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NONTRIVIAL SOLUTIONS FOR A FRACTIONAL SCHRÖDINGER EQUATION VIA CRITICAL POINT THEORY

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ABSTRACT. In this paper we discuss a time-independent fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u) + g(x) \text{ in } \mathbb{R}^N,$$

where $N \geq 2$, $s \in (0,1)$ and $(-\Delta)^s$ stands for the fractional Laplacian. Using the Mountain Pass Theorem, we establish two existence theorems to ensure that the above problem has at least one nontrivial solution.

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1. INTRODUCTION

In this paper we study the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u) + g(x) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $s \in (0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and the conditions on $g(x)$ will be given later. Here the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ of a function $\phi \in \mathcal{S}$ is defined by $\mathcal{F}((-\Delta)^s\phi)(\xi) = |\xi|^{2s}\mathcal{F}(\phi)(\xi)$, $\forall s \in (0,1)$, where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform, i.e.,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp\{-2\pi i\xi \cdot x\}\phi(x)\mathrm{d}x.$$

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If ϕ is smooth enough, it can also be computed by the following singular integral

$$(-\Delta)^s\phi(x) = c_{N,s}P.V.\int_{\mathbb{R}^N}\frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

where $P.V.$ is the principal value and $c_{N,s}$ is a normalization constant.

Nonlinear fractional equations have been studied widely in the literature, and we refer the reader to [6, 2, 1, 7, 8, 9, 10, 5, 3, 4, 11, 12, 13] and the references therein. In [1], Chang and Wang investigated the fractional Laplacian equation $(-\Delta)^\alpha u = g(u)$ in $\mathbb{R}^N$ and obtained a positive ground state under general Berestycki-Lions type assumptions. In [2], Chang obtained the existence of ground state solutions for (1.1) when $f(x,u)$ is asymptotically linear with respect to u at infinity and $g \equiv 0$.

In [3], Wei and Su discussed the semi-linear elliptic PDE driven by the fractional Laplacian

$$\begin{cases}(-\Delta)^s u = f(x,u) \text{ in } \Omega,\\ u = 0, \text{ in } \mathbb{R}^n\backslash\Omega.\end{cases}$$

Using the Mountain Pass Theorem and some other nonlinear analysis methods, the existence and multiplicity of nontrivial solutions for the above equation is established. In particular:

(1) The validity of the (PS) condition without a Ambrosetti-Rabinowitz condition for non-local elliptic equations is discussed. Two nontrivial solutions are given under some weak hypotheses.

(2) Non-local elliptic equations with concave-convex nonlinearities are studied, and the existence of at least six solutions is obtained.

(3) A global result of Ambrosetti-Brezis-Cerami type is given, which shows that the effect of the parameter λ in the nonlinear term changes considerably the nonexistence, existence and multiplicity of solutions.

In [4], Zhang and Yuan investigated the fractional Hamiltonian systems:

$$\begin{cases}{}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t,u(t)),\\ u \in H^\alpha(\mathbb{R},\mathbb{R}^n).\end{cases}$$

They used genus properties in critical point theory to establish infinitely many solutions under the following assumptions:

(FHS)$_1$ $W(t,0)=0$ for all $t\in\mathbb{R}$, $W(t,u)\ge a(t)|u|^\vartheta$, and $|\nabla W(t,u)| \le b(t)|u|^{\vartheta-1}$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$, where $1<\vartheta<2$ is a constant, $a:\mathbb{R}\to\mathbb{R}^+$ is a bounded continuous function, and $b:\mathbb{R}\to\mathbb{R}^+$ is a continuous function such that $b\in L^{2/(2-\vartheta)}(\mathbb{R},\mathbb{R})$.

(FHS)$_2$ There is a constant $1<\sigma\le\vartheta<2$ such that

$$(\nabla W(t,u),u) \le \sigma W(t,u), \text{ for all } t\in\mathbb{R} \text{ and } u\in\mathbb{R}^n\backslash\{0\}.$$

(FHS)$_3$ $W(t,u) = W(t,-u)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$.

In [5], using the Mountain Pass Theorem, Simone Secchi had obtained that (1.1) with $g \equiv 0$ has at least a non-trivial solution when f has subcritical growth and satisfies the famous Ambrosetti-Rabinowitz condition.

Motivated by the works mentioned above, in this paper, we discuss the existence of nontrivial solutions for (1.1). In particular:

(1) With a Ambrosetti-Rabinowitz condition, existence of at least one nontrivial solution is studied.

