NONTRIVIAL SOLUTIONS FOR A FRACTIONAL SCHRODINGER EQUATION VIA CRITICAL POINT THEORY

JIAFA XU\textsuperscript{1}, WEI DONG\textsuperscript{2}, AND DONAL O’REGAN\textsuperscript{3,4}

\textsuperscript{1}School of Mathematical Sciences, Chongqing Normal University,
Chongqing 401331, China
\textit{E-mail:} xujiafa292@sina.com

\textsuperscript{2}Department of Mathematics, Hebei University of Engineering,
Handan 056038, Hebei, China
\textit{E-mail:} wdongau@aliyun.com

\textsuperscript{3}School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway, Ireland
\textsuperscript{4}Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, King Abdulaziz University,
Jeddah, Saudi Arabia
\textit{E-mail:} donal.oregan@nuigalway.ie

\textbf{ABSTRACT.} In this paper we discuss a time-independent fractional Schrödinger equation
\[ (-\Delta)^s u + V(x)u = f(x,u) + g(x) \text{ in } \mathbb{R}^N, \]
where $N \geq 2$, $s \in (0,1)$ and $(-\Delta)^s$ stands for the fractional Laplacian. Using the Mountain Pass Theorem, we establish two existence theorems to ensure that the above problem has at least one nontrivial solution.

\textbf{AMS (MOS) Subject Classification.} 35A15, 35R11.

\section{1. INTRODUCTION}

In this paper we study the fractional Schrödinger equation
\[ (-\Delta)^s u + V(x)u = f(x,u) + g(x) \text{ in } \mathbb{R}^N, \tag{1.1} \]
where $N \geq 2$, $s \in (0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and the conditions on $g(x)$ will be given later. Here the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ of a function $\phi \in \mathcal{S}$ is defined by $\mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi)$, $\forall s \in (0,1)$, where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform, i.e.,
\[ \mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp\{-2\pi i \xi \cdot x\} \phi(x)dx. \]
If $\phi$ is smooth enough, it can also be computed by the following singular integral
\[
(-\Delta)^s \phi(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}} \, dy,
\]
where $P.V.$ is the principal value and $c_{N,s}$ is a normalization constant.

Nonlinear fractional equations have been studied widely in the literature, and we refer the reader to [6, 2, 1, 7, 8, 9, 10, 5, 3, 4, 11, 12, 13] and the references therein. In [1], Chang and Wang investigated the fractional Laplacian equation $(-\Delta)^\alpha u = g(u)$ in $\mathbb{R}^N$ and obtained a positive ground state under general Berestycki-Lions type assumptions. In [2], Chang obtained the existence of ground state solutions for (1.1) when $f(x, u)$ is asymptotically linear with respect to $u$ at infinity and $g \equiv 0$.

In [3], Wei and Su discussed the semi-linear elliptic PDE driven by the fractional Laplacian
\[
\begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Using the Mountain Pass Theorem and some other nonlinear analysis methods, the existence and multiplicity of nontrivial solutions for the above equation is established. In particular:

1. The validity of the (PS) condition without an Ambrosetti-Rabinowitz condition for non-local elliptic equations is discussed. Two nontrivial solutions are given under some weak hypotheses.

2. Non-local elliptic equations with concave-convex nonlinearities are studied, and the existence of at least six solutions is obtained.

3. A global result of Ambrosetti-Brezis-Cerami type is given, which shows that the effect of the parameter $\lambda$ in the nonlinear term changes considerably the nonexistence, existence and multiplicity of solutions.

In [4], Zhang and Yuan investigated the fractional Hamiltonian systems:
\[
\begin{aligned}
\dot{D}_{-\infty}^\alpha (-\infty D_{-\infty}^\alpha u(t)) + L(t)u(t) &= \nabla W(t, u(t)), \\
u &\in H^\alpha(\mathbb{R}, \mathbb{R}^n).
\end{aligned}
\]
They used genus properties in critical point theory to establish infinitely many solutions under the following assumptions:

(FHS)$_1$ $W(t, 0) = 0$ for all $t \in \mathbb{R}$, $W(t, u) \geq a(t)|u|^{\vartheta}$, and $|\nabla W(t, u)| \leq b(t)|u|^\vartheta-1$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$, where $1 < \vartheta < 2$ is a constant, $a : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous function, and $b : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $b \in L^{2/(2-\vartheta)}(\mathbb{R}, \mathbb{R})$.

