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NEGATIVE DEFINITE FUNCTIONS ON DIRECT PRODUCT OF COMMUTATIVE HYPERCOMPLEX SYSTEMS

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ABSTRACT: The main aim of this paper is to explore harmonic properties of functions defined in the product of hypercomplex systems. By means of the generalized translation operators, the precise definition of the product of commutative hypercomplex systems is given and full description for its properties are shown. The definition and some properties of negative definite functions in the product of commutative normal hypercomplex systems are given.

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1. INTRODUCTION

Harmonic analysis in hypercomplex systems (HCSs) dates back to Delsartes and Levitans work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev [1] put HCSs in the right setting for harmonic analysis. Recently, some authors as Zabel and Bin Dehaish [2, 3], Bin Dehaish [4], Okb El Bab, Zabel and Ghany [5] and Okb El Bab, Ghany and Boshnaa [6], studied some important subjects related to harmonic analysis in HCSs. Furthermore, Okb El Bab, Ghany, Hyder and Zakarya [7, 8], studied some important subjects related to a construction of non-Gaussian white noise analysis using the theory of HCSs.

Generalized translation operators (GTOs) were first introduced by Delsarte [9] as an object that generalizes the idea of translation on a group. Later, they were systematically studied by Levitan [10, 11, 12, 13, 14], for some classes of GTOs, he obtained generalizations of harmonic analysis, the Lie theory, the theory of almost periodic functions, the theory of group representations, etc. The fact that GTOs arise in various problems of analysis is explained by Vainerman and Litvinov in [15]. Transformations of Fourier type for which the Plancherel theorem and the inversion formula hold, as a rule, are closely related to families of GTOs. According to Section 1 in [1], each HCSs can be associated with a family of GTOs. So, we begin with recalling some necessary facts deal with theory of GTOs and reviewing the conditions that distinguish the class of HCSs from the class of GTOs [14].

Let $L_1(Q, m)$ be a HCS with basis Q and let Φ be a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \mapsto L_p : \Phi \to \Phi$ is given such that the function $g(p) = (L_p f)(q)$ belongs to Φ for any $f \in \Phi$ and any fixed $q \in Q$.

Definition 1.1. The operators L_p, $p \in Q$ are called GTOs, provided that the following axioms are satisfied.

I. Associativity axiom. The equality

$$\big(L_p^q(L_q f)\big)(r) = \big(L_q^r(L_p f)\big)(r) \tag{1.1}$$

holds for any elements $p, q \in Q$.

II. There exists an element $e \in Q$ such that $L_e = I$, where I is the identity operator in Φ.

Definition 1.2. The GTOs L_p are called commutative if for any $p, q \in Q$, we have $\big(L_p^r(L_q f)\big)(r) = \big(L_q^r(L_p f)\big)(r)$. For commutative GTOs L_p the following equality is satisfied.

$$\big(L_p f\big)(q) = \big(L_q f\big)(p), \quad p, q \in Q. \tag{1.2}$$

In this paper we are interest in the case where Q is locally compact space with regular Borel measure m positive on open sets and bounded GTOs L_p act in the space of functions $\Phi = L_2(Q, m)$.

Definition 1.3. Given an involutive homeomorphism $Q \ni p \mapsto p^* \in Q$. The GTOs L_p are involutive if the equalities

$$\big(L_p f\big)(q) = \overline{\big(L_{q^*} f^*\big)(p^*)}, \quad \big(f \in L_2(Q, m), f^*(p) = \overline{f(p^*)}\,\big), \tag{1.3}$$

and $e^* = e$ hold for almost all $p, q \in Q$.

Definition 1.4. The GTOs L_p preserve positivity if $\big(L_p f\big)(q) \geq 0$ almost everywhere in m whenever $f(q) \geq 0$.

Definition 1.5. The family of operators L_p is called weakly continuous if the operator-valued function $Q \ni p \mapsto L_p$ is weakly continuous.

Definition 1.6. Let L_p^* be the operator adjoint to L_p. The measure m is called strongly invariant if $L_p^* = L_{p^*}$ for all $p \in Q$. We say that the measure m unimodular if $m(A) = m(A^*)$ for all $A \in \mathcal{B}(Q)$ (is the σ-algebra generated by its Borel sets from Q).

Assume that the GTOs L_p satisfy the finiteness condition:

(F) For any $A, B \in \mathcal{B}_0(Q)$, there is a compact set F so large that $\big(L_p f\big)(q) = 0$ for almost all $p \in A$ and $q \in B$ provided that supp $f \cap F = \emptyset$,

where $B_0(Q)$ is the subring of $B(Q)$, consisting of sets with compact closure.

