Communications in Applied Analysis **20** (2016), 239–251

B-SPLINE CURVE AS A MINIMUM OF QUADRATIC FORM AND ITS DERIVATIVES, I

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ABSTRACT. The goal of this short note is to show a new approach for differentiation of b-spline curve.

AMS (MOS) Subject Classification. 41A15, 65Dxx, 65D07.

1. INTRODUCTION

The b-spline curves and surfaces are an essential tool in many engineering software for design and visualization – for example ANSYS ICEM CFD meshing software. Usually such software supports two primary geometry types, faceted and b-spline.

Faceted geometry is made up of lots of triangles (usually the data comes from 2d or 3d scanners or some low end geometry creation tools). Some types here are: STL, VRML and various mesh types, used as geometry.

Higher end CAD tools produce b-spline geometry. The sources would include low end options such as IGES/STEP and high end tools like Solidworks, etc. These are usually represented as large surfaces, each with their own UV-space.

Our goal in the article is to prove that any b-spline curve or surface minimizes positive quadratic operator: appropriate moving least-square error. As a result it is proven a formula for differentiation of b-spline curve.

We refer to Appendix A for some remarks on moving least-square method.

2. PRELIMINARIES: B-SPLINES

In this section we will remind the definition of b-splines generated by control points in $\mathbb{R}^{d+1}$ and definition of moving least-squares approximation for a given data set $\{(\boldsymbol{x}_i, f(\boldsymbol{x}_i)) : \boldsymbol{x}_i \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$.

Let $\{\boldsymbol{p}_i \in \mathbb{R}^{d+1} : i = 0, \ldots, n\}$ be a set of $n+1$ (control) points.

Let r be an integer, $1 \leq r \leq n+1$ (the order of spline).

Received July 12, 2016

We will use uniform knots, without multiplicity: $t_i = i$, $i = 0, \dots, n+r$.

Using Cox-de Boor recursion formula (see [3], [4]), let us define the following basis functions:

$$B_{i,1}(t) = \begin{cases} 1, & \text{if } t_i \le t < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

for $0 \le i \le n+r-1$; and

$$\begin{aligned} B_{i,j}(t) =& \frac{t - t_i}{t_{i+j-1} - t_i} B_{i,j-1}(t) + \frac{t_{i+j} - t}{t_{i+j} - t_{i+1}} B_{i+1,j-1}(t) \\ =& \frac{t-i}{j-1} B_{i,j-1}(t) + \frac{i+j-t}{j-1} B_{i+1,j-1}(t), \end{aligned} \tag{2.2}$$

for $2 \le j \le r$, $0 \le i \le n+r-j$.

The b-spline curve of order r is defined as a linear combination of control points $\boldsymbol{p}_i$:

$$\boldsymbol{\gamma}(t) = \sum_{i=0}^{n} B_{i,r}(t)\boldsymbol{p}_i, \quad t \in [t_{r-1}, t_{n+1}]. \tag{2.3}$$

3. B-SPLINE CURVE AS A MINIMUM OF MOVING LEAST-SQUARES ERROR

Using the definitions and notations introduced in Section 2, our goal is to prove the following theorem.

Theorem 3.1. *Let:*

1. $d = 1$; $n, r \in \mathbb{N} = \{1, 2, \dots\}$, $r \le n+1$; $f : [0, n] \to \mathbb{R}$ *be a continuous function.*
2. $\boldsymbol{P} = \left\{ \boldsymbol{p}_i = \begin{pmatrix} i \\ f(i) \end{pmatrix} : i = 0, \dots, n \right\}$ *be a set of control points.*
3. *Let* $\boldsymbol{\gamma}(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$ *be the b-spline of order* r *and knot vector* $\boldsymbol{T} = \{i : i = 0, \dots, n+r\}$.

Then:

1. $\gamma_1(t) = t - \frac{r}{2}$, $t \in [r, n+r]$
2. *There exists a weight function* W*, such that*

$$\gamma_2(x) = \hat{L}(f)(x), \quad x \in [r-1, n+1],$$

where $\hat{L}(f)(x)$ *is the approximation defined by the moving least-squares method for the data* $\{\boldsymbol{p}_i : i = 0, \dots, n+r\}$.

Proof. We will prove the theorem for the cubic splines, i.e. $r = 4$. The proof for the different orders is similar.