(2) For concave-convex nonlinearities, existence of at least one nontrivial solution is discussed. Also we consider the effect of the parameter λ and the perturbation term g on the existence of solutions.

2. PRELIMINARIES

Throughout this paper, we assume that

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) There exists $r_0 > 0$ such that, for any $M > 0$, $\text{meas}(\{x \in B_{r_0}(y) : V(x) \leq M\}) \to 0$ as $|y| \to \infty$.

Consider the Sobolev space

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\mathrm{d}\xi < \infty\right\},$$

where $\hat{u} = \mathcal{F}(u)$. The norm is defined by

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

In this paper, in view of the presence of the potential $V(x)$, we consider its subspace

$$E = \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x < \infty\right\}.$$

We define the norm in E by

$$\|u\|_E = \left(\int_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\mathrm{d}\xi + \int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Note by Plancherel's theorem we have $\|u\|_2 = \|\hat{u}\|_2$ and

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u(x)|^2\mathrm{d}x &= \int_{\mathbb{R}^N} ((\widehat{-\Delta)^{\frac{s}{2}}u}(\xi))^2\mathrm{d}\xi \\ &= \int_{\mathbb{R}^N} (|\xi|^s\hat{u}(\xi))^2\mathrm{d}\xi \\ &= \int_{\mathbb{R}^N} |\xi|^{2s}\hat{u}^2\mathrm{d}\xi < \infty, \forall u \in H^s(\mathbb{R}^N).\end{aligned}$$

Together with (V_1), we know that $\|\cdot\|_E$ is equivalent to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

In the following, we will use the norm $\|\cdot\|$ in our space E.

Definition 2.1 We say that $u \in E$ is a weak solution of (1.1), if

$$\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi + V(x)u\phi)\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\phi\mathrm{d}x + \int_{\mathbb{R}^N} g(x)\phi\mathrm{d}x,\ \forall\phi \in E.$$

Lemma 2.2 (see [13, Theorem 2.2]) $H^s(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$ and compactly embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$.

Therefore, there exist positive constants τ_r such that

$$\|u\|_r \le \tau_r\|u\|,\ \forall r \in [2, 2_s^*], \tag{2.1}$$

where $\|u\|_r = (\int_{\mathbb{R}^N}|u(x)|^r\mathrm{d}x)^{\frac{1}{r}}$.

The functional associated with (1.1) is defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} g(x)u\mathrm{d}x,$$
$$\forall u \in E, \tag{2.2}$$

where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$.

Lemma 2.3 (see [2, Lemma 2.3]) Assume that (V_1), (V_2) hold. Then μ^* is an eigenvalue of the operator $(-\Delta)^s + V(x)$ and there exists a corresponding eigenfunction φ_1 with $\varphi_1(x) > 0, \|\varphi_1\|_2 = 1$, for any $x \in \mathbb{R}^N$. Furthermore, μ^* can be expressed explicitly by

$$\mu^* = \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2\right)\mathrm{d}x}{\int_{\mathbb{R}^N} u^2\mathrm{d}x}.$$

Lemma 2.4 (Mountain Pass Theorem, see [15, 14, 16]) Let X be a Banach space with the norm $\|\cdot\|_X$. Suppose that $J \in C^1(X,\mathbb{R})$ satisfies the (PS) condition with $J(0) = 0$. In addition,

(i) there are $\rho, \alpha > 0$ such that $J(u) \ge \alpha$ when $\|u\|_X = \rho$,

(ii) there is a $e \in X$, $\|e\|_X > \rho$ such that $J(e) < 0$.

Define $\Gamma = \{\gamma \in C^1([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$. Then

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \ge \alpha$$

is a critical value of $J(u)$.

3. MAIN RESULTS

Theorem 3.1 Let (V_1), (V_2) hold. Suppose that

(H1) There exist $d_1 > 0, d_2 > 0$ and $p \in (2, 2_s^*)$ such that

$$|f(x,s)| \leq d_1|s| + d_2|s|^{p-1}, \ \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

(H2) $\lim_{s\to 0} \frac{f(x,s)}{s} < \mu^*$ uniformly for $x \in \mathbb{R}^N$, where μ^* is defined in Lemma 2.3.

(H3) There are $q > 2$ and $d_3 > 0$ such that $qF(x,s) \leq f(x,s)s + d_3 s^2$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$.

