THE SPECTRUM IN BANACH HYPERALGEBRAS

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\textbf{ABSTRACT:} In this paper, we introduce Banach hyperalgebras and the concept spectrum in hyperalgebras. Moreover, we consider the multiplicative $w$-linear functionals on Banach hyperalgebra $A$ and give some results in this direction.

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\section{1. INTRODUCTION}

The concept of hyperstructure was first introduced by Marty in [5] and has been studied in the following decades to nowadays by many mathematicians. A short review of the theory of hypergroups appears in [2]. The recent books [2], [3] and [12] contain a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, groups, rational algebraic functions and etc.

Vougiouklis in the fourth A.H.A. congress (1990)[11], introduced the notion of $H_v$-structures. The concept of $H_v$-structures constitute a generalization of the well-known algebraic hyperstructures. The principal notions on the $H_v$-structures can be founded in [2], [3], [4], and [11]. In this paper, we introduce the hyperalgebras and the concept of hyperideals of hyperalgebras and we obtain some interesting results.

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The paper is arranged as follows. In Section 2 we define the hypervector spaces and in Section 3 we define the hyperalgebra $A$ and spectrum elements of $A$. Throughout this paper we assume that hyperalgebra $A$ has unit element and we always denote it with $e$. In addition to, 0 will denote zero elements of both field $K$ and hyperalgebra $A$.

## 2. PRELIMINARIES

Let $A$ be a non-empty set, we denote by $P^*(A)$ the set of all non-empty subsets of $A$. Throughout this paper the symbol $K$ will be used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

**Definition 2.1.** [9] Let $K$ be a field and $(A, +)$ be an abelian group. Then a quadruplet $(A, +, \cdot, K)$ is a hypervector space on the field $K$ if the map $:\cdot : K \times A \rightarrow P^*(A)$ satisfies the following conditions:

(i) $\forall \lambda \in K, a, b \in A; \lambda \cdot (a + b) \subseteq \lambda \cdot a + \lambda \cdot b$ (right distributivity);

(ii) $\forall \lambda, \mu \in K, a \in A; (\lambda + \mu) \cdot a \subseteq \lambda \cdot a + \mu \cdot a$ (left distributivity);

(iii) $\forall \lambda, \mu \in K, a \in A; \lambda \cdot (\mu \cdot a) = (\lambda \mu) \cdot a$ (associativity);

(iv) $\forall \lambda \in K, a \in A; \lambda \cdot (-a) = (-\lambda) \cdot a = -(\lambda \cdot a)$;

(v) $\forall a \in A; a \in 1 \cdot a$.

**Remark 2.2.** In the right side of (i) and (ii) the sum is meant in the sense of Frobenius, that is

\[ \lambda \cdot a + \lambda \cdot b = \{m + n : m \in \lambda \cdot a, n \in \lambda \cdot b\}, \]
\[ \lambda \cdot a + \mu \cdot a = \{p + q : p \in \lambda \cdot a, q \in \mu \cdot a\}. \]

Moreover the left side of (iii) meant in the set-theoretical union of all the sets $\lambda \cdot b$, where $b$ runs over the set $\mu \cdot a$.

**Definition 2.3.** [9] A semi-norm $\| \|$ on a hypervector space $(A, +, \cdot, K)$ is a map : $\| \| : A \rightarrow \mathbb{R}$, of $A$ into the positive real number such that:

(i) $\|0\| = 0,$

(ii) $\|a + b\| \leq \|a\| + \|b\| \ \forall a, b \in A,$

(iii) $\sup \|\lambda \cdot a\| = |\lambda| \|a\| \ \forall \lambda \in K, a \in A.$
A semi-norm in $A$ is called a norm if

$$||a|| = 0 \iff a = 0.$$

A hypervector space together with a semi-norm or norm is called a semi-normed or normed hypervector space.

Let $A$ be a hypervector space and $S$ be a nonempty subset of $A$. Then $S$ is said to be a subhypervector space of $A$ if itself is a hypervector space under hyperoperation $\cdot$ restricted to $S$.

### 3. MAIN RESULTS

**Definition 3.1.** A hyperalgebra is a hypervector space $A$ with a multiplication satisfying the following properties:

(i) $\forall a, b, c \in A; (ab)c = a(bc)$, (associativity);

(ii) $\forall a, b, c \in A; (a + b)c = ac + bc$, (left distributivity);

(iii) $\forall a, b, c \in A; a(b + c) = ab + ac$, (right distributivity);

(iv) $\forall a, b \in A; (-a)b = a(-b) = -(ab)$,

(v) $\forall a, b \in A, \text{and } \forall \lambda \in \mathbb{K}; \lambda \cdot (ab) = (\lambda \cdot a)b = a(\lambda \cdot b)$.

