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THE SPECTRUM IN BANACH HYPERALGEBRAS

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ABSTRACT: In this paper, we introduce Banach hyperalgebras and the concept spectrum in hyperalgebras. Moreover, we consider the multiplicative w-linear functionals on Banach hyperalgebra A and give some results in this direction.

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1. INTRODUCTION

The concept of hyperstructure was first introduced by Marty in [5] and has been studied in the following decades to nowadays by many mathematicians. A short review of the theory of hypergroups appears in [2]. The recent books [2], [3] and [12] contain a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, groups, rational algebraic functions and etc.

Vougiouklis in the fourth A.H.A. congress (1990)[11], introduced the notion of H_v-structures. The concept of H_v-structures constitute a generalization of the well-known algebraic hyperstructures. The principal notions on the H_v-structures can be founded in [2], [3], [4], and [11]. In this paper, we introduce the hyperalgebras and the concept of hyperideals of hyperalgebras and we obtain some interesting results.

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The paper is arranged as follows. In Section 2 we define the hypervector spaces and in Section 3 we define the hyperalgebra A and spectrum elements of A. Throughout this paper we assume that hyperalgebra A has unit element and we always denote it with e. In addition to, 0 will denote zero elements of both field $\mathbb{K}$ and hyperalgebra A.

2. PRELIMINARIES

Let A be a non-empty set, we denote by $P^*(A)$ the set of all non-empty subsets of A. Throughout this paper the symbol $\mathbb{K}$ will be used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Definition 2.1. [9] Let $\mathbb{K}$ be a field and $(A, +)$ be an abelian group. Then a quadruplet $(A, +, \cdot, \mathbb{K})$ is a hypervector space on the field $\mathbb{K}$ if the map $\cdot : \mathbb{K} \times A \longrightarrow P^*(A)$ satisfies the following conditions:

(i) $\forall \lambda \in \mathbb{K}, a, b \in A; \lambda \cdot (a + b) \subseteq \lambda \cdot a + \lambda \cdot b$ (right distributivity);

(ii) $\forall \lambda, \mu \in \mathbb{K}, a \in A; (\lambda + \mu) \cdot a \subseteq \lambda \cdot a + \mu \cdot a$ (left distributivity);

(iii) $\forall \lambda, \mu \in \mathbb{K}, a \in A; \lambda \cdot (\mu \cdot a) = (\lambda\mu) \cdot a$ (associativity);

(iv) $\forall \lambda \in \mathbb{K}, a \in A; \lambda \cdot (-a) = (-\lambda) \cdot a = -(\lambda \cdot a)$;

(v) $\forall a \in A; a \in 1 \cdot a$.

Remark 2.2. In the right side of (i) and (ii) the sum is meant in the sense of Frobenius, that is

$$\lambda \cdot a + \lambda \cdot b = \{m + n : m \in \lambda \cdot a, n \in \lambda \cdot b\},$$
$$\lambda \cdot a + \mu \cdot a = \{p + q : p \in \lambda \cdot a, q \in \mu \cdot a\}.$$

Moreover the left side of (iii) meant in the set-theoretical union of all the sets $\lambda \cdot b$, where b runs over the set $\mu \cdot a$.

Definition 2.3. [9] A semi-norm $\|\|$ on a hypervector space $(A, +, \cdot, \mathbb{K})$ is a map : $\|\| : A \longrightarrow \mathbb{R}$, of A into the positive real number such that:

(i) $\|0\| = 0$,

(ii) $\|a + b\| \leq \|a\| + \|b\| \quad \forall a, b \in A$,

(iii) $\sup \|\lambda \cdot a\| = |\lambda|\|a\| \quad \forall \lambda \in \mathbb{K}, a \in A$.

A semi-norm in A is called a norm if

$$\|a\| = 0 \quad \Longleftrightarrow a = 0.$$

A hypervector space together with a semi-norm or norm is called a semi-normed or normed hypervector space.