(H4) There exists $\alpha \in (2, 2_s^*)$ such that $\liminf_{|s|\to\infty} \frac{F(x,s)}{|s|^\alpha} > 0$, uniformly in $x \in \mathbb{R}^N$.

(H5) $g \in L^{p'}(\mathbb{R}^N)$, where $\frac{1}{p'} + \frac{1}{p} = 1$, p is defined by (H1).

There exists $m_0 > 0$ such that for any g with $\|g\|_{p'} \leq m_0$, (1.1) admits a nontrivial solution.

Proof. By Lemma 1 of [17], the assumptions (V_1), (V_2), (H1) and (H5) imply that $J \in C^1(E, \mathbb{R})$ and its derivative is

$$(J'(u), \phi) = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi + V(x)u\phi)\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)\phi \mathrm{d}x - \int_{\mathbb{R}^N} g(x)\phi \mathrm{d}x, \quad (3.1)$$

for all $u, \phi \in E$. Moreover, the critical points of J are weak solutions of (1.1).

To prove the existence of the nontrivial solution for (1.1) we will use Lemma 2.4.

Claim 1. $J(u)$ satisfies the (PS) condition on E.

Let $\{u_n\}$ be a (PS) sequence, i.e., $J(u_n) \to c, J'(u_n) \to 0$ as $n \to \infty$. Next we show $\{u_n\}$ is bounded in E. If not, set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Now

$$\frac{(J'(u_n), u_n)}{\|u_n\|^\alpha} = \frac{\|u_n\|^2}{\|u_n\|^\alpha} - \int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^\alpha}\mathrm{d}x - \frac{\int_{\mathbb{R}^N} g(x)u_n \mathrm{d}x}{\|u_n\|^\alpha},$$

where α is as in (H4). Note that

$$\left| \frac{\int_{\mathbb{R}^N} g(x)u_n \mathrm{d}x}{\|u_n\|^\alpha} \right| \leq \frac{\|g\|_{p'}\|u_n\|_p}{\|u_n\|^\alpha} \leq \frac{\tau_p\|g\|_{p'}\|u_n\|}{\|u_n\|^\alpha} \to 0.$$

Consequently,

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^\alpha}\mathrm{d}x \to 0.$$

On the other hand, $\{v_n\}$ is bounded in E, so we can find some $v \in E$ such that $v_n \rightharpoonup v$ weakly in E, $v_n \to v$ strongly in $L^p(\mathbb{R}^N)$ with $p \in [2, 2_s^*)$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$.

Denote $\Omega = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. If meas$(\Omega) > 0$, then $|v(x)| > 0$ a.e. $x \in \Omega$. By (H1)-(H4), there exist $d_4, d_5 > 0$ such that

$$f(x,u)u \geq d_4|u|^\alpha - d_5u^2 \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^\alpha}\mathrm{d}x \geq \int_{\mathbb{R}^N} \frac{d_4|u_n|^\alpha - d_5u_n^2}{\|u_n\|^\alpha}\mathrm{d}x = d_4\int_{\mathbb{R}^N}|v_n|^\alpha \mathrm{d}x - d_5\int_{\mathbb{R}^N}\frac{|v_n|^2}{\|u_n\|^{\alpha-2}}\mathrm{d}x.$$

Passing to the limit above, we see

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{f(x,u_n)u_n}{\|u_n\|^\alpha}\mathrm{d}x \geq d_4\int_\Omega |v|^\alpha \mathrm{d}x > 0,$$

which leads to a contradiction. Therefore, meas$(\Omega) = 0$ and thus $v(x) = 0$ a.e. $x \in \mathbb{R}^N$.

In addition, by (H3), we see $uf(x,u) - qF(x,u) \geq -d_3u^2$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Therefore,

$$\begin{aligned}
&\frac{1}{\|u_n\|^2}\left(J(u_n) - \frac{1}{q}(J'(u_n),u_n)\right)\\
&= \left(\frac{1}{2} - \frac{1}{q}\right) + \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N}\left[\frac{1}{q}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x\\
&\quad + \frac{1}{\|u_n\|^2}\left(\frac{1}{q} - 1\right)\int_{\mathbb{R}^N} g(x)u_n(x)\mathrm{d}x\\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right) - \frac{d_3}{q}\int_{\mathbb{R}^N}|v_n|^2\mathrm{d}x + \left(\frac{1}{q} - 1\right)\frac{\|g\|_{p'}}{\|u_n\|}\left(\int_{\mathbb{R}^N}|v_n|^p\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