The field $\mathbb{K}$ is called the scalar field of $A$. If $\mathbb{K} = \mathbb{R}$, $A$ is called a real algebra, and if $\mathbb{K} = \mathbb{C}$, a complex algebra.

**Example 3.2.** Let $\mathbb{C}$ be the set of all complex numbers. Then $\mathbb{C}$ is a hyperalgebra on $\mathbb{R}$, with respect usual sum and multiplication, and the following hyperoperation:

$$\cdot : \mathbb{C} \times \mathbb{C} \rightarrow P^*(\mathbb{C})$$

$$t \cdot z = \{re^{i\theta} : 0 \leq r \leq |t||z|, 0 \leq \theta \leq 2\pi\}, \quad \forall t \in \mathbb{R}, z \in \mathbb{C}.$$

**Definition 3.3.** Let $A$ be a hyperalgebra. An element $e \in A$ is called an identity or unit if for every $a \in A, a = ae = ea$. In this case we say that $A$ is an unital hyperalgebra.

**Definition 3.4.** Let $A$ be an unital hyperalgebra. An element $a$ is said to be invertible if it has an inverse in $A$, that is, there exists an element $a^{-1} \in A$ such that

$$aa^{-1} = a^{-1}a = e,$$

where $e$ is the unit element of $A$. In this case we say that $a$ is invertible.
It is clear that no $a \in A$ has more than one inverse. For a hyperalgebra $A$ we let $\text{Inv}(A)$ denote the set of all invertible elements in $A$. The compliment of $\text{Inv}(A)$ in $A$ is called the set of all non-invertible elements in $A$, and denote by $\text{Sing}(A)$.

**Definition 3.5.** A semi-norm $\| \|$ on a hyperalgebra $A$ is called a hyperalgebra semi-norm if it satisfies in definition 2.3 and the following condition:

\[ \|ab\| \leq \|a\| \|b\| \quad \forall a, b \in A. \]

A semi-norm in $A$ is called a norm if

\[ \|a\| = 0 \iff a = 0. \]

A hypealgebra together with a hyperalgebra semi-norm or hyperalgebra norm is called a semi-normed or normed hyperalgebra, moreover if $e \in A$ be a unit element of $A$, we assume that $\|e\| = 1$. A normed hypervector space or normed hyperalgebra which is complete in its norm is called a Banach hypervector space or Banach hyperalgebra, respectively.

**Definition 3.6.** Let $A$ be a hyperalgebra and $\| \|$ be a norm on $A$. Then $(A, \| \|)$ is called a normed hyperalgebra.

Let $(A, \| \|)$ be a normed hyperalgebra, we have the equalities:

\[ \|a\| = \sup \| -1 \cdot a \| = \sup \| 1 \cdot (-a) \| = \| -a \| \]

that is, $\|a\| = \| -a \|$, for every $a \in A$.

\[ 0 = \|0\| = \|a - a\| \leq \|a\| + \| -a \| = \|a\| + \|a\| = 2\|a\|. \]

therefore $\|a\| \geq 0$, for every $a \in A$.

\[ \sup \|\lambda \cdot 0\| = \|\lambda\| \|0\| = \|\lambda\| \times 0 = 0, \quad \forall \lambda \in \mathbb{K} \]

and

\[ \sup \|0 \cdot a\| = \sup \|0 \cdot 0\| = \sup \|\lambda \cdot 0\| = 0, \quad \forall a \in A, \lambda \in \mathbb{K}. \]

**Lemma 3.7.** Let $(A, \| \|)$ be a normed hyperalgebra. If for every $\lambda \in \mathbb{K}, a \in A$, $\sup \|\lambda \cdot a\| = 0$. Then $\lambda \cdot a = \{0\}$. Moreover, $\lambda \cdot 0$ and $0 \cdot a$ are singleton sets.

**Proof.** Since for every $\lambda \in \mathbb{K}$ and $a \in A$, $\sup \|\lambda \cdot a\| = 0$, then for every $t \in \lambda \cdot a$, $\|t\| = 0$. Therefore $t = 0$.

The proof of the lemmas 3.8 until 3.10 is similar case which $A$ is a Banach algebra, (see [13]).
Lemma 3.8. Let $A$ be a Banach hyperalgebra. If $a$ is in $A$ with $\|a\| < 1$, then $e - a$ is in $\text{Inv}(A)$.

Proof. The proof is similar case which $A$ is a Banach algebra, (see [13] Proposition 2.1).

Lemma 3.9. Let $(A, \|\|)$ be a Banach hyperalgebra. Then $\text{Inv}(A)$, the set of invertible elements of $A$ is an open set in $A$.

Proof. The proof is similar case which $A$ is a Banach algebra (see [13], Proposition 2.2).