(FHS)$_2$ There is a constant $1 < \sigma \leq \vartheta < 2$ such that
\[
|\nabla W(t, u)| \leq \sigma W(t, u), \text{ for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \setminus \{0\}.
\]

(FHS)$_3$ $W(t, u) = W(t, -u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$. 
In [5], using the Mountain Pass Theorem, Simone Secchi had obtained that (1.1) with \( g \equiv 0 \) has at least a non-trivial solution when \( f \) has subcritical growth and satisfies the famous Ambrosetti-Rabinowitz condition.

Motivated by the works mentioned above, in this paper, we discuss the existence of nontrivial solutions for (1.1). In particular:

1. With an Ambrosetti-Rabinowitz condition, existence of at least one nontrivial solution is studied.
2. For concave-convex nonlinearities, existence of at least one nontrivial solution is discussed. Also we consider the effect of the parameter \( \lambda \) and the perturbation term \( g \) on the existence of solutions.

2. PRELIMINARIES

Throughout this paper, we assume that

\((V_1)\) \( V \in C(\mathbb{R}^N, \mathbb{R}) \), \( V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0 \).

\((V_2)\) There exists \( r_0 > 0 \) such that, for any \( M > 0 \), \( \text{meas}\{x \in B_{r_0}(y) : V(x) \leq M\} \rightarrow 0 \) as \( |y| \rightarrow \infty \).

Consider the Sobolev space

\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\xi|^{2s} \hat{u}^2 + \hat{u}^2) d\xi < \infty \right\}, \]

where \( \hat{u} = \mathcal{F}(u) \). The norm is defined by

\[ \|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} (|\xi|^{2s} \hat{u}^2 + \hat{u}^2) d\xi \right)^{\frac{1}{2}}. \]

In this paper, in view of the presence of the potential \( V(x) \), we consider its subspace

\[ E = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\}. \]

We define the norm in \( E \) by

\[ \|u\|_E = \left( \int_{\mathbb{R}^N} (|\xi|^{2s} \hat{u}^2 + \hat{u}^2) d\xi + \int_{\mathbb{R}^N} V(x) u^2 dx \right)^{\frac{1}{2}}. \]

Note by Plancherel’s theorem we have \( \|u\|_2 = \|\hat{u}\|_2 \) and

\[ \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} \mathcal{F}(u)(\xi))^2 d\xi = \int_{\mathbb{R}^N} (|\xi|^{s} \hat{u}(\xi))^2 d\xi = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}^2 d\xi < \infty, \forall u \in H^s(\mathbb{R}^N). \]
Together with \((V_1)\), we know that \(\| \cdot \|_E\) is equivalent to the norm
\[
\| u \| = \left( \int_{\mathbb{R}^N} \left( (\Delta)\frac{u}{2} \right)^2 + V(x)u^2 \right) dx \right)^{\frac{1}{2}}.
\]
In the following, we will use the norm \(\| \cdot \|\) in our space \(E\).

**Definition 2.1** We say that \(u \in E\) is a weak solution of (1.1), if
\[
\int_{\mathbb{R}^N} \left( (\Delta) s^2 u \phi + V(x)u \phi \right) dx = \int_{\mathbb{R}^N} f(x, u) \phi dx + \int_{\mathbb{R}^N} g(x) \phi dx, \quad \forall \phi \in E.
\]

**Lemma 2.2** (see [13, Theorem 2.2]) \(H^s(\mathbb{R}^N)\) is continuously embedded into \(L^p(\mathbb{R}^N)\) for \(p \in [2, 2^*_s]\) and compactly embedded into \(L^p(\mathbb{R}^N)\) for \(p \in [2, 2^*_s)\), where \(2^*_s = \frac{2N}{N - 2s}\).

Therefore, there exist positive constants \(\tau_r\) such that
\[
\| u \|_r \leq \tau_r \| u \|, \quad \forall r \in [2, 2^*_s], \quad (2.1)
\]
where \(\| u \|_r = \left( \int_{\mathbb{R}^N} |u(x)|^r dx \right)^{\frac{1}{r}}\).