Let A be a hypervector space and S be a nonempty subset of A. Then S is said to be a subhypervector space of A if itself is a hypervector space under hyperoperation " $\cdot$ " restricted to S.

3. MAIN RESULTS

Definition 3.1. A hyperalgebra is a hypervector space A with a multiplication satisfying the following properties:

(i) $\forall a, b, c \in A; (ab)c = a(bc)$, (associativity);

(ii) $\forall a, b, c \in A;\ (a + b)c = ac + bc$, (left distributivity);

(iii) $\forall a, b, c \in A;\ a(b + c) = ab + ac$, (right distributivity);

(iv) $\forall a, b \in A; (-a)b = a(-b) = -(ab)$,

(v) $\forall a, b \in A$, and $\forall \lambda \in \mathbb{K};\ \lambda \cdot (ab) = (\lambda \cdot a)b = a(\lambda \cdot b)$.

The field $\mathbb{K}$ is called the scalar field of A. If $\mathbb{K} = \mathbb{R}$, A is called a real algebra, and if $\mathbb{K} = \mathbb{C}$, a complex algebra.

Example 3.2. Let $\mathbb{C}$ be the set of all complex numbers. Then $\mathbb{C}$ is a hyperalgebra on $\mathbb{R}$, with respect usual sum and multiplication, and the following hyperoperation:

$$\cdot : \mathbb{C} \times \mathbb{C} \longrightarrow P^*(\mathbb{C})$$

$$t \cdot z = \{re^{i\theta} : 0 \leq r \leq |t||z|, 0 \leq \theta \leq 2\pi\}, \qquad \forall t \in \mathbb{R}, z \in \mathbb{C}.$$

Definition 3.3. Let A be a hyperalgebra. An element $e \in A$ is called an identity or unit if for every $a \in A, a = ae = ea$. In this case we say that A is an unital hyperalgebra.

Definition 3.4. Let A be an unital hyperalgebra. An element a is said to be invertible if it has an inverse in A, that is, there exists an element $a^{-1} \in A$ such that

$$aa^{-1} = a^{-1}a = e,$$

where e is the unit element of A. In this case we say that a is invertible.

It is clear that no $a \in A$ has more than one inverse. For a hyperalgebra A we let $Inv(A)$ denote the set of all invertible elements in A. The compliment of $Inv(A)$ in A is called the set of all non-invertible elements in A, and denote by $Sing(A)$.

Definition 3.5. A semi-norm $\|\|$ on a hyperalgebra A is called a hyperalgebra semi-norm if it satisfies in definition 2.3 and the following condition:

$$\|ab\| \leq \|a\|\|b\| \qquad \forall a, b \in A.$$

A semi-norm in A is called a norm if

$$\|a\| = 0 \quad \Longleftrightarrow a = 0.$$

A hypealgebra together with a hyperalgebra semi-norm or hyperalgebra norm is called a semi-normed or normed hyperalgebra, moreover if $e \in A$ be a unit element of A, we assume that $\|e\| = 1$. A normed hypervector space or normed hyperalgebra which is complete in its norm is called a Banach hypervector space or Banach hyperalgebra, respectively.

Definition 3.6. Let A be a hyperalgebra and $\|\|$ be a norm on A. Then $(A, \|\|)$ is called a normed hyperalgebra.

Let $(A, \|\|)$ be a normed hyperalgebra, we have the equalities:

$$\|a\| = \sup \| - 1 \cdot a\| = \sup \|1 \cdot (-a)\| = \| - a\|$$

that is, $\|a\| = \| - a\|$, for every $a \in A$.

$$0 = \|0\| = \|a - a\| \leq \|a\| + \| - a\| = \|a\| + \|a\| = 2\|a\|.$$

therefore $\|a\| \geq 0$, for every $a \in A$.