Lemma 3.10. Let $(A, \|\|)$ be a Banach hyperalgebra. Then the inversion $a \mapsto a^{-1}$ is continuous on $\text{Inv}(A)$.

Proof. The proof is similar case which $A$ is a Banach algebra, (see [13] Proposition 2.3).

Lemma 3.11. Let $A$ be a hyperalgebra and $a \in A$, $\lambda \in K$. Then there exists $z \in \lambda \cdot a$ such that $\|z\| = \sup \|\lambda \cdot a\|$.

Proof. If $\lambda = 0$ then we can take $z = 0$. Now we assume $\lambda \neq 0$, for every $a \in A$ we have

$$a \in 1 \cdot a = \lambda^{-1} \lambda \cdot a = \lambda^{-1} \cdot (\lambda \cdot a) = \cup_{x \in \lambda \cdot a} \lambda^{-1} \cdot x$$

so that there exists $z \in \lambda \cdot a$ such that $a \in \lambda^{-1} \cdot z$. Hence

$$\|a\| \leq |\lambda|^{-1} \|z\| \Rightarrow \|a\| |\lambda| \leq \|z\|.$$ 

On the other hand, $z \in \lambda \cdot a$ then $\|z\| \leq \|a\| |\lambda|$. Thus $\|z\| = |\lambda| \|a\|$.

Remark 3.12. Throughout this paper, we denote $z$ be obtained in the lemma 3.11 by $z_{\lambda \cdot a}$, also if $a = e$ we denote by $z_{\lambda}$.

Let $A$ be a hypervector space and $S$ be a nonempty subset of $A$. Then $S$ is said to be a weak subhypervector space of $A$ if for every $a, b \in S$ and $\lambda \in C$, $a + b \in S$, $z_{\lambda \cdot a} \in S$.

Definition 3.13. Let $A$ be a hyperalgebra. A mapping $\varphi : A \rightarrow C$ is said to be weak linear functional (or simply, w-linear functional) if

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$ \quad $\forall a, b \in A$

(ii) $\varphi(z_{\lambda \cdot a}) = \lambda \varphi(a)$ \quad $\forall a \in A, \lambda \in K$

Definition 3.14. A w-linear functional $\varphi$ on a hyperalgebra $A$ is multiplicative if $\varphi$ is nontrivial and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a$ and $b$ in $A$. We let $M^w_A$ denote the set of all multiplicative w-linear functional on $A$. 
Theorem 3.15. Let $\varphi$ be a multiplicative w-linear functional on hyperalgebra $A$. Then $\varphi(e) = 1$. Moreover if $a \in A$ is invertible then $\varphi(a) \neq 0$.

Proof. The proof is similar case which $A$ is a Banach algebra.

Theorem 3.16. Let $\varphi$ be a multiplicative w-linear functional on Banach hyperalgebra $A$. Then $\varphi$ is continuous.

Proof. If $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$ then for all $a \in A$ that $|a| < 1$ we have $e - \lambda^{-1} \cdot a \subseteq Inv(A)$ (lemma 3.8), by theorem 3.15, $0 \notin \varphi(e - \lambda^{-1} \cdot a)$ for every $\varphi \in M^w_A$.

On the other hand for every $\varphi \in M^w_A$, we have $1 - \lambda^{-1} \varphi(a) \in \varphi(e - \lambda^{-1} \cdot a)$, so $\varphi(a) \neq \lambda$.

Hence we proved that if $|a| < 1$ then $|\varphi(a)| \leq 1$, that is, $\varphi$ is continuous at zero, therefore $\varphi$ is continuous on $A$.

Definition 3.17. Let $A$ is a hyperalgebra, we say $A$ is normal hyperalgebra if

(i) $\forall \lambda \in \mathbb{K}, a, b \in A; z_{\lambda(a+b)} = z_{\lambda a} + z_{\lambda b}$.

(ii) $\forall \lambda, \mu \in \mathbb{K}, a \in A; z_{(\lambda+\mu) a} = z_{\lambda a} + z_{\mu a}$.

Definition 3.18. Suppose $A$ is a hyperalgebra and $a$ is in $A$. the spectrum of $a$, denoted by $\sigma_A(a)$ or simply by $\sigma(a)$, is the set of all complex numbers $\lambda$ such that $z_{\lambda - a} \in Sing(A)$, that is, $\sigma(a) = \{\lambda \in \mathbb{C} : z_{\lambda - a} \in Sing(A)\}$. The compliment of $\sigma(a)$ in $\mathbb{C}$ is called the resolvent set of $a$, and denote by $\rho(a)$.

Lemma 3.19. Let $A$ be a Banach hyperalgebra and $a \in A$ be invertible, then $(\lambda \cdot a)^{-1} \cap \lambda^{-1} \cdot a^{-1} = \{z_{\lambda a}\}$ for every $\lambda \in \mathbb{K}$.