$\vdots$	$\vdots$			
$t_{i_0-4} < t_{i_0-3}$	$B_{i_0-4,1}(t) = 0$	$\vdots$		
		$B_{i_0-4,2}(t) = 0$	$\vdots$	
$t_{i_0-3} < t_{i_0-2}$	$B_{i_0-3,1}(t) = 0$		$B_{i_0-4,3}(t) = 0$	$\vdots$
		$B_{i_0-3,2}(t) = 0$		$B_{i_0-4,4}(t) = 0$
$t_{i_0-2} < t_{i_0-1}$	$B_{i_0-2,1}(t) = 0$		$B_{i_0-3,2}(t) = 0$	
		$B_{i_0-2,2}(t) = 0$		$\boldsymbol{B_{i_0-3,4}(t) > 0}$
$t_{i_0-1} < t_{i_0}$	$B_{i_0-1,1}(t) = 0$		$\boldsymbol{B_{i_0-2,3}(t) > 0}$	
		$\boldsymbol{B_{i_0-1,2}(t) > 0}$		$\boldsymbol{B_{i_0-2,4}(t) > 0}$
$\boldsymbol{t_{i_0} < t < t_{i_0+1}}$	$\boldsymbol{B_{i_0,1}(t) = 1}$		$\boldsymbol{B_{i_0-1,3}(t) > 0}$	
		$\boldsymbol{B_{i_0,2}(t) > 0}$		$\boldsymbol{B_{i_0-1,4}(t) > 0}$
$t_{i_0+1} < t_{i_0+2}$	$B_{i_0+1,1}(t) = 0$		$\boldsymbol{B_{i_0,3}(t) > 0}$	
		$B_{i_0+1,2}(t) = 0$		$\boldsymbol{B_{i_0,4}(t) > 0}$
$t_{i_0+2} < t_{i_0+3}$	$B_{i_0+2,1}(t) = 0$		$B_{i_0+1,3}(t) = 0$	
		$B_{i_0+2,2}(t) = 0$		$B_{i_0+1,4}(t) = 0$
$t_{i_0+3} < t_{i_0+4}$	$B_{i_0+3,1}(t) = 0$		$B_{i_0+2,3}(t) = 0$	$\vdots$
		$B_{i_0+3,2}(t) = 0$	$\vdots$	
$t_{i_0+4} < t_{i_0+5}$	$B_{i_0+4,1}(t) = 0$	$\vdots$		
$\vdots$	$\vdots$			

TABLE 1. Cox-de Boor recursion algorithm

The b-spline curve of degree 3 (i.e. order $r = 4$), defined using knots $\boldsymbol{T}$ and control points $\boldsymbol{P}$, is

$$\gamma(t) = \sum_{i=0}^{n} B_{i,4}(t)\boldsymbol{p}_i = \begin{pmatrix} \sum_{i=0}^{n} B_{i,4}(t)i \\ \sum_{i=0}^{n} B_{i,4}(t)f(i) \end{pmatrix}, \quad t \in [t_{r-1}, t_{n+1}] \equiv [3, n+1],$$

Let us set

$$\gamma_1(t) = \sum_{i=0}^{n} B_{i,4}(t)i, \quad t \in [3, n+1]$$

and

$$\gamma_2(t) = \sum_{i=0}^{n} B_{i,4}(t)f(x_i), \quad x \in [3, n+1].$$

The following properties of b-spline basis functions $B_{i,j}(t)$ are well known:

(BS-0) By the direc calculation (see formulas (2.1), (2.2), and the schema illustrated in Table 1, $r = 4$):

$$B_{i,4}(t) = \begin{cases} 0, & \text{if } t < i, \\ \frac{1}{6}(t-i)^3, & \text{if } i \le t < i+1, \\ -\frac{2}{3}(t-i-1)^3 + \frac{1}{6}(t-i)^3, & \text{if } i+1 \le t < i+2, \\ (t-i-2)^3 - \frac{2}{3}(t-i-1)^3 & \\ +\frac{1}{6}(t-i)^3, & \text{if } i+2 \le t < i+3, \\ -\frac{2}{3}(t-i-3)^3 + (t-i-2)^3 & \\ -\frac{2}{3}(t-i-1)^3 + \frac{1}{6}(t-i)^3, & \text{if } i+3 \le t < i+4, \\ 0, & \text{if } i+4 \le t. \end{cases}$$

