2\sigma_2}{1-\delta_1\sigma_2}}}{(\delta_1\sigma_2 B_{12*})^{\frac{\delta_1}{1-\delta_1\sigma_2}}} = 1,$$

and therefore $R_1 > 1 > r_1 > 0$.

The second inequality in (2.6) holds directly by the choice of R_2, and it would remain to prove that $R_2 = \frac{\rho_2^*}{r_{10}^{\delta_2}} > 1$. To the end, it follows from (2.2) that

$$R_2 = \frac{\rho_2^*}{r_{10}^{\delta_2}} > \frac{(\delta_2\sigma_1 B_{11*})^{\delta_2} (\rho_2^*)^{\frac{\delta_2\sigma_1}{1-\delta_2\sigma_1}}}{(\delta_2\sigma_1 B_{11*})^{\frac{\delta_2}{1-\delta_2\sigma_1}}} > \frac{(\delta_2\sigma_1 B_{11*})^{\delta_2} (\delta_2\sigma_1 B_{11*})^{\frac{\delta_2^2\sigma_1}{1-\delta_2\sigma_1}}}{(\delta_2\sigma_1 B_{11*})^{\frac{\delta_2}{1-\delta_2\sigma_1}}} = 1,$$

and therefore $R_2 > 1 > r_2 > 0$, this completes the proof. □

Example 2.5. *Let us consider the second order periodic boundary value problem*

$$\begin{cases} x''(t) + \frac{1}{4}x(t) = f_1(t,y) - e_1, \\ y''(t) + \frac{1}{5}y(t) = f_2(t,x) - e_2, \end{cases} \quad t \in (0,\pi), \tag{2.7}$$

$$\begin{aligned} x(0) &= x(\pi), \quad x'(0) = x'(\pi), \\ y(0) &= y(\pi), \quad y'(0) = y'(\pi), \end{aligned}$$

where

$$\begin{aligned} f_1(t,y) &= \frac{7-t}{y^{\frac{1}{7}}}, \quad y \in (0,\infty), \quad a.e.\ t \in [0,\pi], \\ f_2(t,x) &= \frac{5-t}{x^{\frac{1}{5}}}, \quad x \in (0,\infty), \quad a.e.\ t \in [0,\pi], \end{aligned}$$

and

$$\begin{aligned} e_1 &\in \left(0, \frac{3}{4}\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}}\right], \\ e_2 &\in \left(0, \frac{9}{5}\left(\frac{1}{16}\right)^{\frac{10}{9}}\right], \end{aligned}$$

are a constants.

It is easy to check that (2.7) is equivalent to the operator equation

$$\begin{aligned} (A_1y)(t) &= \int_0^\pi G_1(t,s) f_1(s,y(s))\,ds + \int_0^\pi G_1(t,s)(-e_1)\,ds, \qquad t \in [0,\pi], \\ (A_2x)(t) &= \int_0^\pi G_2(t,s) f_2(s,x(s))\,ds + \int_0^\pi G_2(t,s)(-e_2)\,ds, \qquad t \in [0,\pi], \end{aligned}$$

where

$$G_1(t,s) = \begin{cases} \sin\frac{\pi-t+s}{2} + \sin\frac{t-s}{2}, & 0 \le s \le t \le \pi, \\ \sin\frac{\pi-s+t}{2} + \sin\frac{s-t}{2}, & 0 \le t \le s \le \pi, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)}\sin\frac{\pi-t+s}{\sqrt{5}} + \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)}\sin\frac{t-s}{\sqrt{5}}, & 0 \le s \le t \le \pi, \\ \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)}\sin\frac{\pi-s+t}{\sqrt{5}} + \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)}\sin\frac{s-t}{\sqrt{5}}, & 0 \le t \le s \le \pi. \end{cases}$$

Clearly, $G_i(t,s) > 0$ *for all* $(t,s) \in [0,\pi] \times [0,\pi]$ *and* f_i *satisfies (A2).*

Let

$$\begin{aligned} b_{11}(t) &= \frac{1}{2},\ b_{12}(t) = 1,\ b_{21}(t) = 7,\ b_{22}(t) = 5,\ c_1(t) = 9,\ c_2(t) = 7, \\ \alpha_1 &= \alpha_2 = \frac{1}{2},\ \beta_1 = \frac{1}{8},\ \beta_2 = \frac{1}{6},\ \mu_1 = \frac{1}{9},\ \mu_2 = \frac{1}{7},\ \nu_1 = \frac{1}{5},\ \nu_2 = \frac{1}{2}. \end{aligned}$$