Lemma 1.1. *[1] If weakly continuous GTOs L_p are commutative, then relation (1.2) holds for almost all p and q.*

Lemma 1.2. *[1] Let m be a measure strongly invariant with respect to the GTOs $L_p\,(p \in Q)$ which preserve the unit element and satisfy the finiteness condition **(F)**, Then*

$$\int_Q \big(L_p f\big)(q)dq = \int_Q f(q)dq, \qquad (p \in Q \text{ and } f \in L_{2,0}), \tag{1.4}$$

where $L_{2,0}$ is the subspace of finite functions from $L_2(Q, m)$.

Theorem 1.3. *[1] There exists a one-to-one correspondence between normal HCSs with basis unity e and weakly continuous families of bounded involutive GTOs L_p*

satisfying the finiteness condition, preserving positivity in the space $L_2(Q,m)$ with unimodular strongly invariant measure m, and preserving the unit element. Convolution in the HCS $L_1(Q,m)$ and the corresponding family of GTOs L_p satisfy the relation

$$\big(f*g\big)(p)=\int_Q \big(L_pf\big)(q)\,g(q^*)dq=\big(L_pf,g^*\big),\ \ \big(f,g\in L_2(Q,m)\big). \tag{1.5}$$

Moreover, the HCS $L_1(Q,m)$ is commutative if and only if the GTOs $L_p,\ p\in Q$ are commutative.

This work can be immediately generalized to a direct product of any finite number of HCSs. While, the case of infinite number of HCSs is still open. Moreover, it is fairly easy to observe that all our results for direct product of HCSs can be easily investigated for direct products of semigroups and hypergroups (See [16, 17]).

This paper is organized as follows: In Section 2, we give the basic definition of the direct product of HCSs and discuss its objects like convolution, characters, normality and commutativity preserving, and give an example to improve the concept of direct product of HCSs. In Section 3, we study the negative definite functions on the direct product of commutative normal HCSs and we proved some facts related to this subject. Section 4 is employed for conclusion.

2. DIRECT PRODUCT OF HYPERCOMPLEX SYSTEMS

Suppose that $L_{p_i}(i=1,2)$ be GTOs associated with normal HCSs with basis unity $L_1(Q_1,m_1)$ and $L_1(Q_2,m_2)$ respectively. We denote, $\mathbf{H}_1:=L_1(Q_1,m_1)$ and $\mathbf{H}_2:=L_1(Q_2,m_2)$. The direct product of the GTOs L_{p_1} and L_{p_2} $(p_1\in Q_1,\ p_2\in Q_2)$ is defined as the operator-valued function

$$Q_1\times Q_2\ni(p_1,p_2)\mapsto L_{(p_1,p_2)}=L_{p_1}\otimes L_{p_2}:\mathbf{H}_1\otimes\mathbf{H}_2\to\mathbf{H}_1\otimes\mathbf{H}_2. \tag{2.1}$$

It is clear that the operators $L_{(p_1,p_2)}$ $((p_1,p_2)\in Q_1\times Q_2)$ form in $\mathbf{H}_1\otimes\mathbf{H}_2$ a family of GTOs satisfying the conditions of Theorem 1.3. The HCSs $\mathbf{H}_1\otimes\mathbf{H}_2$ constructed from the GTOs $L_{(p_1,p_2)}$ is called the direct product of the HCSs $L_1(Q_1,m_1)$ and $L_1(Q_2,m_2)$. To denote the operation of taking the direct product, we write

$$\mathbf{H}_1\otimes\mathbf{H}_2=L_1(Q_1\times Q_2,m_1\otimes m_2)=L_1(Q_1,m_1)\otimes L_1(Q_2,m_2).$$

The following Lemma shows that the operation of taking the direct product preserves commutativity

Lemma 2.1. *The direct product of two commutative HCSs is commutative.*

Proof. Let $\mathbf{H}_1, \mathbf{H}_2$ be two commutative HCSs and L_p, L_q be the corresponding GTOs, respectively. According to the above Theorem, it is sufficient to prove that the GTOs $L_{(p_1,p_2)} = L_{p_1} \otimes L_{p_2}$, $L_{(q_1,q_2)} = L_{q_1} \otimes L_{q_2}$ $\big((p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2\big)$ are commutative. So from definition 1.2 and Eq.(1.2) we have,

$$\big(L_{(p_1,p_2)}\big(L_{(q_1,q_2)}f\big)\big)(r_1,r_2) = \big(L_{(q_1,q_2)}\big(L_{(p_1,p_2)}f\big)\big)(r_1,r_2), \tag{2.2}$$

and hence, the Lemma is proved. □

There are important concepts related to any HCSs like structure measure, multiplicative measure, characters and normality. Now, we transfer these concepts to the direct product of two commutative HCSs.

