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BOUNDED ORBITS AND G-CONTRACTIVE FIXED POINTS

T. PHANEENDRA[1] AND S. SARAVANAN[2]

[1,2]Department of Mathematics
School of Advanced Sciences,
VIT University
Vellore, 632014, Tamil Nadu, INDIA

ABSTRACT: Let f be a self-map on a metric space (X, d) and $x_0 \in X$. The orbit $O_f(x_0)$ at x_0 is the sequence of f-iterates $\langle x_0, fx_0, ..., f^n x_0, ...\rangle$. A fixed point p of f is a contractive fixed point if every $O_f(x_0)$ converges to p. The existence of contractive fixed points for self-maps in metric spaces was investigated by Edelstein [1], Leader and Hoyle in [2], and by Reich [11]. The notion of G-contractive fixed point in a generalized metric space was introduced by the first author and Kumara Swamy in [8] in 2013. This paper devotes to the study of bounded orbits and contractive fixed points for certain self-maps in G-metric space, without using the iterations.

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1. INTRODUCTION

Let f be a self-map on a metric space (X, d) and $x_0 \in X$. The orbit $O_f(x_0)$ at x_0 is the sequence of f-iterates $\langle x_0, fx_0, ..., f^n x_0, ...\rangle$. A fixed point p of f is a contractive fixed point, if every $O_f(x_0)$ converges to p. Since convergent sequences in metric spaces have unique limits, a contractive fixed point must be a unique fixed point. However, a unique fixed point need not be a contractive fixed point.

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Example 1.1. Consider $X = [0, \infty)$ with usual metric $d(x, y) = |x - y|$ for all $x \in X$. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & x < 1 \\ 2x & x \geq 1. \end{cases} \tag{1}$$

We see that $x = 0$ is the unique fixed point of f. But for $0 \leq x < 1$,

$$O_f(x_0) = \langle x_0, \frac{x_0}{2}, \frac{x_0}{2^2}, ..., \frac{x_0}{2^n}, ...\rangle$$

converges to 0, while for $x \geq 1$,

$$O_f(x_0) = \langle x_0, 2x_0, 2^2x_0, , ..., 2^n x_0, ...\rangle \to \infty.$$

In other words, 0 is not a contractive fixed point of f.

Contractive fixed points for self-maps in a metric space were obtained by Edelstein [1], Leader and Hoyle in [2], and by Reich [11].

This paper obtains G-contractive fixed points for certain self-maps on a G-metric space, without using the usual iterative procedure.

2. PRELIMINARIES AND NOTATION

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y)$
$= G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then G is called a G-metric on X and the pair (X, G), a G-metric space. Axiom (G5) is referred to as the rectangle inequality (of G). This notion was introduced by Mustafa and Sims [6] in 2006. If $x, y, z \in X$ are such that $G(x, y, z) = 0$, it is obvious that $x = y = z$. In particular, if $x, y \in X$ are such that $G(x, y, y) = 0$,then $x = y$. Also

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{2}$$

We use the following notions, developed in [6]:

Definition 2.1. Let (X, G) be a G-metric space. A G-ball in X is defined by

$$B_G(x, r) = \{y \in X : G(x, y, y) < r\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X.

Also

$$\rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y \in X. \tag{3}$$

induces a metric on X, and the G-metric topology coincides with the metric topology induced by the metric ρ_G.

Definition 2.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be *G-convergent* with limit $p \in X$ if it converges to p in the G-metric topology $\tau(G)$.

Lemma 2.1. *The following statements are equivalent in a G-metric space (X, G):*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$*,*

(b) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$*,*

(c) $\lim_{n\to\infty} G(x_n, p, p) = 0$*.*

Definition 2.3. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy if $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

Definition 2.4. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in X converges in it.

Definition 2.5. A fixed point p of f on a G-metric space (X, G) is a G-contractive fixed point of it if the orbital sequence $x_0, fx_0, ..., f^n x_0, ...$ at each $x_0 \in X$ is G-convergent with limit p.

Example 2.1. Let (X, d) be a metric space where $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Define $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ for all $x, y, z \in X$. Then (X, G) is a G-metric space. Consider $f : X \to X$ by $fx = x/2$ for all $x \in X$. Then 0 is the only fixed point of f. Also for each $x_0 \in X$, we have $O_f(x_0) = \{x_0, x_0/2, x_0/2^2, ...\}$, Since for any $x_0 \in X$, $x_0/2^n \to 0$ as $n \to \infty$, we see that $O_f(x_0) \to 0$ for all x_0. In other words, every f-orbit converges to the fixed point 0. Therefore, 0 is the G-contractive fixed point.

The notion of G-contractive fixed point was introduced by the first author and Kumara Swamy in [8] in 2013. It was shown in [8] that the unique fixed point of the self-map f with the following choices is a G-contractive fixed point.