Proof. First we prove $(\lambda \cdot e)^{-1} \cap \lambda^{-1} \cdot e \subseteq \{z_{\lambda}\}$. Let there exists $b \neq z_{\lambda}$ such that $b^{-1} \in (\lambda \cdot e)^{-1} \cap \lambda^{-1} \cdot e$, and $\|b\| < |\lambda|$. Thus

$$1 = \|bb^{-1}\| \leq \|b\| \|b^{-1}\| < |\lambda| \|\lambda^{-1}\| = 1,$$

this is a contradiction. Now Let $b \in ((\lambda \cdot e)a)^{-1} \cap \lambda^{-1} \cdot a^{-1}$ therefore $b = (ta)^{-1}, b = (sa)^{-1}$ for some $t \in (\lambda \cdot e)$ and $s \in \lambda \cdot e$, hence $(t - s)a^{-1} = 0$, because $a$ is invertible.

It followed $t - s = 0$ then $t = s$.

Since $a$ is invertible then $e = aa^{-1} = a^{-1}a$, and on the other hand

$$e \in 1 \cdot e = (\lambda \cdot a)(\lambda^{-1} \cdot a^{-1}).$$

Therefore $e = yz$ for some $y \in \lambda \cdot a$ and by statements above $y = z_{\lambda a}$.

Theorem 3.20. Let $A$ be a Banach normal hyperalgebra and $a$ is in $A$. Then $\sigma(a)$ is nonempty.
The function $f$ is defined on the entire complex plan since $\sigma(a)$ is empty. Fix $\lambda_0 \in \mathbb{C}$,

$$\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = (\lambda - \lambda_0)^{-1} F((z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1})$$

$$= F((z_{\lambda-\lambda_0})^{-1}((z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1})),$$

Since

$$(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1} = (z_\lambda - a)^{-1}(z_\lambda - z_{\lambda_0})(z_{\lambda_0} - a)^{-1}$$

hence

$$F((z_{\lambda-\lambda_0})^{-1}((z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1}) = F((z_{\lambda-\lambda_0})^{-1}((z_\lambda - a)^{-1}(z_\lambda - z_{\lambda_0})(z_{\lambda_0} - a)^{-1})$$

$$\frac{(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1}}{\lambda - \lambda_0} = F((z_{\lambda_0} - a)^{-1})$$

By the continuity of $F$ and inversion, we obtain

$$\lim_{\lambda \to \lambda_0} \frac{(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1}}{\lambda - \lambda_0} = -F((z_{\lambda_0} - a)^{-1})$$

and hence

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = -F((z_{\lambda_0} - a)^{-1}).$$

We see that $f$ is an entire function. On the other hand

$$z_{\lambda(e - z_{\lambda-1} - a)} = z_\lambda - z_{\lambda_0}^{-1} - a = z_\lambda - a.$$ 

Therefore $(z_\lambda - a)^{-1} \in (\lambda \cdot (e - z_{\lambda-1} - a))^{-1} \cap (\lambda^{-1} \cdot (e - z_{\lambda-1} - a))^{-1}$. Furthermore

$$|f(\lambda)| \leq \|F\|(z_\lambda - a)^{-1} \leq \|F\||\lambda|^{-1}(e - z_{\lambda-1} - a)^{-1}.$$

Thus $f(\lambda)$ is a bounded entire function which converges to 0 as $|\lambda| \to +\infty$. By Liouville theorem, $f$ must be identically zero. Since $F$ is arbitrary, the Hahn-Banach extension theorem implies that $(z_\lambda - a)^{-1} = 0$, which is impossible because 0 is never invertible in a Banach hyperalgebra.

**Theorem 3.21.** Let $A$ be a Banach normal hyperalgebra. Then $\sigma(a)$ is bounded in $\mathbb{C}$ and is contained in the closed disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. 


Proof. If $|\lambda| > \|a\|$ then $1 > \|a\|/|\lambda| = \sup \|\lambda^{-1} \cdot a\| = \sup \|e - (e - \lambda^{-1} \cdot a)\|$, so $(e - \lambda^{-1} \cdot a) \subseteq \text{Inv}(A)$ (lemma 3.8).

On the other hand

\[ z_\lambda - a = z_\lambda - z_{1 \cdot a} = z_\lambda - z_{\lambda^{-1} \cdot a} = z_{\lambda(e - \lambda^{-1} \cdot a)} \in \text{Inv}(A) \]

so that $z_\lambda - a \in \text{Inv}(A)$, for every $\lambda$ such that $|\lambda| > \|a\|$, that is, $\lambda$ in $\rho(a)$ so $\sigma(a)$ is bounded. \qed

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