$$\sup \|\lambda \cdot 0\| = \|\lambda\|\|0\| = \|\lambda\| \times 0 = 0, \quad \forall \lambda \in \mathbb{K}$$

and

$$\sup \|0 \cdot a\| = \sup \|0 \cdot 0\| = \sup \|\lambda \cdot 0\| = 0, \quad \forall a \in A, \lambda \in \mathbb{K}.$$

Lemma 3.7. *Let $(A, \|\|)$ be a normed hyperalgebra. If for every $\lambda \in \mathbb{K}, a \in A$, $\sup \|\lambda \cdot a\| = 0$. Then $\lambda \cdot a = \{0\}$. Moreover, $\lambda \cdot 0$ and $0 \cdot a$ are singleton sets.*

Proof. Since for every $\lambda \in \mathbb{K}$ and $a \in A$, $\sup \|\lambda \cdot a\| = 0$, then for every $t \in \lambda \cdot a$, $\|t\| = 0$. Therefore $t = 0$. □

The proof of the lemmas 3.8 until 3.10 is similar case which A is a Banach algebra, (see [13]).

Lemma 3.8. *Let A be a Banach hyperalgebra. If a is in A with $\|a\| < 1$, then $e - a$ is in $Inv(A)$.*

Proof. The proof is similar case which A is a Banach algebra, (see [13] Proposition 2.1). □

Lemma 3.9. *Let $(A, \|\|)$ be a Banach hyperalgebra. Then $Inv(A)$, the set of invertible elements of A is an open set in A.*

Proof. The proof is similar case which A is a Banach algebra (see [13], Proposition 2.2). □

Lemma 3.10. *Let $(A, \|\|)$ be a Banach hyperalgebra. Then the inversion $a \longrightarrow a^{-1}$ is continuous on $Inv(A)$.*

Proof. The proof is similar case which A is a Banach algebra, (see [13] Proposition 2.3). □

Lemma 3.11. *Let A be a hyperalgebra and $a \in A, \lambda \in \mathbb{K}$. Then there exists $z \in \lambda \cdot a$ such that $\|z\| = \sup \|\lambda \cdot a\|$.*

Proof. If $\lambda = 0$ then we can take $z = 0$. Now we assume $\lambda \neq 0$, for every $a \in A$ we have

$$a \in 1 \cdot a = \lambda^{-1}\lambda \cdot a = \lambda^{-1} \cdot (\lambda \cdot a) = \cup_{x \in \lambda \cdot a} \lambda^{-1} \cdot x$$

so that there exists $z \in \lambda \cdot a$ such that $a \in \lambda^{-1} \cdot z$. Hence

$$\|a\| \leq |\lambda|^{-1}\|z\| \Rightarrow \|a\||\lambda| \leq \|z\|.$$

On the other hand, $z \in \lambda \cdot a$ then $\|z\| \leq \|a\||\lambda|$. Thus $\|z\| = |\lambda|\|a\|$. □

Remark 3.12. Throughout this paper, we denote z be obtained in the lemma 3.11 by $z_{\lambda \cdot a}$, also if $a = e$ we denote by z_λ.

Let A be a hypervector space and S be a nonempty subset of A. Then S is said to be a weak subhypervector space of A if for every $a, b \in S$ and $\lambda \in \mathbb{C}$, $a+b \in S$, $z_{\lambda \cdot a} \in S$.

Definition 3.13. Let A be a hyperalgebra. A mapping $\varphi : A \longrightarrow \mathbb{C}$ is said to be weak linear functional (or simply, w-linear functional) if
(i) $\varphi(a + b) = \varphi(a) + \varphi(b) \quad \forall a, b \in A$
(ii) $\varphi(z_{\lambda \cdot a}) = \lambda\varphi(a) \quad \forall a \in A, \lambda \in \mathbb{K}$

Definition 3.14. A w-linear functional φ on a hyperalgebra A is multiplicative if φ is nontrivial and $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b in A. We let M_A^w denote the set of all multiplicative w-linear functional on A.