(a) $G(fx, fy, fz) \leq qG(x, y, z)$ for all $x, y, z \in X$, where $0 \leq q < 1$,

(b) $G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) + eG(x, y, z)$ for all $x, y, z \in X$, where a, b, c and e are nonnegative real numbers with $a+b+c+e < 1$.

In a recent paper [9], the authors obtained a contractive fixed point of a Chatterjee-type contraction f on a complete G-metric space (X, G), which satisfies the condition

$$G(fx, fy, fz) \leq \gamma[G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)], \quad \text{for all } x, y, z \in X, \tag{4}$$

where $0 < \gamma < 1/3$. In continuation to the work, the authors obtained G-contractive fixed points under some general contraction type conditions in a G-metric space in [10]. Further, the authors of the present papers [12] proved the following result:

Theorem 2.1. *Let X be a complete G-metric space and f, a self-map on X satisfying the condition:*

$$G(fx, fy, fz) \leq aG(x, y, z) + b \max\{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\} \quad \textit{for all } x, y, z \in X, \tag{5}$$

where a and b are nonnegative real numbers with $a + b < 1$, which are not both zero. Then f has a unique fixed point p. Further, p will be a G-contractive fixed point if $a + 3b < 1$.

We demonstrate below that the G-contractive fixed point can be obtained under the condition that $a + b < 1$. Consequently, the condition $a + 3b < 1$ becomes insignificant in this regard.

Proof. Writing $x = f^{n-1}x_0, y = z = p$ in (5), we get

$$\begin{aligned} G(f^n x_0, p, p) &= G(ff^{n-1}x_0, fp, fp) \\ &\leq aG(f^{n-1}x_0, p, p) \\ &\quad + b \max\{G(f^{n-1}x_0, f^n x_0, f^n x_0), G(p, fp, fp), G(p, fp, fp)\} \\ &= aG(f^{n-1}x_0, p, p) + bG(f^{n-1}x_0, f^n x_0, f^n x_0). \end{aligned} \tag{6}$$

Now with $x = f^{n-2}x_0$, $y = z = f^{n-1}x_0$ in (5), we see that

$$\begin{aligned} G(f^{n-1}x_0, f^n x, f^n x_0) \leq aG(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) \\ + b \max\{G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0), \\ G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-1}x_0, f^n x_0, f^n x_0) \end{aligned}$$

$$= aG(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + bM, \tag{7}$$

where $M = \max\{G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0), G(f^{n-1}x_0, f^n x_0, f^n x_0)\}$.

Case (a): Let $M = G(f^{n-1}x_0, f^n x_0, f^n x_0)$. Then (7) becomes

$$G(f^{n-1}x_0, f^n x, f^n x_0) \le aG(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + bG(f^{n-1}x_0, f^n x_0, f^n x_0)$$

or

$$G(f^{n-1}x_0, f^n x, f^n x_0) \le \left(\tfrac{a}{1-b}\right) G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0).$$

This in general gives

$$G(f^{n-1}x_0, f^n x, f^n x_0) \le \left(\tfrac{a}{1-b}\right)^{n-1} G(x_0, fx_0, fx_0).$$

Inserting this in (6), we obtain that

$$G(f^n x_0, p, p) \le aG(f^{n-1}x_0, p, p) + b\left(\tfrac{a}{1-b}\right)^{n-1} G(x_0, fx_0, fx_0). \tag{8}$$

By induction, we get

$$\begin{aligned} G(f^{n-1}x_0, f^n x, f^n x_0) &\le a^{n-1}G(fx_0, p, p) \\ &\quad + b(1 + a + a^2 + \cdots + a^{n-2})\left(\tfrac{a}{1-b}\right)^{n-1} G(x_0, fx_0, fx_0) \\ &\le a^{n-1}G(fx_0, p, p) + \left(\tfrac{b}{1-a}\right)\left(\tfrac{a}{1-b}\right)^{n-1} G(x_0, fx_0, fx_0). \end{aligned} \tag{9}$$

Case (b): Suppose that $M = G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0)$. Then (7) can be written as

$$\begin{aligned} G(f^{n-1}x_0, f^n x, f^n x_0) &\le aG(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + bG(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) \\ &= (a+b)G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) \end{aligned}$$

so that

$$G(f^{n-1}x_0, f^n x, f^n x_0) \le (a+b)^{n-1} G(x_0, fx_0, fx_0).$$

Inserting this in (6), we obtain that

$$G(f^n x_0, p, p) \le aG(f^{n-1}x_0, p, p) + b(a+b)^{n-1} G(x_0, fx_0, fx_0).$$

By induction, we get

$$\begin{aligned} G(f^{n-1}x_0, f^n x, f^n x_0) &\le a^{n-1}G(fx_0, p, p) \\ &\quad + b(1 + a + a^2 + \cdots + a^{n-2})(a+b)^{n-1} G(x_0, fx_0, fx_0) \\ &\le a^{n-1}G(fx_0, p, p) + (a+b)^{n-1}\tfrac{b}{1-a} G(x_0, fx_0, fx_0). \end{aligned} \tag{10}$$