Let Q_1 and Q_2 be complete separable locally compact metric spaces. $\mathcal{B}(Q_1 \times Q_2)$ is the σ-algebra of Borel subsets from $Q_1 \times Q_2$, and $\mathcal{B}_0(Q_1 \times Q_2)$ be the subring of $\mathcal{B}(Q_1 \times Q_2)$ which consists of sets with compact closure. We will consider the Borel measures, that is, positive regular measures on $\mathcal{B}(Q_1 \times Q_2)$, finite on compact sets. The spaces of continuous functions, of finite continuous functions, of continuous functions vanishing at infinity and of bounded functions are denoted by $C(Q_1 \times Q_2)$, $C_0(Q_1 \times Q_2)$, $C_\infty(Q_1 \times Q_2)$ and $C_b(Q_1 \times Q_2)$, respectively.

Let $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$ and let $\mathcal{K}_{(A_1 \times A_2)}$ and $\mathcal{K}_{(B_1 \times B_2)}$ be the characteristic functions of $A_1 \times A_2, B_1 \times B_2$, respectively. By using Eq.(1.5), we can set up the structure measure of the HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ as follows

$$\begin{aligned} c\big(A_1 \times A_2, B_1 \times B_2, (r_1,r_2)\big) &= \mathcal{K}_{(A_1 \times A_2)} * \mathcal{K}_{(B_1 \times B_2)}(r_1,r_2) \\ &= \int\limits_{Q_1 \times Q_2} \big(L_{(r_1,r_2)}\mathcal{K}_{(A_1 \times A_2)}\big)(q_1,q_2)\mathcal{K}_{(B_1 \times B_2)}(q_1^*,q_2^*)d(q_1,q_2), \end{aligned} \tag{2.3}$$

where $(r_1,r_2) \in Q_1 \times Q_2$. This structure measure is said to be commutative whenever

$$c\Big(A_1 \times A_2, B_1 \times B_2, (r_1,r_2)\Big) = c\Big(B_1 \times B_2, A_1 \times A_2, (r_1,r_2)\Big). \tag{2.4}$$

A regular Borel measure $m := m_1 \otimes m_2$ on $\mathcal{B}_0(Q_1 \times Q_2)$ is called multiplicative if

$$\int\limits_{Q_1 \times Q_2} c\Big(A_1 \times A_2, B_1 \times B_2, (r_1,r_2)\Big)dm(r_1,r_2) = m(A_1 \times A_2)m(B_1 \times B_2). \tag{2.5}$$

By using Eq.(1.5), we can define the convolution in $\mathbf{H}_1 \otimes \mathbf{H}_2$ as follows

$$\begin{aligned} \big(f * g\big)(p_1,p_2) &= \int\limits_{Q_1 \times Q_2} \big(L_{(p_1,p_2)}f\big)(q_1,q_2)g(q_1^*,q_2^*)d(q_1,q_2) \\ &= \big(L_{(p_1,p_2)}f, g^*\big)_{(\mathbf{H}_1 \otimes \mathbf{H}_2)_2}, \end{aligned} \tag{2.6}$$

where $f, g \in L_2\big(Q_1 \times Q_2, m_1 \otimes m_2\big) = L_2(Q_1,m_1) \otimes L_2(Q_2,m_2) := (\mathbf{H}_1 \otimes \mathbf{H}_2)_2$.

A non zero measurable and bounded almost everywhere function $Q_1 \times Q_2 \ni (r_1, r_2) \mapsto \chi(r_1, r_2) \in \mathbb{C}$ is said to be a character of HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$, if the equality

$$\int\limits_{Q_1 \times Q_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big)\chi(r_1, r_2) dm(r_1, r_2) = \chi(A_1 \times A_2)\chi(B_1 \times B_2) \quad (2.7)$$

holds for any $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$. Every direct product of HCSs has at least one character, namely, the function $\chi = 1$. A non zero measurable complex-valued function $\chi(r_1, r_2)$, $(r_1, r_2) \in Q_1 \times Q_2$ is called a generalized character of the HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$, if the equality (2.7) holds.

The HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ is said to be normal, if there exists an involution homomorphism $Q_1 \times Q_2 \ni (r_1, r_2) \mapsto (r_1^*, r_2^*) \in Q_1 \times Q_2$, such that $m(E_1 \times E_2) = m(E_1^* \times E_2^*)$ $\big(E_1 \times E_2 \in \mathcal{B}(Q_1 \times Q_2)\big)$ and for all $A_1 \times A_2, B_1 \times B_2, C_1 \times C_2 \in \mathcal{B}_0(Q_1 \times Q_2)$, we have

$$\begin{aligned} c\big(A_1 \times A_2, B_1 \times B_2, C_1 \times C_2\big) &= c\big(C_1 \times C_2, B_1^* \times B_2^*, A_1 \times A_2\big), \\ &= c\big(A_1^* \times A_2^*, C_1 \times C_2, B_1 \times B_2\big), \end{aligned} \quad (2.8)$$

where

$$c\big(A_1 \times A_2, B_1 \times B_2, C_1 \times C_2\big) = \int\limits_{C_1 \times C_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) dm(r_1, r_2). \quad (2.9)$$

A normal HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ possesses a basis unity if there exists a point $(e_1, e_2) \in Q_1 \times Q_2$ such that $(e_1, e_2) = (e_1^*, e_2^*)$ and

$$c\big(A_1 \times A_2, B_1 \times B_2, (e_1 \times e_2)\big) = m\big((A_1^* \times A_2^*) \cap (B_1 \times B_2)\big), \quad (2.10)$$

where $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}(Q_1 \times Q_2)$.

A normal HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ is called Hermitian if $(r_1^*, r_2^*) = (r_1, r_2)$ fore all $(r_1, r_2) \in Q_1 \times Q_2$. Every Hermitian direct product of HCSs is commutative. We should remark that, for a normal HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$, the mapping

$$\mathbf{H}_1 \otimes \mathbf{H}_2 \ni f(r_1, r_2) \mapsto f^*(r_1, r_2) \in \mathbf{H}_1 \otimes \mathbf{H}_2 \quad (2.11)$$

is an involution in the Banach algebra $\mathbf{H}_1 \otimes \mathbf{H}_2$. A character χ of a normal HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ is said to be Hermitian if

$$\chi(r_1^*, r_2^*) = \overline{\chi(r_1, r_2)}, \quad (r_1, r_2) \in Q_1 \times Q_2. \quad (2.12)$$

Denote the families of characters, of generalized characters and of bounded Hermitian characters by $\mathbf{X}$, $\mathbf{X}_g$ and $\mathbf{X}_h$, respectively.

The following result gives us the criterium of the generalized characters of a normal commutative direct product of HCSs.

Lemma 2.2. *In order that a function* $\chi(r_1,r_2) \in C(Q_1 \times Q_2)$ *be a generalized character of the normal commutative direct product of HCSs* $\boldsymbol{H}_1 \otimes \boldsymbol{H}_2$ *with basis unity* (e_1, e_1) *it is necessary and sufficient that the equality*

$$\big(L_{(p_1,p_2)}\chi\big)(q_1,q_2) = \chi(p_1,p_2)\chi(q_1,q_2), \tag{2.13}$$

hold for almost all $(p_1,p_2),(q_1,q_2) \in (Q_1 \times Q_2)$.

Proof. Assume that a function $\chi \in X_g$. Then, we have

$$\begin{aligned}
\chi(A_1 \times A_2)\chi(B_1 \times B_2) &= \int\limits_{Q_1\times Q_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1,r_2)\big)\chi(r_1,r_2)d(r_1,r_2) \\
&= \int\limits_{Q_1\times Q_2}\int\limits_{B_1^*\times B_2^*} \big(L_{(r_1,r_2)}\mathcal{K}_{(A_1\times A_2)}\big)(s_1,s_2)d(s_1,s_2)\chi(r_1,r_2)d(r_1,r_2) \\
&= \int\limits_{B_1\times B_2}\int\limits_{Q_1\times Q_2} \big(L_{(s_1^*,s_2^*)}\mathcal{K}_{(A_1\times A_2)}\big)(r_1,r_2)\chi(r_1,r_2)d(r_1,r_2)d(s_1,s_2) \\
&= \int\limits_{B_1\times B_2}\int\limits_{A_1\times A_2} \big(L_{(s_1,s_2)}\chi\big)(r_1,r_2)d(r_1,r_2)d(s_1,s_2),
\end{aligned} \tag{2.14}$$

for any $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$, which yields to Eq. (2.13). The converse statement can be proved by analogy. □

Practically, to illustrate the concept of direct product of HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$, we give an example as follows:

Example 2.1. Let $Q_1 = G_1, Q_2 = G_2$ be commutative locally compact groups. It is easy to see that $Q_1 \times Q_2 = G_1 \times G_2$ is commutative locally compact group with unity (e_1, e_2), where e_1 and e_2 are the unities of G_1 and G_2, respectively. Consider its group algebra, i.e., a set $L_1(G_1 \times G_2, m)$ of functions defined on the group $G_1 \times G_2$ and summable with respect to the Haar measure $m := m_1 \otimes m_2$. So, we can define the involution

$$G_1 \times G_2 \ni (p_1,p_2) \longmapsto (p_1^*,p_2^*) \in G_1 \times G_2. \tag{2.15}$$

In this case, where

$$\big(L_{(p_1,p_2)}f\big)(q_1,q_2) = f\big((q_1,q_2)(p_1,p_2)\big), \quad (p_1,p_2),(q_1,q_2) \in G_1 \times G_2, \tag{2.16}$$

we have the convolution

$$\big(f * g\big)(p_1,p_2) = \int\limits_{Q_1\times Q_2} f\big((q_1,q_2)(p_1,p_2)\big)g(q_1^*,q_2^*)d(q_1,q_1) \tag{2.17}$$

Also, the structure measure has the form,

$$c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) = m\big((A_1^{-1} \times A_2^{-1})(r_1, r_2) \cap (B_1 \times B_2)\big), \qquad (2.18)$$

where $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}(G_1 \times G_2), (r_1, r_2) \in G_1 \times G_2$. Thus, we obtain the direct product of the commutative HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$. This direct product is also commutative and with basis unity (e_1, e_2). In particular, if $G_1 \times G_2 = \mathbb{R} \times \mathbb{R}$ is an additive groups of all real numbers. For such HCSs it is possible to introduce product of GTOs $L_{(p_1,p_2)}$:

$$\mathbb{R} \times \mathbb{R} \ni (p_1, p_2) \longmapsto \big(L_{(p_1,p_2)} f\big)(q_1, q_2) \in \mathbb{C}, \quad f \in C(\mathbb{R} \times \mathbb{R}),$$

where $\big(L_{(p_1,p_2)} f\big)(q_1, q_2) = f\big((q_1, q_2) + (p_1, p_2)\big)$. By using the operators $L_{(p_1,p_2)}$, one can rewrite the involution and convolution as follows respectively:

$$\mathbb{R} \times \mathbb{R} \ni (p_1, p_2) \longmapsto (p_1^*, p_2^*) := (p_1^{-1}, p_2^{-1}) \in \mathbb{R} \times \mathbb{R}, \qquad (2.19)$$

$$\begin{aligned} (f * g)(p_1, p_2) &= \int\limits_{\mathbb{R}\times\mathbb{R}} f(q_1, q_2)\big(L_{(q_1^*, q_2^*)} g\big)(p_1, p_2) d(q_1, q_1) \\ &= \int\limits_{\mathbb{R}\times\mathbb{R}} f(q_1, q_2) g\big((p_1, p_2) - (q_1, q_2)\big) d(q_1, q_1), \end{aligned} \qquad (2.20)$$

where $(q_1^*, q_2^*) = (-q_1, -q_2) = -(q_1, q_2)$ in additive groups $\mathbb{R} \times \mathbb{R}$, $f, g \in L_1(\mathbb{R} \times \mathbb{R}, d(t_1, t_2))$ and the functions $\chi(t_1, t_2) = e^{i(t_1,t_2)(s_1,s_2)}$, $\big((s_1, s_2) \in \mathbb{R}\times\mathbb{R}\big)$ are characters.

Actually, there are many examples can be modified to the case of direct product of the HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$. For more details see [1, 19].

3. NEGATIVE DEFINITE FUNCTIONS ON DIRECT PRODUCT OF HCS'S

In this section, we present a concept of negative definite functions on a commutative normal direct product of HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ with basis unity (e_1, e_2). Negative definite functions were studied in [21], [16] and [2] for commutative groups, semigroups and HCSs respectively. The definition of such functions on $Q_1 \times Q_2$ is a natural generalization of that defined on a commutative hypergroups.

Definition 3.1. A continuous bounded function $\Psi : Q_1 \times Q_2 \to \mathbb{C}$ is called negative definite if for any $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\,(n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \left[\Psi(r_1^{(i)}, r_2^{(i)}) + \overline{\Psi(r_1^{(j)}, r_2^{(j)})} - \Big(L_{\big((r_1^*)^{(j)}, (r_2^*)^{(j)}\big)} \Psi\Big)(r_1^{(i)}, r_2^{(i)})\right] c_i \overline{c_j} \geq 0. \qquad (3.1)$$

Remark. Obviously, the following relations hold for a negative definite function Ψ.