Theorem 3.15. *Let φ be a multiplicative w-linear functional on hyperalgebra A. Then $\varphi(e) = 1$. Moreover if $a \in A$ is invertible then $\varphi(a) \neq 0$.*

Proof. The proof is similar case which A is a Banach algebra. □

Theorem 3.16. *Let φ be a multiplicative w-linear functional on Banach hyperalgebra A. Then φ is continuous.*

Proof. If $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$ then for all $a \in A$ that $\|a\| < 1$ we have $e - \lambda^{-1} \cdot a \subseteq Inv(A)$ (lemma 3.8), by theorem 3.15, $0 \notin \varphi(e - \lambda^{-1} \cdot a)$ for every $\varphi \in M_A^w$.
On the other hand for every $\varphi \in M_A^w$, we have $1 - \lambda^{-1}\varphi(a) \in \varphi(e - \lambda^{-1} \cdot a)$, so $\varphi(a) \neq \lambda$.

Hence we proved that if $\|a\| < 1$ then $|\varphi(a)| \leq 1$, that is, φ is continuous at zero, therefore φ is continuous on A. □

Definition 3.17. Let A is a hyperalgebra, we say A is normal hyperalgebra if
(i) $\forall \lambda \in \mathbb{K}, a, b \in A; z_{\lambda\cdot(a+b)} = z_{\lambda\cdot a} + z_{\lambda\cdot b}$.
(ii) $\forall \lambda, \mu \in \mathbb{K}, a \in A; z_{(\lambda+\mu)\cdot a} = z_{\lambda\cdot a} + z_{\mu\cdot a}$.

Definition 3.18. Suppose A is a hyperalgebra and a is in A. the spectrum of a, denoted by $\sigma_A(a)$ or simply by $\sigma(a)$, is the set of all complex numbers λ such that $z_\lambda - a \in Sing(A)$, that is, $\sigma(a) = \{\lambda \in \mathbb{C} : z_\lambda - a \in Sing(A)\}$. The compliment of $\sigma(a)$ in $\mathbb{C}$ is called the resolvent set of a, and denote by $\rho(a)$.

Lemma 3.19. *Let A be a Banach hyperalgebra and $a \in A$ be invertible, then $(\lambda \cdot a)^{-1} \cap \lambda^{-1} \cdot a^{-1} = \{z_{\lambda\cdot a}\}$ for every $\lambda \in \mathbb{K}$.*

Proof. First we prove $(\lambda \cdot e)^{-1} \cap \lambda^{-1} \cdot e \subseteq \{z_\lambda\}$. Let there exists $b \neq z_\lambda$ such that $b^{-1} \in (\lambda \cdot e)^{-1} \cap \lambda^{-1} \cdot e$, and $\|b\| < |\lambda|$. Thus

$$1 = \|bb^{-1}\| \leq \|b\|\|b^{-1}\| < |\lambda||\lambda^{-1}| = 1,$$

this is a contradiction. Now Let $b \in ((\lambda \cdot e)a)^{-1} \cap \lambda^{-1} \cdot a^{-1}$ therefore $b = (ta)^{-1}, b = (sa)^{-1}$ for some $t \in (\lambda \cdot e)$ and $s \in \lambda \cdot e$, hence $(t - s)a^{-1} = \underline{0}$, because a is invertible. It followed $t - s = \underline{0}$ then $t = s$.
Since a is invertible then $e = aa^{-1} = a^{-1}a$, and on the other hand

$$e \in 1 \cdot e = (\lambda \cdot a)(\lambda^{-1} \cdot a^{-1}).$$

Therefore $e = yz$ for some $y \in \lambda \cdot a$ and by statements above $y = z_{\lambda\cdot a}$.