Proceeding the limit as $n \to \infty$ in (9) or (10), we obtain that $G(f^n x_0, p, p) \to 0$. Thus $f^n x_0 \to p$ for each $x_0 \in X$, in view of Lemma 2.1. In other words, p is a G-contractive fixed point of f. □

In this paper, we establish that the unique fixed point obtained in some earlier fixed point theorems will be a G-contractive fixed point. Since every convergent sequence is bounded in a G-metric space, it immediately follows that each $O_f(x_0)$ is bounded. However, we obtain the boundedness in a way, independent of the convergence criterion.

3. MAIN RESULTS

We begin with the following result.

Theorem 3.1 (Mohanta [3], Theorem 3.1). *Let (X, G) be a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq aG(x, y, z) + bG(x, fx, fx) + cG(y, fy, fy) + dG(z, fz, fz) \\ + e \max\big\{G(x, fy, fy), G(y, fx, fx), G(y, fz, fz), G(z, fy, fy), \\ G(z, fx, fx), G(x, fz, fz)\big\} \text{ for all } x, y, z \in X, \end{aligned} \tag{11}$$

where $a + b + c + d + 2e < 1$ Then f has a unique fixed point p.

Employing the usual f-iterations: $x_0, fx_0, f^2x_0, ...$, Mohanta [3] first proved that each orbit $O_f(x_0)$ converges to some $p \in X$, and then p is a unique fixed point. Thus, p is a contractive fixed point. This proof inevitably depends on the choice of $O_f(x_0)$.

Our proof is in a different direction: With the help of (11), (G5) and induction on n, we establish first that each orbit $O_f(x_0)$ converges to $p \in X$ (See Lemma 3.1). To obtain a unique fixed point, we introduce a set S of nonnegative real numbers, which will have the infimum say, a. Then we show that $a = 0$, which results in a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X (which is not necessarily $O_f(x_0)$), satisfying some asymptotic condition. We, then, just use (11) and the rectangle inequality (G5), to prove that $\langle x_n\rangle_{n=1}^{\infty}$ as a Cauchy sequence, which in turn converges to some p. The limit p is shown to be a unique fixed point of f. This elegant proof is more analytical, and does not depend on the structure of $O_f(x_0)$, unlike that of [3].

Our proof needs the following lemma:

Lemma 3.1. *Let p be a unique fixed point of the self-map f on (X, G), satisfying (11). Then $O_f(x_0) \to p$ as $n \to \infty$ for all $x_0 \in X$.*

Proof. In fact, inserting $x = f^{n-1}x_0, y = z = p$ in (11) and using (G5), we get

$$G(f^n x_0, p, p) = G(f^n x_0, fp, fp)$$

$$\begin{aligned}
&\leq aG(f^{n-1}x_0,p,p)+bG(f^{n-1}x_0,f^nx_0,f^nx_0)\\
&\qquad + cG(p,fp,fp)+dG(p,fp,fp)\\
&\qquad + e\max\big\{G(f^{n-1}x_0,fp,fp),G(p,f^nx_0,f^nx_0),G(p,fp,fp),\\
&\qquad\quad G(p,fp,fp),G(p,f^nx_0,f^nx_0),G(f^{n-1}x_0,fp,fp)\big\}\\
&\leq aG(f^{n-1}x_0,p,p)+b[G(f^{n-1}x_0,f^nx_0,f^nx_0)+0+0\\
&\qquad + e\max\big\{G(f^{n-1}x_0,p,p),G(p,f^nx_0,f^nx_0)\big\}\\
&= aG(f^{n-1}x_0,p,p)+bG(f^{n-1}x_0,f^nx_0,f^nx_0)\\
&\qquad + e\max\big\{G(f^{n-1}x_0,p,p),2G(f^nx_0,p,p)\big\}\cdot \qquad (12)
\end{aligned}$$