The functional associated with (1.1) is defined by
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( (\Delta)\frac{u}{2} \right)^2 + V(x)u^2 \right) dx - \int_{\mathbb{R}^N} F(x, u) dx - \int_{\mathbb{R}^N} g(x) u dx, \quad \forall u \in E, \quad (2.2)
\]
where \(F(x, u) = \int_0^u f(x, s) ds\).

**Lemma 2.3** (see [2, Lemma 2.3]) Assume that \((V_1), (V_2)\) hold. Then \(\mu^*\) is an eigenvalue of the operator \((-\Delta)^s + V(x)\) and there exists a corresponding eigenfunction \(\varphi_1\) with \(\varphi_1(x) > 0, \|\varphi_1\|_2 = 1\), for any \(x \in \mathbb{R}^N\). Furthermore, \(\mu^*\) can be expressed explicitly by
\[
\mu^* = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( (\Delta)\frac{u}{2} \right)^2 + V(x)u^2 \right) dx}{\int_{\mathbb{R}^N} u^2 dx}.
\]

**Lemma 2.4** (Mountain Pass Theorem, see [15, 14, 16]) Let \(X\) be a Banach space with the norm \(\| \cdot \|_X\). Suppose that \(J \in C^1(X, \mathbb{R})\) satisfies the (PS) condition with \(J(0) = 0\). In addition,

(i) there are \(\rho, \alpha > 0\) such that \(J(u) \geq \alpha \) when \(\| u \|_X = \rho\),

(ii) there is a \(e \in X\), \(\| e \|_X > \rho\) such that \(J(e) < 0\).

Define \(\Gamma = \{ \gamma \in C^1([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \}\). Then
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)) \geq \alpha
\]
is a critical value of \(J(u)\).
3. MAIN RESULTS

**Theorem 3.1** Let \((V_1), (V_2)\) hold. Suppose that

(H1) There exist \(d_1 > 0, d_2 > 0\) and \(p \in (2, 2^*_a)\) such that

\[
|f(x, s)| \leq d_1|s| + d_2|s|^{p-1}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}.
\]

(H2) \(\lim_{s \to 0} \frac{f(x, s)}{s} < \mu^*\) uniformly for \(x \in \mathbb{R}^N\), where \(\mu^*\) is defined in Lemma 2.3.

(H3) There are \(q > 2\) and \(d_3 > 0\) such that \(qF(x, s) \leq f(x, s)s + d_3s^2\) for all \((x, s) \in \mathbb{R}^N \times \mathbb{R}\).

(H4) There exists \(\alpha \in (2, 2^*_a)\) such that \(\liminf_{|s| \to \infty} \frac{F(x, s)}{|s|^\alpha} > 0\), uniformly in \(x \in \mathbb{R}^N\).

(H5) \(g \in L^{p'}(\mathbb{R}^N)\), where \(\frac{1}{p'} + \frac{1}{p} = 1\), \(p\) is defined by (H1).

There exists \(m_0 > 0\) such that for any \(g\) with \(\|g\|_{p'} \leq m_0\), (1.1) admits a nontrivial solution.

**Proof.** By Lemma 1 of [17], the assumptions \((V_1), (V_2), (H1)\) and (H5) imply that \(J \in C^1(E, \mathbb{R})\) and its derivative is

\[
(J'(u), \phi) = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} \phi + V(x)u\phi)dx - \int_{\mathbb{R}^N} f(x, u)\phi dx
\]

for all \(u, \phi \in E\). Moreover, the critical points of \(J\) are weak solutions of (1.1).

To prove the existence of the nontrivial solution for (1.1) we will use Lemma 2.4.