(i) $\Psi(e_1, e_2) \geq 0,$

(ii) $\overline{\Psi(r_1, r_2)} = \Psi(r_1^*, r_2^*),$

(iii) $\big(L_{(r_1^*, r_2^*)}\Psi\big)(r_1, r_2) \in \mathbb{R},$

(iv) $\Psi(r_1, r_2) + \Psi(r_1^*, r_2^*) \geq \big(L_{(r_1^*, r_2^*)}\Psi\big)(r_1, r_2).$

We note that $\Psi = \Psi^*$ and $\operatorname{Re}\Psi$ is non negative if $\big(L_{(r_1^*, r_2^*)}\Psi\big)(r_1, r_2) \geq 0$. Let us abbreviate the set of negative definite functions on $Q_1 \times Q_2$ by $\mathcal{N}(Q_1 \times Q_2)$.

Theorem 3.1. *A function $\Psi : Q_1 \times Q_2 \to \mathbb{C}$ is negative definite if and only if the following conditions are satisfied.*

(i) $\Psi(e_1, e_2) \geq 0$, *Ψ is continuous bounded function,*

(ii) $\overline{\Psi(r_1, r_2)} = \Psi(r_1^*, r_2^*)$, *for each $(r_1, r_2) \in Q_1 \times Q_2$,*

(iii) *If $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\,(n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$ with*

$$\sum_{i=1}^{n} c_i = 0, \tag{3.2}$$

then the following relation is holds.

$$\sum_{i,j=1}^{n} \Big[\Big(L_{\big((r_1^*)^{(j)}, (r_2^*)^{(j)}\big)}\Psi\Big)(r_1^{(i)}, r_2^{(i)})\Big] c_i \overline{c_j} \leq 0. \tag{3.3}$$

Proof. Assume first that $\Psi \in \mathcal{N}(Q_1 \times Q_2)$. It is clear that ***(i)*** and ***(ii)*** are satisfied. Let $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\,(n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$ be such that the relation in Eq.(3.2) holds. Then, by Definition 3.1, we have the following

$$\begin{aligned}
0 &\leq \sum_{i,j=1}^{n} \Big[\Psi(r_1^{(i)}, r_2^{(i)}) + \overline{\Psi(r_1^{(j)}, r_2^{(j)})} - \Big(L_{\big((r_1^*)^{(j)}, (r_2^*)^{(j)}\big)}\Psi\Big)(r_1^{(i)}, r_2^{(i)})\Big] c_i \overline{c_j} \\
&= \overline{\Big(\sum_{j=1}^{n} c_j\Big)}\Big(\sum_{i=1}^{n} \Psi(r_1^{(i)}, r_2^{(i)}) c_i\Big) + \Big(\sum_{i=1}^{n} c_i\Big)\overline{\Big(\sum_{j=1}^{n} \Psi(r_1^{(j)}, r_2^{(j)}) c_j\Big)} \\
&- \sum_{i,j=1}^{n} \Big(L_{\big((r_1^*)^{(j)}, (r_2^*)^{(j)}\big)}\Psi\Big)(r_1^{(i)}, r_2^{(i)}) c_i \overline{c_j} \\
&= -\sum_{i,j=1}^{n} \Big(L_{\big((r_1^*)^{(j)}, (r_2^*)^{(j)}\big)}\Psi\Big)(r_1^{(i)}, r_2^{(i)}) c_i \overline{c_j}.
\end{aligned} \tag{3.4}$$

Then, ***(iii)*** is satisfied.

Conversely, suppose that Ψ satisfies ***(i)***-***(iii)***, and consider $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\,(n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$. For the $(n+1)$-tuples $(e_i, e_j), (r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\,(n \in \mathbb{N}))$ and $c_0, c_1, c_2, ..., c_n \in \mathbb{C}$, where

$$\sum_{i=0}^{n} c_i = 0, \qquad \text{i.e.,} \quad c_0 = -\sum_{i=1}^{n} c_i,$$

we obtain by ***(iii)***

$$\begin{aligned}
0 &\geq \sum_{i,j=0}^{n} \left[\left(L_{\left((r_1^*)^{(j)}, (r_2^*)^{(j)}\right)} \Psi \right) (r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j} \\
&= \sum_{i,j=1}^{n} \left[\left(L_{\left((r_1^*)^{(j)}, (r_2^*)^{(j)}\right)} \Psi \right) (r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j} + \overline{c_0} \sum_{i=1}^{n} \left(\Psi(r_1^{(i)}, r_2^{(i)}) \right) c_i \\
&+ \ c_0 \sum_{j=1}^{n} \Psi\left((r_1^*)^{(j)}, (r_2^*)^{(j)} \right) \overline{c_j} + \Psi(e_1, e_2)|c_0|^2 \\
&= \sum_{i,j=1}^{n} \left[\left(L_{\left((r_1^*)^{(j)}, (r_2^*)^{(j)}\right)} \Psi \right) (r_1^{(i)}, r_2^{(i)}) - \Psi(r_1^{(i)}, r_2^{(i)}) - \Psi\left((r_1^*)^{(j)}, (r_2^*)^{(j)} \right) \right] c_i \overline{c_j} \\
&\quad + \Psi(e_1, e_2)|c_0|^2.
\end{aligned} \tag{3.5}$$