(BS-1) If $t \in (i, i+j)$, then $B_{i,j}(t) > 0$.
(BS-2) If $t \in [0, i] \cup [i+j, n+j]$, then $B_{i,j}(t) = 0$.
(BS-3) $\sum_{i=0}^{n} B_{i,j}(t) = 1$, for any $t \in (j-1, n+1)$.
(BS-4) $B_{i,j}(t)$ has C^{j-2} continuity at each knot.
(BS-5) By the simple substitutions in the formulas in (BS-0):

$$\begin{aligned} B_{i,j}(t+i) =& B_{k,j}(t+k), \qquad t \in (0, j), \\ B_{i-2,j}(t) =& B_{i,j}(t+2), \qquad t \in (i-2, i+j-2). \end{aligned}$$

Let us prove the first statement of Theorem 3.1.

From conditions (2) and (3), if we set $x_i = i$, $i = 0, \dots, n$, $t_i = i$, $i = 0, \dots, n+4$, then $x_{i-2} = i - 2 = \gamma_1(t_i)$, $i = 2, \dots, n+2$.

Indeed, using the definition of b-spline function, let t be a fixed point in $[3, n+1]$ and let i_0 be an integer such that

$$3 \le i_0 \le t < i_0 + 1 < n + 1.$$

Then $B_{i,4}(t) = 0$, if $i \le i_0 - 4$ or $i \ge i_0 + 1$. Therefore

$$\begin{aligned} \gamma_1(t) =& \sum_{i=0}^{n} B_{i,4}(t) i \\ =& \sum_{i=i_0-3}^{i_0} B_{i,4}(t) i \end{aligned}$$

$$
= (i_0 - 3)\begin{cases}
0, & \text{if } t < i_0 - 3,\\
\frac{1}{6}\left(t + 3 - i_0\right)^3, & \text{if } t < -2 + i_0,\\
-\frac{2}{3}\left(t + 2 - i_0\right)^3 + \frac{1}{6}\left(t + 3 - i_0\right)^3, & \text{if } t < -1 + i_0,\\
\left(t + 1 - i_0\right)^3 - \frac{2}{3}\left(t + 2 - i_0\right)^3 + \frac{1}{6}\left(t + 3 - i_0\right)^3, & \text{if } t < i_0,\\
-\frac{2}{3}\left(t - i_0\right)^3 + \left(t + 1 - i_0\right)^3 & \\
-\frac{2}{3}\left(t + 2 - i_0\right)^3 + \frac{1}{6}\left(t + 3 - i_0\right)^3, & \text{if } t < 1 + i_0,\\
0, & \text{if } 1 + i_0 \le t,
\end{cases}
$$

$$
+ (i_0 - 2)\begin{cases}
0, & \text{if } t < -2 + i_0,\\
\frac{1}{6}\left(t + 2 - i_0\right)^3, & \text{if } t < -1 + i_0,\\
-\frac{2}{3}\left(t + 1 - i_0\right)^3 + \frac{1}{6}\left(t + 2 - i_0\right)^3, & \text{if } t < i_0,\\
\left(t - i_0\right)^3 - \frac{2}{3}\left(t + 1 - i_0\right)^3 & \\
+\frac{1}{6}\left(t + 2 - i_0\right)^3, & \text{if } t < 1 + i_0,\\
-\frac{2}{3}\left(t - 1 - i_0\right)^3 + \left(t - i_0\right)^3 & \\
-\frac{2}{3}\left(t + 1 - i_0\right)^3 + \frac{1}{6}\left(t + 2 - i_0\right)^3, & \text{if } t < 2 + i_0,\\
0, & \text{if } 2 + i_0 \le t,
\end{cases}
$$

$$
+ (i_0 - 1)\begin{cases}
0, & \text{if } t < -1 + i_0,\\
\frac{1}{6}\left(t + 1 - i_0\right)^3, & \text{if } t < i_0,\\
-\frac{2}{3}\left(t - i_0\right)^3 + \frac{1}{6}\left(t + 1 - i_0\right)^3, & \text{if } t < 1 + i_0,\\
\left(t - 1 - i_0\right)^3 - \frac{2}{3}\left(t - i_0\right)^3 & \\
+\frac{1}{6}\left(t + 1 - i_0\right)^3, & \text{if } t < 2 + i_0,\\
-\frac{2}{3}\left(t - 2 - i_0\right)^3 + \left(t - 1 - i_0\right)^3 & \\
-\frac{2}{3}\left(t - i_0\right)^3 + \frac{1}{6}\left(t + 1 - i_0\right)^3, & \text{if } t < 3 + i_0,\\
0, & \text{if } 3 + i_0 \le t,
\end{cases}
$$