**Claim 1.** \(J(u)\) satisfies the (PS) condition on \(E\).

Let \(\{u_n\}\) be a (PS) sequence, i.e., \(J(u_n) \to c, J'(u_n) \to 0\) as \(n \to \infty\). Next we show \(\{u_n\}\) is bounded in \(E\). If not, set \(v_n = \frac{u_n}{\|u_n\|}\), then \(\|v_n\| = 1\). Now

\[
\frac{(J'(u_n), u_n)}{\|u_n\|^\alpha} = \frac{\|u_n\|^2}{\|u_n\|^\alpha} - \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n dx}{\|u_n\|^\alpha} - \int_{\mathbb{R}^N} \frac{g(x,u_n)}{\|u_n\|^\alpha} dx,
\]

where \(\alpha\) is as in (H4). Note that

\[
\left| \int_{\mathbb{R}^N} \frac{g(x)u_n dx}{\|u_n\|^\alpha} \right| \leq \frac{\|g\|_{p'} \|u_n\|}{\|u_n\|^\alpha} \leq \tau_p \|g\|_{p'} \|u_n\| \to 0.
\]

Consequently,

\[
\int_{\mathbb{R}^N} \frac{f(x, u_n) u_n dx}{\|u_n\|^\alpha} \to 0.
\]

On the other hand, \(\{v_n\}\) is bounded in \(E\), so we can find some \(v \in E\) such that \(v_n \rightharpoonup v\) weakly in \(E\), \(v_n \to v\) strongly in \(L^p(\mathbb{R}^N)\) with \(p \in [2, 2^*_a)\), \(v_n(x) \to v(x)\) for a.e. \(x \in \mathbb{R}^N\).
Denote $\Omega = \{ x \in \mathbb{R}^N : v(x) \neq 0 \}$. If $\text{meas}(\Omega) > 0$, then $|v(x)| > 0$ a.e. $x \in \Omega$. By (H1)-(H4), there exist $d_4, d_5 > 0$ such that
\[ f(x, u)u \geq d_4 |u|^{\alpha} - d_5 u^2 \quad \text{for all} \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}. \]
Hence
\[
\int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^\alpha} \, dx \geq \int_{\mathbb{R}^N} \frac{d_4 |u_n|^{\alpha} - d_5 u_n^2}{\|u_n\|^\alpha} \, dx = d_4 \int_{\mathbb{R}^N} |u_n|^{\alpha} \, dx - d_5 \int_{\mathbb{R}^N} \frac{|v_n|^2}{\|u_n\|^{\alpha-2}} \, dx.
\]
Passing to the limit above, we see
\[
0 = \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^\alpha} \, dx \geq d_4 \int_{\Omega} |\xi| \, dx > 0,
\]
which leads to a contradiction. Therefore, $\text{meas}(\Omega) = 0$ and thus $v(x) = 0$ a.e. $x \in \mathbb{R}^N$.

In addition, by (H3), we see $uf(x, u) - qF(x, u) \geq -d_3 u^2$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$. Therefore,
\[
\frac{1}{\|u_n\|^2} \left( J(u_n) - \frac{1}{q} \langle J'(u_n), u_n \rangle \right) = \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \frac{1}{q} f(x, u_n)u_n - F(x, u_n) \right) \, dx \\
+ \frac{1}{\|u_n\|^2} \left( \frac{1}{q} - 1 \right) \int_{\mathbb{R}^N} g(x)u_n(x) \, dx \\
\geq \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{d_3}{q} \int_{\mathbb{R}^N} |v_n|^2 \, dx + \left( \frac{1}{q} - 1 \right) \frac{\|g\|_{p'}}{\|u_n\|} \left( \int_{\mathbb{R}^N} |v_n|^{p'} \, dx \right)^{\frac{1}{p'}}.
\]
Also passing to the limit above, we see $0 \geq \frac{1}{2} - \frac{1}{q} > 0$. This is a contradiction. We conclude that $\{u_n\}$ is bounded in $E$. Thus there exists $u_0 \in E$ such that, passing to a subsequence if necessary, we have $u_n \rightharpoonup u_0$ weakly in $E$, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ with $p \in [2, 2^*_N)$, $u_n(x) \to u_0(x)$ for a.e. $x \in \mathbb{R}^N$. Hence, we see
\[
\langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \\
= \|u_n - u_0\|^2 - \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u_0))(u_n - u_0) \, dx. \tag{3.2}
\]
Clearly, $(J'(u_n) - J'(u_0), u_n - u_0) \to 0$ as $n \to \infty$ and the Lebesgue’s dominated convergence theorem enables us to obtain
\[
\left| \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u_0))(u_n - u_0) \, dx \right| \leq \int_{\mathbb{R}^N} \|f(x, u_n) - f(x, u_0)\|\|u_n - u_0\| \, dx \\
\leq \|f(x, u_n) - f(x, u_0)\|_{p'}\|u_n - u_0\|_p \to 0
\]
as $n \to \infty$, with the fact that
\[
\|f(x, u_n) - f(x, u_0)\|_{p'} \leq 4^{p'-1} \int_{\mathbb{R}^N} [d_1'(|u_n|^{p'} + |u_0|^{p'}) + d_2'(|u_n|^p + |u_0|^p)] \, dx < \infty.
\]
Hence we have \( \|u_n - u_0\|^2 \to 0 \), i.e., \( \{u_n\} \) has a convergent subsequence in \( E \), so \( J \) satisfies the (PS) condition. Therefore, Claim 1 is true.