Also passing to the limit above, we see $0 \geq \frac{1}{2} - \frac{1}{q} > 0$. This is a contradiction. We conclude that $\{u_n\}$ is bounded in E. Thus there exists $u_0 \in E$ such that, passing to a subsequence if necessary, we have $u_n \rightharpoonup u_0$ weakly in E, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ with $p \in [2, 2_s^*)$, $u_n(x) \to u_0(x)$ for a.e. $x \in \mathbb{R}^N$. Hence, we see

$$\begin{aligned}
&(J'(u_n) - J'(u_0), u_n - u_0)\\
&\qquad = \|u_n - u_0\|^2 - \int_{\mathbb{R}^N}(f(x,u_n) - f(x,u_0))(u_n - u_0)\mathrm{d}x. \quad (3.2)
\end{aligned}$$

Clearly, $(J'(u_n) - J'(u_0), u_n - u_0) \to 0$ as $n \to \infty$ and the Lebesgue's dominated convergence theorem enables us to obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u_0))(u_n - u_0)\mathrm{d}x\right| &\leq \int_{\mathbb{R}^N}|(f(x,u_n) - f(x,u_0))||(u_n - u_0)|\mathrm{d}x\\
&\leq \|f(x,u_n) - f(x,u_0)\|_{p'}\|u_n - u_0\|_p \to 0
\end{aligned}$$

as $n \to \infty$, with the fact that

$$\|f(x,u_n) - f(x,u_0)\|_{p'}^{p'} \leq 4^{p'-1}\int_{\mathbb{R}^N}[d_1^{p'}(|u_n|^{p'} + |u_0|^{p'}) + d_2^{p'}(|u_n|^p + |u_0|^p)]\mathrm{d}x < \infty.$$

Hence we have $\|u_n - u_0\|^2 \to 0$, i.e., $\{u_n\}$ has a convergent subsequence in E, so J satisfies the (PS) condition. Therefore, Claim 1 is true.

Claim 2. There exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\| = \rho$.

By (H1) and (H2), there exist $\delta_0 > 0, d_6 > 0$ such that

$$F(x,u) \leq \frac{\mu^* - \delta_0}{2}|u|^2 + d_6|u|^p,\ (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Consequently,

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N}\left[\frac{\mu^* - \delta_0}{2}|u|^2 + d_6|u|^p\right]\mathrm{d}x - \tau_p\|g\|_{p'}\|u\| \\ &\geq \frac{\delta_0}{2\mu^*}\|u\|^2 - d_6\tau_p^p\|u\|^p - \tau_p\|g\|_{p'}\|u\|. \end{aligned}$$

Taking $\rho = \sqrt[p-2]{\frac{\delta_0}{4\mu^*(d_6\tau_p^p + \tau_p)}}$, $m_0 = \rho^{p-1}$, Claim 2 holds with $\alpha = \frac{\delta_0}{4\mu^*}\left[\frac{\delta_0}{4\mu^*(d_6\tau_p^p + \tau_p)}\right]^{\frac{2}{p-2}}$.

Claim 3. There exists $e \in E$ with $\|e\| > \rho$ such that $J(e) < 0$.

By (H4), there exist $l > 0$ and $d_l > 0$ such that $F(x,u) \geq d_l|u|^\alpha$ for $x \in \mathbb{R}^N$ and $|u| \geq l$. Note from (H1), we get

$$|F(x,u)| \leq \int_0^1 |f(x,ut)u|\mathrm{d}t \leq \int_0^1 (d_1|ut| + d_2|ut|^{p-1})|u|\mathrm{d}t \leq d_{l1}(u^2 + |u|^p),$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, for some appropriate positive constant d_{l1}. Hence, $|F(x,u)| \leq d_{l1}(1 + l^{p-2})u^2$ for $x \in \mathbb{R}^N$, $|u| \leq l$, which yields that

$$F(x,u) \geq d_l|u|^\alpha - d_{l2}u^2,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $d_{l2} = d_{l1}(1 + l^{p-2})$. Consequently,

$$\begin{aligned} J(s\varphi_1) &= \frac{s^2}{2}\|\varphi_1\|^2 - \int_{\mathbb{R}^N} F(x, s\varphi_1(x))\mathrm{d}x - s\int_{\mathbb{R}^N} g(x)\varphi_1(x)\mathrm{d}x \\ &\leq \frac{s^2}{2}\|\varphi_1\|^2 - s^\alpha d_l\int_{\mathbb{R}^N}|\varphi_1|^\alpha\mathrm{d}x + s^2 d_{l2}\int_{\mathbb{R}^N}\varphi_1^2\mathrm{d}x - s\int_{\mathbb{R}^N} g(x)\varphi_1(x)\mathrm{d}x. \end{aligned}$$