Hence, using ***(i)*** and ***(ii)***, we have

$$\sum_{i,j=1}^{n} \left[\Psi(r_1^{(i)}, r_2^{(i)}) + \overline{\Psi(r_1^{(j)}, r_2^{(j)})} - \left(L_{\left((r_1^*)^{(j)}, (r_2^*)^{(j)}\right)} \Psi \right) (r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j}$$

$$\geq \Psi(e_1, e_2)|c_0|^2 \geq 0. \tag{3.6}$$

Therefore, $\Psi \in \mathcal{N}(Q_1 \times Q_2)$. $\square$

Now we give the relation and important properties of negative definite and positive definite functions $\Psi \in \mathcal{N}(Q_1 \times Q_2)$, $\Theta \in \mathcal{P}(Q_1 \times Q_2)$ respectively, as follows.

Corollary 3.2. *Let Ψ and Θ be functions on $Q \times Q$. Then, the following properties are satisfied*

(a) *If $\Psi \in \mathcal{N}(Q_1 \times Q_2)$, then $Q \times Q \ni (r_1, r_2) \mapsto \Psi(r_1, r_2) - \Psi(e_1, e_2)$ is negative definite.*

(b) *If $\Theta \in \mathcal{P}(Q_1 \times Q_2)$, then $Q \times Q \ni (r_1, r_2) \mapsto \Theta(e_1, e_2) - \Theta(r_1, r_2)$ is negative definite.*

Proof. **(a)** Let $\Psi \in \mathcal{N}(Q_1 \times Q_2)$, by using Theorem 3.1, conditions ***(i)*** and ***(ii)*** are clearly satisfied. For the condition ***(iii)***, let $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j =$

$1, 2, ..., n\, (n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$, such that $\sum_{i=1}^{n} c_i = 0$. Then, we have

$$\begin{aligned}
&\sum_{i,j=1}^{n} \left[L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \left(\Psi(r_1^{(i)}, r_2^{(i)}) - \Psi(e_1, e_2) \right) \right] c_i \overline{c_j} \\
&= \sum_{i,j=1}^{n} \left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Psi \right)(r_1^{(i)}, r_2^{(i)}) c_i \overline{c_j} - \Psi(e_1, e_2) \left| \sum_{i=1}^{n} c_i \right|^2 \\
&= \sum_{i,j=1}^{n} \left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Psi \right)(r_1^{(i)}, r_2^{(i)}) c_i \overline{c_j} \leq 0 \qquad (3.7)
\end{aligned}$$

which proves the negative definiteness of $\Psi(r_1, r_2) - \Psi(e_1, e_2)$.

(b) Let $\Theta \in \mathcal{P}(Q_1 \times Q_2)$, $(r_i, r_j), ..., (r_n, r_n) \in Q_1 \times Q_2$, $(i, j = 1, 2, ..., n\, (n \in \mathbb{N}))$ and $c_1, c_2, ..., c_n \in \mathbb{C}$, such that $\sum_{i=1}^{n} c_i = 0$. Then, we have

$$\begin{aligned}
&\sum_{i,j=1}^{n} \left[\left(\Theta(e_1, e_2) - L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Theta \right)(r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j} \\
&= -\sum_{i,j=1}^{n} \left[\left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Theta \right)(r_1^{(i)}, r_2^{(i)}) - \Theta(e_1, e_2) \right] c_i \overline{c_j} \\
&= -\sum_{i,j=1}^{n} \left[\left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Theta \right)(r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j} + \Theta(e_1, e_2) \left| \sum_{i=1}^{n} c_i \right|^2 \\
&= -\sum_{i,j=1}^{n} \left[\left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Theta \right)(r_1^{(i)}, r_2^{(i)}) \right] c_i \overline{c_j} \leq 0. \qquad (3.8)
\end{aligned}$$