**Claim 2.** There exist \( \alpha > 0 \) and \( \rho > 0 \) such that \( J(u) \geq \alpha > 0 \) for any \( u \in E \) with \( \|u\| = \rho \).

By (H1) and (H2), there exist \( \delta_0 > 0, d_6 > 0 \) such that
\[
F(x, u) \leq \frac{\mu^* - \delta_0}{2} |u|^2 + d_6 |u|^p, \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}.
\]
Consequently,
\[
J(u) \geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} \left[ \frac{\mu^* - \delta_0}{2} |u|^2 + d_6 |u|^p \right] \, dx - \tau_p \|g\|_{p'} \|u\|
\geq \frac{\delta_0}{2\mu^*} \|u\|^2 - d_6 \tau_p \|u\|^p - \tau_p \|g\|_{p'} \|u\|.
\]
Taking \( \rho = r^{\frac{\delta_0}{4\mu^*(d_6\tau_p + \tau_p)}} \), \( m_0 = \rho^{p-1} \), Claim 2 holds with \( \alpha = \frac{\delta_0}{4\mu^*(d_6\tau_p + \tau_p)} \left[ \rho \right]^{\frac{p}{p-2}} \).

**Claim 3.** There exists \( e \in E \) with \( \|e\| > \rho \) such that \( J(e) < 0 \).

By (H4), there exist \( l > 0 \) and \( d_l > 0 \) such that \( F(x, u) \geq d_l |u|^{\alpha} \) for \( x \in \mathbb{R}^N \) and \( |u| \geq l \). Note from (H1), we get
\[
|F(x, u)| \leq \int_0^1 |f(x, ut)| \, dt \leq \int_0^1 (d_1 |ut| + d_2 |ut|^{p-1}) \, |u| \, dt \leq d_1 (u^2 + |u|^p),
\]
for all \( (x, u) \in \mathbb{R}^N \times \mathbb{R} \), for some appropriate positive constant \( d_1 \). Hence, \( |F(x, u)| \leq d_1 (1 + l^{p-2}) u^2 \) for \( x \in \mathbb{R}^N, |u| \leq l \), which yields that
\[
F(x, u) \geq d_l |u|^{\alpha} - d_2 l^2, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R},
\]
where \( d_2 = d_1 (1 + l^{p-2}) \). Consequently,
\[
J(s \varphi_1) = \frac{s^2}{2} \|\varphi_1\|^2 - \int_{\mathbb{R}^N} F(x, s \varphi_1(x)) \, dx - s \int_{\mathbb{R}^N} g(x) \varphi_1(x) \, dx
\leq \frac{s^2}{2} \|\varphi_1\|^2 - s^\alpha d_1 \int_{\mathbb{R}^N} |\varphi_1|^\alpha \, dx + s^2 d_2 \int_{\mathbb{R}^N} \varphi_1^2 \, dx - s \int_{\mathbb{R}^N} g(x) \varphi_1(x) \, dx.
\]
Since \( \alpha > 2 \) in (H4), \( J(s \varphi_1) \to -\infty \) as \( s \to +\infty \). Thus there exists \( s_1 > 0 \) so large that \( \|s_1 \varphi_1\| > \rho \) and \( J(s_1 \varphi_1) < 0 \). This shows Claim 3.