$$
+ i_0\begin{cases}
0, & \text{if } t < i_0,\\
\frac{1}{6}\left(t - i_0\right)^3, & \text{if } t < 1 + i_0,\\
-\frac{2}{3}\left(t - 1 - i_0\right)^3 + \frac{1}{6}\left(t - i_0\right)^3, & \text{if } t < 2 + i_0,\\
\left(t - 2 - i_0\right)^3 - \frac{2}{3}\left(t - 1 - i_0\right)^3 & \\
+\frac{1}{6}\left(t - i_0\right)^3, & \text{if } t < 3 + i_0,\\
-\frac{2}{3}\left(t - 3 - i_0\right)^3 + \left(t - 2 - i_0\right)^3 & \\
-\frac{2}{3}\left(t - 1 - i_0\right)^3 + \frac{1}{6}\left(t - i_0\right)^3, & \text{if } t < 4 + i_0,\\
0, & \text{if } 4 + i_0 \le t,
\end{cases}
$$

$$
\begin{aligned}
&= (i_0-3)\left(-\frac{2}{3}(t-i_0)^3+(t+1-i_0)^3-\frac{2}{3}(t+2-i_0)^3+\frac{1}{6}(t+3-i_0)^3\right)\\
&\quad+(i_0-2)\left((t-i_0)^3-\frac{2}{3}(t+1-i_0)^3+\frac{1}{6}(t+2-i_0)^3\right)\\
&\quad+(i_0-1)\left(-\frac{2}{3}(t-i_0)^3+\frac{1}{6}(t+1-i_0)^3\right)\\
&\quad+i_0\frac{1}{6}(t-i_0)^3\\
&=t-2.
\end{aligned}
$$

The statement (1) of Theorem 3.1 has been proven.

To prove the second statement of Theorem 3.1, let again t be a fixed point in the interval $(3, n+1)$ and i_0 be an integer such that $3 \le i_0 < t < i_0+1 \le n+1$. Then

$$
\gamma_2(t) = \sum_{i=0}^{n} B_{i,4}(t) f(i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(t) f(i). \tag{3.1}
$$

because if $i = 1, \ldots, i_0-4, i_0+1, \ldots, n$, then $B_{i,4}(x) = 0$, see (BS-1) and (BS-2).

On the other hand, let us consider the moving least-squares problem for the given data $\{(i, f(i)) : i = 0, \ldots, n\}$. Let us set $l = 1$, and

$$
W(x) = \begin{cases}
0, & \text{if } x < -2,\\
\frac{1}{6}(x+2)^3, & \text{if } -2 \le x < -1,\\
-\frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } -1 \le x < 0,\\
\frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } 0 \le x < 1,\\
-\frac{1}{6}(x-2)^3, & \text{if } 1 \le x < 2,\\
0, & \text{if } 2 \le x,
\end{cases}
$$

see Figure 1.

Then for any $i = 0, \ldots, n$, we have (see also (BS-5)):

$$
W(|x|) = W(x) = B_{i,4}(x+i+2) = B_{i-2,4}(x+i), \quad x \in [-2, 2].
$$

Hence $W(|x-i|) = B_{i-2,4}(x)$, $x \in [i-2, i+2]$.

The least-squares error is (see also conditions (H1): $p_1(x) = 1$, so $p(x) = p$ has to be a constant)

$$
\sum_{i=1}^{n+r} W(x-i)\,(p-f(i))^2 = \sum_{i=i_0-1}^{i_0+2} W(x-i)\,(p-f(i))^2,
$$

because $W(x-i) > 0$, iff $i_0 - 1 \le i \le i_0 + 2$.