Theorem 3.20. *Let A be a Banach normal hyperalgebra and a is in A. Then $\sigma(a)$ is nonempty.*

Proof. If a is not invertible, $0 \in \sigma(a)$. Thus we may as well assume that a is invertible. Attempting arrive at a contradiction, we assume that $\sigma(a)$ is empty. Without loss of generality, we may assume that z_λ is unique. Fix a bounded w-linear functional F on A and consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(\lambda) = F((z_\lambda - a)^{-1})$$

The function f is defined on the entire complex plan since $\sigma(a)$ is empty. Fix $\lambda_0 \in \mathbb{C}$,

$$\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = (\lambda - \lambda_0)^{-1} F((z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1})$$
$$= F(z_{(\lambda-\lambda_0)^{-1}\cdot((z_\lambda - a)^{-1} - (z_\lambda - a)^{-1})}).$$

Since

$$(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1} = (z_\lambda - a)^{-1}(z_\lambda - z_{\lambda_0})(z_{\lambda_0} - a)^{-1}$$

hence

$$F(z_{(\lambda-\lambda_0)^{-1}\cdot((z_\lambda - a)^{-1} - (z_\lambda - a)^{-1})}) = F(z_{(\lambda-\lambda_0)^{-1}\cdot((z_\lambda - a)^{-1}(z_\lambda - z_{\lambda_0})(z_{\lambda_0} - a)^{-1}})$$

$$\frac{(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1}}{\lambda - \lambda_0} = F(z_{(z_\lambda - a)^{-1}(z_{\lambda_0} - a)^{-1}})$$

By the continuity of F and inversion, we obtain

$$\lim_{\lambda \to \lambda_0} \frac{(z_\lambda - a)^{-1} - (z_{\lambda_0} - a)^{-1}}{\lambda - \lambda_0} = -F(z_{\lambda_0} - a)^{-2}$$

and hence

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = -F(z_{\lambda_0} - a)^{-2}.$$

We see that f is an entire function. On the other hand

$$z_{\lambda\cdot(e - z_{\lambda^{-1}\cdot a})} = z_\lambda - z_{\lambda\lambda^{-1}\cdot a} = z_\lambda - a.$$

Therefore $(z_\lambda - a)^{-1} \in (\lambda \cdot (e - z_{\lambda^{-1}\cdot a}))^{-1} \cap (\lambda^{-1} \cdot (e - z_{\lambda^{-1}\cdot a})^{-1})$. Furthermore

$$|f(\lambda)| \leq \|F\| \|(z_\lambda - a)^{-1}\| \leq \|F\| |\lambda|^{-1} \|(e - z_{\lambda^{-1}\cdot a})^{-1}\|.$$

Thus $f(\lambda)$ is a bounded entire function which converges to 0 as $|\lambda| \to +\infty$. By Liouville theorem, f must be identically zero. Since F is arbitrary, the Hahn-Banach extension theorem implies that $(z_\lambda - a)^{-1} = 0$, which is impossible because 0 is never invertible in a Banach hyperalgebra. □

Theorem 3.21. *Let A be a Banach normal hyperalgebra. Then $\sigma(a)$ is bounded in $\mathbb{C}$ and is contained in the closed disk $\{\lambda \in \mathbb{C} \ : \ |\lambda| \leq \|a\|\}$.*

Proof. If $|\lambda| > \|a\|$ then $1 > \|a\|/|\lambda| = \sup\|\lambda^{-1} \cdot a\| = \sup\|e - (e - \lambda^{-1} \cdot a)\|$, so $(e - \lambda^{-1} \cdot a) \subseteq Inv(A)$ (lemma 3.8).
On the other hand

$$z_\lambda - a = z_\lambda - z_{1\cdot a} = z_\lambda - z_{\lambda\lambda^{-1}\cdot a} = z_{\lambda(e-\lambda^{-1}\cdot a)} \in Inv(A)$$

so that $z_\lambda - a \in Inv(A)$, for every λ such that $|\lambda| > \|a\|$, that is, λ in $\rho(a)$ so $\sigma(a)$ is bounded. □

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