Therefore, it follows from Lemma 2.4 that there exists \( u_1 \in E \) such that \( u_1 \) is a solution of (1.1). Furthermore, \( J(u_1) \geq \alpha > 0 \). This completes the proof. \( \square \)

**Theorem 3.2** Let \( g \in L^2(\mathbb{R}^N) \). Suppose (V1), (V2) and
\[
(H6) \quad f(x, u) = \lambda h_1(x) |u|^{q-2} u + h_2(x) |u|^{r-2} u \quad \text{with} \quad 1 < q < 2 < r < 2^*_s \quad \text{for} \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}, \quad \text{in which} \quad h_1 \in L^{q_0}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \quad \text{and} \quad h_2 \in L^{\infty}(\mathbb{R}^N) \quad \text{where} \quad q_0 = 2/(2 - q). \quad \text{In addition, there exists a non-empty open domain} \quad \Omega \subset \mathbb{R}^N \quad \text{such that} \quad h_2 > 0 \quad \text{in} \quad \Omega.
\]

Then there exist \( \lambda_0, m_0 > 0 \) such that for all \( \lambda \in (0, \lambda_0), (1.1) \) has a nontrivial weak solution in \( E \) when \( \|g\|_2 \leq m_0 \).
By the Hölder inequality, we have
\[ J(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\mathbb{R}^N} h_1(x)|u|^q dx - \frac{1}{r} \int_{\mathbb{R}^N} h_2(x)|u|^r dx - \int_{\mathbb{R}^N} g(x)u dx, \]
and for any \( \phi \in E \), we have
\[ (J'(u), \phi) = \int_{\mathbb{R}^N} (-(\Delta)^{\frac{\nu}{2}} u(-\Delta)^{\frac{\nu}{2}} \phi + V(x) u \phi) dx \]
\[ - \int_{\mathbb{R}^N} \lambda h_1 |u|^{q-2} u \phi dx - \int_{\mathbb{R}^N} h_2 |u|^{r-2} u \phi dx - \int_{\mathbb{R}^N} g(x) \phi dx. \]
By the Hölder inequality, we have
\[ \int_{\mathbb{R}^N} |h_1| |u|^q dx \leq \|h_1\|_{q_0} \|u\|_{q_0}^q \leq V_0^{-\frac{q}{2}} \|h_1\|_{q_0} \|u\|^q \text{ with } q_0 = 2/(2-q). \]
On the other hand, by (2.1), we have
\[ \int_{\mathbb{R}^N} |h_2| |u|^r dx \leq \|h_2\|_{\infty} \|u\|^r \leq \tau_r^r \|h_2\|_{\infty} \|u\|^r. \]
Similarly, we have by the Young inequality with \( \varepsilon \),
\[ \int_{\mathbb{R}^N} |g| |u| dx \leq \|g\|_2 \|u\|_2 \leq V_0^{-\frac{1}{2}} \|g\|_2 \|u\| \leq \frac{1}{4} \|u\|^2 + C_\varepsilon \|g\|_2^2. \]
Therefore,
\[ J(u) \geq \frac{1}{4} \|u\|^2 - \lambda \beta_1 \|u\|^q - \beta_2 \|u\|^r - C_\varepsilon \|g\|_2^2 \]
with \( \beta_1 = \frac{1}{q} V_0^{-\frac{q}{2}} \|h_1\|_{q_0}, \beta_2 = \frac{1}{r} \tau_r^r \|h_2\|_{\infty}. \) (3.3)

Let
\[ G(z) = \lambda \beta_1 z^{q-2} + \beta_2 z^{r-2}, z > 0. \]
We show \( G(z_0) < \frac{1}{4} \) for some \( z_0 > 0 \). Note that \( G(z) \to +\infty \) as \( z \to 0^+ \) or \( z \to +\infty \). Then \( G(z) \) has a minimum at \( z_0 > 0 \). In order to find \( z_0 \), note
\[ G'(z_0) = \lambda \beta_1 (q-2) z_0^{q-3} + \beta_2 (r-2) z_0^{r-3} = 0 \text{ and } z_0 = \lambda^{1/(r-q)} \left( \frac{\beta_1 (2-q)(r-2)}{\beta_2 (2-q)(r-2)} \right)^{1/(r-q)} > 0. \]
Thus \( G(z_0) = \lambda^{(r-2)/(r-q)} (\beta_1 \beta_2^{(q-2)/(r-q)} + \beta_2 \beta_0^{(r-2)/(r-q)}) \) with \( \beta_0 = \frac{\beta_1 (2-q) \beta_2 (r-2)}{\beta_1 (2-q) \beta_2 (r-2)}. \) This shows that there exists \( \lambda_0 > 0 \) such that \( 0 < \lambda < \lambda_0 \) and \( G(z_0) < \frac{1}{4}. \) Hence, (3.3) implies that there exists \( m_0, \alpha > 0 \) such that \( J(u) \geq \alpha \) with \( z_0 = \|u\| \) and \( \|g\|_2 \leq m_0. \) As a result, (i) of Lemma 2.4 holds.