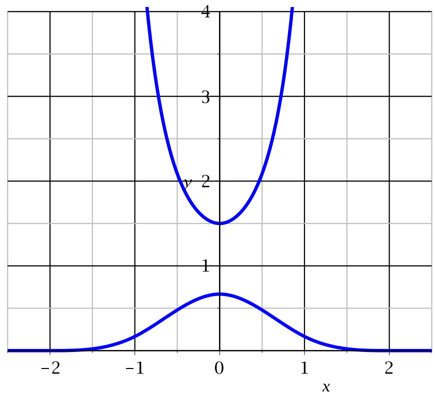

FIGURE 1. The graphics of $W(x),\ x \in (-2.5, 2.5)$ and $w(x),\ x \in (-1, 1)$

It is not hard to compute (see Theorem 3.2 from Appendix A)

$$E = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{c} = \begin{pmatrix} 1 \end{pmatrix},$$

$$D = 2 \begin{pmatrix} w(|x-(i_0-1)|) & 0 & 0 & 0 \\ 0 & w(|x-i_0|) & 0 & 0 \\ 0 & 0 & w(|x-(i_0+1)|) & 0 \\ 0 & 0 & 0 & w(|x-(i_0+2)|) \end{pmatrix},$$

where $w(x) = \dfrac{1}{W(x)}$, and

$$\begin{aligned} E^t D^{-1} E =& \frac{1}{2}\left(\frac{1}{w(|x-(i_0-1))|} + \frac{1}{w(|x-i_0|)} + \frac{1}{w(|x-(i_0+1)|)} \right. \\ & \left. + \frac{1}{w(|x-(i_0+2)|)} \right) \\ =& \frac{1}{2}\left(B_{i_0-3,4}(x) + B_{i_0-2,4}(x) + B_{i_0-1,4}(x) + B_{i_0,4}(x) \right) \\ =& \frac{1}{2}, \quad \text{because } x \in [i_0, i_0+1], \text{ see (BS-3)}, \\ \boldsymbol{a} =& D^{-1} E \left(E^t D^{-1} E \right)^{-1} \boldsymbol{c} = \begin{pmatrix} B_{i_0-3,4}(x) \\ B_{i_0-2,4}(x) \\ B_{i_0-1,4}(x) \\ B_{i_0,4}(x) \end{pmatrix}. \end{aligned}$$

Hence, by Theorem 3.2 from Appendix A, we have

$$\hat{L}(f)(x) = \sum_{i=1}^{4} a_i f(x_i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(x) f(i),$$

i.e. we received b-spline (3.1). □

Our goal below is to calculate the first derivative of γ_2. Again, we will consider cubic b-splines.

Let t_0 be a fixed point in the interval $(0, n+4)$ and i_0 be an integer such that $3 \le i_0 < t < i_0 + 1 \le n+1$; $x_0 = t_0 - 2 \in (3, n+1)$. Then

$$\hat{L}(f)(x) = \sum_{i=1}^{4} a_i f(x_i),$$

where:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \; E = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; \boldsymbol{c} = \begin{pmatrix} 1 \end{pmatrix},$$

$$D = 2\begin{pmatrix} w(|x-(i_0-1)|) & 0 & 0 & 0 \\ 0 & w(|x-i_0|) & 0 & 0 \\ 0 & 0 & w(|x-(i_0+1)|) & 0 \\ 0 & 0 & 0 & w(|x-(i_0+2)|) \end{pmatrix},$$

$$D^{-1} = \frac{1}{2}\begin{pmatrix} W(x-(i_0-1)) & 0 & 0 & 0 \\ 0 & W(x-i_0) & 0 & 0 \\ 0 & 0 & W(x-(i_0+1)) & 0 \\ 0 & 0 & 0 & W(x-(i_0+2)) \end{pmatrix},$$

$w(x) = \dfrac{1}{W(x)}$, and (see Theorem 3.2)

$$\boldsymbol{a} = B_0 \boldsymbol{c} \quad \text{and} \quad B_0 = D^{-1}E\left(E^t D^{-1} E\right)^{-1}.$$

Let us set:

$$A_0 = D^{-1}E\left(E^t D^{-1} E\right)^{-1} E^t, \quad A_1 = A_0 - I,$$

and let H be a diagonal matrix, such that $\dfrac{dD^{-1}}{dx} = -HD^{-1}$, see Remark 3.3 from the appendix. In fact, we may set

$$H = -\frac{dD^{-1}}{dx} D.$$

Using Corollary 3.5 from the applendix, we have

$$\frac{d\boldsymbol{a}(x)}{dx} = A_1 H \boldsymbol{a}(x) = -A_1 \frac{dD^{-1}}{dx} D\boldsymbol{a}(x). \tag{3.2}$$