Because $\Theta \in \mathcal{P}(Q_1 \times Q_2)$, and since the function $\Theta(e_1, e_2) - \Theta(r_1, r_2)$ clearly satisfies (i) and (ii) of Theorem 3.1. Then, it is negative definite. □

Theorem 3.3. *Let $\Psi \in \mathcal{N}(Q_1 \times Q_2)$ with $Re\Psi \geq 0$. Then*

$$\sqrt{\left(L_{(r_1,r_2)} \Psi \right)(s_1, s_2)} \leq \sqrt{|\Psi(r_1, r_2)|} \sqrt{|\Psi(s_1, s_2)|}, \qquad (r_1, r_2), (s_1, s_2) \in (Q_1 \times Q_2). \quad (3.9)$$

Proof. Let $\Psi \in \mathcal{N}(Q_1 \times Q_2)$, as similarly in Theorem 3.7. [2]. Then the $n \times n$ matrix $\left(\Psi(r_1^{(i)}, r_2^{(i)}) + \overline{\Psi(r_1^{(j)}, r_2^{(j)})} - \left(L_{\left((r_1^*)^{(j)},(r_2^*)^{(j)}\right)} \Psi \right)(r_1^{(i)}, r_2^{(i)}) \right)$ is positive Hermitian for any $i, j = 1, ..., n$.

Take $n = 2$, and $(r_1, r_2), (s_1, s_2) \in (Q_1 \times Q_2)$. Since the matrix

$$\begin{pmatrix} \Psi(r_1, r_2) + \overline{\Psi(r_1, r_2)} - (L_{(r_1^*, r_2^*)} \Psi)(r_1, r_2) & \Psi(r_1, r_2) + \overline{\Psi(s_1, s_2)} - (L_{(s_1^*, s_2^*)} \Psi)(r_1, r_2) \\ \Psi(s_1, s_2) + \overline{\Psi(r_1, r_2)} - (L_{(r_1^*, r_2^*)} \Psi)(s_1, s_2) & \Psi(s_1, s_2) + \overline{\Psi(r_1, r_2)} - (L_{(s_1^*, s_2^*)} \Psi)(s_1, s_2) \end{pmatrix} \quad (3.10)$$

has non-negative determinant, according to the above properties from i,...,iv, using $(L_{(r_1^*,r_2^*)}\Psi)(s_1,s_2) = (L_{(s_1^*,s_2^*)}\Psi)(r_1,r_2)$, we get

$$\Big|\Psi(r_1,r_2) + \overline{\Psi(s_1,s_2)} - (L_{(s_1^*,s_2^*)}\Psi)(r_1,r_2)\Big|^2 \\ \leq \Big(2Re\Psi(r_1,r_2) - (L_{(r_1^*,r_2^*)}\Psi)(r_1,r_2)\Big). \quad (3.11)$$

So, we have

$$\begin{aligned} \Big(2Re\Psi(s_1,s_2) - (L_{(s_1^*,s_2^*)}\Psi)(s_1,s_2)\Big) &\leq 4Re\Psi(r_1,r_2)Re\Psi(s_1,s_2) \\ &\leq 4\Big|\Psi(r_1,r_2)\Big|\Big|\Psi(s_1,s_2)\Big|. \end{aligned} \quad (3.12)$$

Then

$$\Big|(L_{(s_1^*,s_2^*)}\Psi)(r_1,r_2) - \Psi(r_1,r_2) - \overline{\Psi(s_1,s_2)}\Big| \leq 2\sqrt{|\Psi(r_1,r_2)|}\sqrt{|\Psi(s_1,s_2)|} \quad (3.13)$$

and

$$\Big|(L_{(s_1^*,s_2^*)}\Psi)(r_1,r_2)\Big| \leq \left(\sqrt{|\Psi(r_1,r_2)|} + \sqrt{|\Psi(s_1,s_2)|}\right)^2. \quad (3.14)$$

Then the relation (3.9) is satisfied. Hence, the proof of the Theorem is completes. □

4. CONCLUSION

A direct product of two HCSs is precisely defined via the theory of GTOs. We showed that, under some conditions, the properties of commutativity, normality are preserved under the operation of taking the direct product. Furthermore we given an example to improve the concept of direct product of HCSs. Also, we transferred the objects of harmonic analysis, in particular, the criteria of negative definite functions on the direct product of two HCSs. In our paper We intrusted with a concept of negative definite functions on a commutative normal direct product of HCSs.

This work can be immediately generalized to a direct product of any finite number of HCSs. While, the case of infinite number of HCSs is still open. Moreover, it is fairly easy to observe that all our results for direct product of HCSs can be easily investigated for direct products of semigroups and hypergroups (See [16, 17]).

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