Since $\alpha > 2$ in (H4), $J(s\varphi_1) \to -\infty$ as $s \to +\infty$. Thus there exists $s_1 > 0$ so large that $\|s_1\varphi_1\| > \rho$ and $J(s_1\varphi_1) < 0$. This shows Claim 3.

Therefore, it follows from Lemma 2.4 that there exists $u_1 \in E$ such that u_1 is a solution of (1.1). Furthermore, $J(u_1) \geq \alpha > 0$. This completes the proof. □

Theorem 3.2 Let $g \in L^2(\mathbb{R}^N)$. Suppose (V_1), (V_2) and

(H6) $f(x,u) = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u$ with $1 < q < 2 < r < 2_s^*$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, in which $h_1 \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h_2 \in L^\infty(\mathbb{R}^N)$ with $q_0 = 2/(2-q)$. In addition, there exists a non-empty open domain $\Omega \subset \mathbb{R}^N$ such that $h_2 > 0$ in Ω.

Then there exist $\lambda_0, m_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (1.1) has a nontrivial weak solution in E when $\|g\|_2 \leq m_0$.

Proof. By Theorem 3.1 and [18, Lemma 2.1], we easily find $J \in C^1(E,\mathbb{R})$ has the following form

$$J(u)=\frac{1}{2}\|u\|^2-\frac{\lambda}{q}\int_{\mathbb{R}^N}h_1(x)|u|^q\mathrm{d}x-\frac{1}{r}\int_{\mathbb{R}^N}h_2(x)|u|^r\mathrm{d}x-\int_{\mathbb{R}^N}g(x)u\mathrm{d}x,$$

and for any $\phi \in E$, we have

$$\begin{aligned}(J'(u),\phi)=&\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi+V(x)u\phi)\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\lambda h_1|u|^{q-2}u\phi\mathrm{d}x-\int_{\mathbb{R}^N}h_2|u|^{r-2}u\phi\mathrm{d}x-\int_{\mathbb{R}^N}g(x)\phi\mathrm{d}x.\end{aligned}$$

By the Hölder inequality, we have

$$\int_{\mathbb{R}^N}|h_1||u|^q\mathrm{d}x\le\|h_1\|_{q_0}\|u\|_2^q\le V_0^{-\frac{q}{2}}\|h_1\|_{q_0}\|u\|^q \text{ with } q_0=2/(2-q).$$

On the other hand, by (2.1), we have

$$\int_{\mathbb{R}^N}|h_2(x)||u|^r\mathrm{d}x\le\|h_2\|_\infty\|u\|_r^r\le\tau_r^r\|h_2\|_\infty\|u\|^r.$$

Similarly, we have by the Young inequality with ε,

$$\int_{\mathbb{R}^N}|g(x)||u|\mathrm{d}x\le\|g\|_2\|u\|_2\le V_0^{-\frac{1}{2}}\|g\|_2\|u\|\le\frac{1}{4}\|u\|^2+C_\varepsilon\|g\|_2^2.$$

Therefore,

$$J(u)\ge\frac{1}{4}\|u\|^2-\lambda\beta_1\|u\|^q-\beta_2\|u\|^r-C_\varepsilon\|g\|_2^2$$
$$\text{with}\quad \beta_1=\frac{1}{q}V_0^{-\frac{q}{2}}\|h_1\|_{q_0},\beta_2=\frac{1}{r}\tau_r^r\|h_2\|_\infty. \quad (3.3)$$

Let

$$G(z)=\lambda\beta_1z^{q-2}+\beta_2z^{r-2}, z>0.$$

We show $G(z_0)<\frac{1}{4}$ for some $z_0>0$. Note that $G(z)\to+\infty$ as $z\to 0^+$ or $z\to+\infty$. Then $G(z)$ has a minimum at $z_0>0$. In order to find z_0, note

$$G'(z_0)=\lambda\beta_1(q-2)z_0^{q-3}+\beta_2(r-2)z_0^{r-3}=0 \text{ and } z_0=\lambda^{1/(r-q)}\left(\frac{\beta_1(2-q)}{\beta_2(r-2)}\right)^{1/(r-q)}>0.$$