But

$$A_1 = D^{-1}E\left(E^t D^{-1} E\right)^{-1} E^t - I$$

$$
\begin{aligned}
&= D^{-1} E \left(\frac{1}{2}\right)^{-1} E^t - I \\
&= 2D^{-1} E E^t - I.
\end{aligned}
$$

Therefore

$$
\frac{d\boldsymbol{a}(x)}{dx} = -\left(2D^{-1}EE^t - I\right)\frac{dD^{-1}}{dx} D\boldsymbol{a}(x). \tag{3.3}
$$

APPENDIX A: MOVING LEAST-SQUARES METHOD

Let:

1. $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$.
2. $\boldsymbol{x}_i \in \mathcal{D}$, $i = 0, \ldots, m$; $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, if $i \neq j$.
3. $f : \mathcal{D} \to \mathbb{R}$ be a continuous function.
4. $p_i : \mathcal{D} \to \mathbb{R}$ be continuous functions, $i = 1, \ldots, l$. The functions $\{p_1, \ldots, p_l\}$ are linearly independent in $\mathcal{D}$ and let $\mathcal{P}_l$ be their linear span.
5. $W : (0, \infty) \to (0, \infty)$ is a strictly positive functions.

Usually the basis in $\mathcal{P}_l$ is constructed by monomials. For example: $p_l(\boldsymbol{x}) = x_1^{k_1} \ldots x_d^{k_d}$, where $\boldsymbol{x} = (x_1, \ldots, x_d)$, $k_1, \ldots k_d \in \mathbb{N}$, $k_1 + \cdots + k_d \leq l - 1$. In the case $d = 1$, the standard basis is $\{1, x, \ldots, x^{l-1}\}$.

Following [5], [6], [7], [8], we use the following definition: The *moving least-squares approximation* of order l at a fixed point $\boldsymbol{x}$ is the value of $p^*(\boldsymbol{x})$, where $p^* \in \mathcal{P}_l$ is minimizing the least-squares error

$$
\sum_{i=1}^{m} W(\|\boldsymbol{x} - \boldsymbol{x}_i\|)\,(p(\boldsymbol{x}) - f(\boldsymbol{x}_i))^2 \tag{3.4}
$$

among all $p \in \mathcal{P}_l$.

The approximation is "local" if weight function $W(s)$ is fast decreasing as s tends to infinity. Interpolation is achieved if $W(0) = \infty$. We define additional function $w : [0, \infty) \to [0, \infty)$, such that:

$$
w(s) = \begin{cases} \dfrac{1}{W(s)}, & \text{if } (s > 0) \text{ or } (s = 0 \text{ and } W(0) < \infty), \\ 0, & \text{if } (s = 0 \text{ and } W(0) = \infty). \end{cases}
$$

Some examples of $W(s)$ and $w(s)$, $s \geq 0$:

$$
\begin{aligned}
&W(s) = e^{-\alpha^2 s^2} && \text{exp-weight,} \\
&W(s) = s^{-\alpha^2} && \text{Shepard weights,} \\
&w(s) = s^2 e^{-\alpha^2 s^2} && \text{McLain weight,} \\
&w(s) = e^{\alpha^2 s^2} - 1 && \text{see Levin's works.}
\end{aligned}
$$

Here and below: the superscript ^{t} denotes transpose of matrix; I is the identity matrix.

Let us introduce the matrices:

$$E = \begin{pmatrix} p_1(\boldsymbol{x}_1) & p_2(\boldsymbol{x}_1) & \cdots & p_l(\boldsymbol{x}_1) \\ p_1(\boldsymbol{x}_2) & p_2(\boldsymbol{x}_2) & \cdots & p_l(\boldsymbol{x}_2) \\ \vdots & \vdots & & \vdots \\ p_1(\boldsymbol{x}_m) & p_2(\boldsymbol{x}_m) & \cdots & p_l(\boldsymbol{x}_m) \end{pmatrix}, \ \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, \ \boldsymbol{c} = \begin{pmatrix} p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ \vdots \\ p_l(\boldsymbol{x}) \end{pmatrix}$$

$$D = 2 \begin{pmatrix} w(\|\boldsymbol{x}-\boldsymbol{x}_1\|) & 0 & \cdots & 0 \\ 0 & w(\|\boldsymbol{x}-\boldsymbol{x}_2\|) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & w(\|\boldsymbol{x}-\boldsymbol{x}_m\|) \end{pmatrix}.$$

We assume the following conditions (H1):

(H1.1) $1 \in \mathcal{P}_l$.
(H1.2) $1 \le l \le m$.
(H1.3) $\operatorname{rank}(E^t) = l$.
(H1.4) w is a smooth function.