Choose \( \varphi_2 \in C_{c,0}^\infty(\Omega), \varphi_2 \geq 0, \varphi_2 \not\equiv 0 \) in \( \Omega. \) We know that \( h_2 > 0 \) in \( \Omega. \) Then
\[ J(t \varphi_2) = \frac{t^2}{2} \|\varphi_2\|^2 - \frac{\lambda t^q}{q} \int_{\Omega} h_1(x)|\varphi_2|^q dx - \frac{t^r}{r} \int_{\Omega} h_2(x)|\varphi_2|^r dx - t \int_{\Omega} g(x)\varphi_2 dx \to -\infty \]
as \( t \to +\infty \) with \( 1 < q < 2 < r. \) Thus, there exists \( t_2 \) large enough, such that \( J(t_2 \varphi_2) < 0. \) Let \( e = t_2 \varphi_2 \in E \) and then \( J(e) < 0, \) so (ii) of Lemma 2.4 also holds.
Finally, we prove that $J$ satisfies the (PS) condition. Suppose that $\{u_n\}_{n=1}^\infty$ is a (PS) sequence, i.e., $J(n_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. We claim that $\{u_n\}$ is bounded in $E$. For $n$ large enough, we see
\[
c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{r}(J'(u_n), u_n)
\]
\[
= \left(\frac{1}{2} - \frac{1}{r}\right)\|u_n\|^2 + \lambda \left(\frac{1}{r} - \frac{1}{q}\right)\int_{\mathbb{R}^N} h_1(x)|u_n|^q dx + \left(\frac{1}{r} - 1\right)\int_{\mathbb{R}^N} g(x)u_n dx
\]
\[
\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_n\|^2 + \lambda \left(\frac{1}{r} - \frac{1}{q}\right) V_0^{-\frac{q}{2}}\|h_1\|_{q_0} \|u_n\|^q + \left(\frac{1}{r} - 1\right) V_0^{-\frac{1}{2}}\|g\|_2 \|u_n\|.
\]
Since $1 < q < 2 < r$, we see $\{u_n\}$ is bounded in $E$, as required. Consequently, there exists $u \in E$ such that, passing to a subsequence if necessary, we have $u_n \rightharpoonup u$ weakly in $E$, $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ with $p \in [2, 2^*_s)$, $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$.

As mentioned above, in order to have $u_n \to u$ strongly in $E$, it sufficient to show that
\[
\int_{\mathbb{R}^N} f(x, u_n)(u_n - u)dx = \int_{\mathbb{R}^N} (\lambda h_1(x)|u_n|^{q-2}u_n + h_2(x)|u_n|^{r-2}u_n)(u_n - u)dx \to 0.
\]
Indeed,
\[
\int_{\mathbb{R}^N} |h_1||u_n|^{q-1}|u_n - u|dx \leq \|h_1\|_{q_0} \|u_n\|^{q-1}_2 \|u_n - u\|_2 \to 0,
\]
and
\[
\int_{\mathbb{R}^N} |h_2||u_n|^{r-1}|u_n - u|dx \leq \|h_2\|_\infty \|u_n\|^{r-1}_r \|u_n - u\|_r \to 0.
\]
Therefore, $J$ satisfies the (PS) condition, as claimed. All the assumptions in Lemma 2.4 are satisfied for $J$. Then there exists $u_2 \in E$ such that $u_2$ is a solution of (1.1) and $J(u_2) \geq \alpha > 0$. This completes the proof. \)

ACKNOWLEDGMENTS

This work was supported by the Natural Science Foundation of Chongqing Normal University (15XLB011).

REFERENCES