Then

$$\sigma_1 = \sigma_2 = \frac{1}{2}, \quad \delta_1 = \frac{1}{5}, \quad \delta_2 = \frac{1}{2},$$

and

$$\begin{aligned}
\frac{\frac{1}{2}}{y^{\frac{1}{2}}} &\leq \frac{7-t}{y^{\frac{1}{7}}} \leq \frac{7}{y^{\frac{1}{8}}}, \quad y \in [1,\infty), \quad a.e.\ t \in [0,\pi], \\
\frac{\frac{1}{2}}{y^{\frac{1}{9}}} &\leq \frac{7-t}{y^{\frac{1}{7}}} \leq \frac{9}{y^{\frac{1}{5}}}, \quad y \in (0,1), \quad a.e.\ t \in [0,\pi], \\
\frac{1}{x^{\frac{1}{2}}} &\leq \frac{5-t}{x^{\frac{1}{5}}} \leq \frac{5}{x^{\frac{1}{6}}}, \quad x \in [1,\infty), \quad a.e.\ t \in [0,\pi], \\
\frac{1}{x^{\frac{1}{7}}} &\leq \frac{5-t}{x^{\frac{1}{5}}} \leq \frac{7}{x^{\frac{1}{2}}}, \quad x \in (0,1), \quad a.e.\ t \in [0,\pi].
\end{aligned}$$

Thus, the condition (A4) is satisfied. By simple computations, we get

$$\begin{aligned}
B_{11}(t) &= \int_0^\pi G_1(t,s)\, b_{11}(s)\, ds = 2,\ B_{12}(t) = \int_0^\pi G_2(t,s)\, b_{12}(s)\, ds = 5, \\
B_{21}(t) &= \int_0^\pi G_1(t,s)\, b_{21}(s)\, ds = 28,\ B_{22}(t) = \int_0^\pi G_2(t,s)\, b_{22}(s)\, ds = 25, \\
C_1(t) &= \int_0^\pi G_1(t,s)\, c_1(s)\, ds = 36,\ C_2(t) = \int_0^\pi G_2(t,s)\, c_2(s)\, ds = 35, \\
B_{11*} &= B_{11}^* = 2,\ B_{12*} = B_{12}^* = 5,\ B_{21*} = B_{21}^* = 28,\ B_{22*} = B_{22}^* = 25, \\
C_{1*} &= C_1^* = 36,\ C_{2*} = C_2^* = 35, \\
(\delta_1\sigma_2 B_{12*})^{\delta_1} &= \frac{1}{\sqrt[5]{2}},\ (\delta_1\sigma_2 B_{12*})^{\frac{1}{\sigma_1}} = \frac{1}{4}, \\
(\delta_2\sigma_1 B_{11*})^{\delta_2} &= \frac{\sqrt{2}}{2},\ (\delta_2\sigma_1 B_{11*})^{\frac{1}{\sigma_2}} = \frac{1}{4}, \\
\rho_1^* &= C_1^* + B_{21}^* = 64 > \frac{1}{\sqrt[5]{2}} = \max\left\{(\delta_1\sigma_2 B_{12*})^{\delta_1}, (\delta_1\sigma_2 B_{12*})^{\frac{1}{\sigma_1}}\right\}, \\
\rho_2^* &= C_2^* + B_{22}^* = 60 > \frac{\sqrt{2}}{2} = \max\left\{(\delta_2\sigma_1 B_{11*})^{\delta_2}, (\delta_2\sigma_1 B_{11*})^{\frac{1}{\sigma_2}}\right\},
\end{aligned}$$

and therefore the condition (2.1), (2.2) is satisfied. Moreover,

$$\begin{aligned}
\gamma_{1*}(t) &= \int_0^\pi G_1(t,s)(-e_1)\, ds = -4e_1, \\
\gamma_{2*}(t) &= \int_0^\pi G_2(t,s)(-e_2)\, ds = -5e_2,
\end{aligned}$$

and so

$$\gamma_1^* = \gamma_{1*} = -4e_1 < 0,$$

$$\gamma_2^* = \gamma_{2*} = -5e_2 < 0.$$

Finally

$$\gamma_{1*} = -4e_1 \geq -4.\frac{3}{4}\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}} = -3\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}} = \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}}\delta_2\sigma_1\right]^{\frac{1}{1-\delta_2\sigma_1}}\left(1-\frac{1}{\delta_2\sigma_1}\right),$$

$$\gamma_{2*} = -5e_2 \geq -5.\frac{9}{5}\left(\frac{1}{16}\right)^{\frac{10}{9}} = -9\left(\frac{1}{16}\right)^{\frac{10}{9}} = \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}}\delta_1\sigma_2\right]^{\frac{1}{1-\delta_1\sigma_2}}\left(1-\frac{1}{\delta_1\sigma_2}\right).$$

Consequently Theorem 2.4 yields that (2.7) has a positive T-periodic solution.

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