Theorem 3.2 (see [6]). *Let the conditions (H1) hold true.*

Then:

1. *The matrix $E^t D^{-1} E$ is non-singular.*
2. *The approximation defined by the moving least-squares method is*

$$\hat{L}(f) = \sum_{i=1}^{m} a_i f(\boldsymbol{x}_i), \tag{3.5}$$

where

$$\boldsymbol{a} = B_0 \boldsymbol{c} \quad \text{and} \quad B_0 = D^{-1}E\left(E^t D^{-1} E\right)^{-1}. \tag{3.6}$$

3. *If $w(0) = 0$, then the approximation is interpolatory.*

Let us set

$$A_0 = D^{-1}E\left(E^t D^{-1} E\right)^{-1} E^t, \quad A_1 = A_0 - I,$$

and let us introduce the following hypothesis (H2):

(H2) There exist matrix H such that

$$\frac{dD}{dx} = HD.$$

Remark 3.3. *If the matrix D is non-singular, the condition (H2) is equivalent to*

$$\frac{dD^{-1}}{dx} = -HD^{-1}.$$

Indeed

$$\begin{aligned}\frac{dD^{-1}}{dx} &= -D^{-1}\frac{dD}{dx}D^{-1}\\ &= -D^{-1}HDD^{-1}\\ &= -D^{-1}H = -HD^{-1}.\end{aligned}$$

Remark 3.4. *It is not difficult to construct the matrix H in the case of exp-weight. For example if $m = 4$, then*

$$D = \begin{pmatrix} e^{-\alpha^2(x-x_1)^2} & 0 & 0 & 0\\ 0 & e^{-\alpha^2(x-x_2)^2} & 0 & 0\\ 0 & 0 & e^{-\alpha^2(x-x_3)^2} & 0\\ 0 & 0 & 0 & e^{-\alpha^2(x-x_4)^2}\end{pmatrix}$$

and

$$\frac{dD}{dx} = HD,$$

where

$$H = -2\alpha^2\begin{pmatrix} x-x_1 & 0 & 0 & 0\\ 0 & x-x_2 & 0 & 0\\ 0 & 0 & x-x_3 & 0\\ 0 & 0 & 0 & x-x_4\end{pmatrix}.$$

Corollary 3.5. *Let the conditions (H1) and (H2) hold true and let $d = 1$.*

Then

$$\frac{d\boldsymbol{a}(x)}{dx} = A_1H\boldsymbol{a}(x) + B_0\frac{d}{dx}\boldsymbol{c}(x), \tag{3.7}$$

and

$$E^t\frac{d\boldsymbol{a}(x)}{dx} = \frac{d}{dx}\boldsymbol{c}. \tag{3.8}$$

Proof. We have

$$\begin{aligned}\frac{d\boldsymbol{a}(x)}{dx} =& \frac{d}{dx}\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c}\right)\\ =& \left(\frac{d}{dx}D^{-1}\right)E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(\frac{d}{dx}\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c}\\ &+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\ =& -HD^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c}\\ &+ D^{-1}E\left(-\left(E^tD^{-1}E\right)^{-1}\left(\frac{d}{d\alpha}E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c}\\ &+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\ =& -H\boldsymbol{a}\end{aligned}$$

$$
\begin{aligned}
&+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tHD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c}\\
&+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\
=&D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tH\right)\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c}\\
&- H\boldsymbol{a}\\
&+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\
=&\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t - I\right)H\boldsymbol{a}\\
&+ D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\
=&A_1H\boldsymbol{a} + B_0\frac{d}{dx}\boldsymbol{c}.
\end{aligned}
$$

Multiplying the equation (3.7) by E^t on the left, yields

$$
\begin{aligned}
E^t\frac{d\boldsymbol{a}(x)}{dx} =&E^t\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t - I\right)H\boldsymbol{a}\\
&+ E^tD^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\
=&\left(\left(E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}E^t - E^t\right)H\boldsymbol{a}\\
&+ \left(E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\frac{d}{dx}\boldsymbol{c}\\
=&\frac{d}{dx}\boldsymbol{c}.
\end{aligned}
$$

Corollary 3.5 has been proven. □

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