Thus $G(z_0)=\lambda^{(r-2)/(r-q)}(\beta_1\beta_0^{(q-2)/(r-q)}+\beta_2\beta_0^{(r-2)/(r-q)})$ with $\beta_0=\frac{\beta_1(2-q)}{\beta_2(r-2)}$. This shows that there exists $\lambda_0>0$ such that $0<\lambda<\lambda_0$ and $G(z_0)<\frac{1}{4}$. Hence, (3.3) implies that there exists $m_0,\alpha>0$ such that $J(u)\ge\alpha$ with $z_0=\|u\|$ and $\|g\|_2\le m_0$. As a result, (i) of Lemma 2.4 holds.

Choose $\varphi_2\in C_0^\infty(\Omega)$, $\varphi_2\ge 0$, $\varphi_2\not\equiv 0$ in Ω. We know that $h_2>0$ in Ω. Then

$$J(t\varphi_2)=\frac{t^2}{2}\|\varphi_2\|^2-\frac{\lambda t^q}{q}\int_\Omega h_1(x)|\varphi_2|^q\mathrm{d}x-\frac{t^r}{r}\int_\Omega h_2(x)|\varphi_2|^r\mathrm{d}x-t\int_\Omega g(x)\varphi_2\mathrm{d}x\to-\infty$$

as $t\to+\infty$ with $1<q<2<r$. Thus, there exists t_2 large enough, such that $J(t_2\varphi_2)<0$. Let $e=t_2\varphi_2\in E$ and then $J(e)<0$, so (ii) of Lemma 2.4 also holds.

Finally, we prove that J satisfies the (PS) condition. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a (PS) sequence, i.e., $J(n_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. We claim that $\{u_n\}$ is bounded in E. For n large enough, we see

$$\begin{aligned} c+1+\|u_n\| &\geq J(u_n) - \frac{1}{r}(J'(u_n), u_n) \\ &= \left(\frac{1}{2}-\frac{1}{r}\right)\|u_n\|^2 + \lambda\left(\frac{1}{r}-\frac{1}{q}\right)\int_{\mathbb{R}^N} h_1(x)|u_n|^q \mathrm{d}x + \left(\frac{1}{r}-1\right)\int_{\mathbb{R}^N} g(x)u_n \mathrm{d}x \\ &\geq \left(\frac{1}{2}-\frac{1}{r}\right)\|u_n\|^2 + \lambda\left(\frac{1}{r}-\frac{1}{q}\right)V_0^{-\frac{q}{2}}\|h_1\|_{q_0}\|u_n\|^q + \left(\frac{1}{r}-1\right)V_0^{-\frac{1}{2}}\|g\|_2\|u_n\|. \end{aligned}$$

Since $1<q<2<r$, we see $\{u_n\}$ is bounded in E, as required. Consequently, there exists $u \in E$ such that, passing to a subsequence if necessary, we have $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ with $p \in [2, 2_s^*)$, $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$.

As mentioned above, in order to have $u_n \to u$ strongly in E, it sufficient to show that

$$\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\mathrm{d}x = \int_{\mathbb{R}^N} (\lambda h_1(x)|u_n|^{q-2}u_n + h_2(x)|u_n|^{r-2}u_n)(u_n-u)\mathrm{d}x \to 0.$$

Indeed,

$$\int_{\mathbb{R}^N} |h_1||u_n|^{q-1}|u_n-u|\mathrm{d}x \leq \|h_1\|_{q_0}\|u_n\|_2^{q-1}\|u_n-u\|_2 \to 0,$$

and

$$\int_{\mathbb{R}^N} |h_2||u_n|^{r-1}|u_n-u|\mathrm{d}x \leq \|h_2\|_{\infty}\|u_n\|_r^{r-1}\|u_n-u\|_r \to 0.$$

Therefore, J satisfies the (PS) condition, as claimed. All the assumptions in Lemma 2.4 are satisfied for J. Then there exists $u_2 \in E$ such that u_2 is a solution of (1.1) and $J(u_2) \geq \alpha > 0$. This completes the proof. □

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