Again writing $x=f^{n-2}x_0$, $y=z=f^{n-1}x_0$ in (11), we see that

$$\begin{aligned}
G(f^{n-1}x_0,f^nx_0,f^nx_0) &\leq aG(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)+bG(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)\\
&\quad + cG(f^{n-1}x_0,f^nx_0,f^nx_0)+dG(f^{n-1}x_0,f^nx_0,f^nx_0)\\
&\quad + e\max\big\{G(f^{n-2}x_0,f^nx_0,f^nx_0),G(f^{n-1}x_0,f^{n-1}x_0,f^{n-1}x_0),\\
&\quad G(f^{n-1}x_0,f^nx_0,f^nx_0),G(f^{n-1}x_0,f^nx_0,f^nx_0),\\
&\quad G(f^{n-1}x_0,f^{n-1}x_0,f^{n-1}x_0),G(f^{n-2}x_0,f^nx_0,f^nx_0)\}\\
&\leq (a+b)G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)+\\
&\quad + (c+d)G(f^{n-1}x_0,f^nx_0,f^nx_0)\\
&\quad + e[G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)+G(f^{n-1}x_0,f^nx_0,f^nx_0)]
\end{aligned}$$

or

$$G(f^{n-1}x_0,f^nx_0,f^nx_0)\leq\left(\tfrac{a+b+e}{1-c-d-e}\right)G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0).$$

Inserting this in (12), we get

$$\begin{aligned}
G(f^nx_0,p,p)&\leq aG(f^{n-1}x_0,p,p)\\
&\quad + b\left(\tfrac{a+b+e}{1-c-d-e}\right)G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)+eN, \qquad (13)
\end{aligned}$$

where $N=\max\big\{G(f^{n-1}x_0,p,p),2G(f^nx_0,p,p)\big\}$.

Case (a): If $N=2G(f^nx_0,p,p)$, it follows from (13) that

$$\begin{aligned}
G(f^nx_0,p,p)&\leq aG(f^{n-1}x_0,p,p)+b\left\{\tfrac{a+b+e}{1-c-d-e}\right\}G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0)\\
&\quad + 2eG(f^nx_0,p,p)\\
&\leq\left\{\tfrac{a}{1-2e}\right\}G(f^{n-1}x_0,p,p)\\
&\quad + \left\{\tfrac{b}{1-2e}\cdot\tfrac{a+b+e}{1-c-d-e}\right\}G(f^{n-2}x_0,f^{n-1}x_0,f^{n-1}x_0).
\end{aligned}$$

By induction, we get

$$\begin{aligned} G(f^n x_0, p, p) &\le \left(\tfrac{a}{1-2e}\right)^{n-1} G(fx_0, p, p) + \left(\tfrac{b}{1-2e}\right) \Bigg[1 + \left(\tfrac{a}{1-2e}\right) \\ &\qquad + \cdots + \left(\tfrac{a}{1-2e}\right)^{n-2}\Bigg] \left(\tfrac{a+b+e}{1-c-d-e}\right)^{n-1} G(x_0, fx_0, fx_0) \\ &\le \left(\tfrac{a}{1-2e}\right)^{n-1} G(fx_0, p, p) \\ &\qquad + \left(\tfrac{b}{1-2e-a}\right) \left(\tfrac{a+b+e}{1-c-d-e}\right)^{n-1} G(x_0, fx_0, fx_0). \end{aligned} \tag{14}$$

Case (b): If $N = G(f^{n-1}x_0, p, p)$, then (13) implies that

$$\begin{aligned} G(f^n x_0, p, p) &\le aG(f^{n-1}x_0, p, p) + b\left\{\tfrac{a+b+e}{1-c-d-e}\right\} G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) \\ &\qquad + eG(f^{n-1}x_0, p, p) \\ &\le (a+e)G(f^{n-1}x_0, p, p) \\ &\qquad + b\left(\tfrac{a+b+e}{1-c-d-e}\right) G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) \text{ for all } n. \end{aligned}$$

By induction, we get

$$\begin{aligned} G(f^n x_0, p, p) &\le (a+e)^{n-1} G(fx_0, p, p) + b\Bigg[1 + (a+e) \\ &\qquad + \cdots + (a+e)^{n-2}\Bigg] \left(\tfrac{a+b+e}{1-c-d-e}\right)^{n-1} G(x_0, fx_0, fx_0) \\ &\le (a+e)^{n-1} G(fx_0, p, p) \\ &\qquad + \left(\tfrac{b}{1-e-a}\right) \left(\tfrac{a+b+e}{1-c-d-e}\right)^{n-1} G(x_0, fx_0, fx_0). \end{aligned} \tag{15}$$

Proceeding $n \to \infty$ in (14) or (15), we finally obtain that $G(f^n x_0, p, p) \to 0$. Thus $f^n x_0 \to p$ for each $x_0 \in X$. That is p is a G-contractive fixed point. □

From this proof, it easily follows that each $O_f(x_0)$ is bounded, since every convergent sequence is bounded in a G-metric space. However, we shall obtain the boundedness of the orbit independently as follows:

Lemma 3.2. *First we prove that* $O_f(x_0) = \{x_0\} \cup \langle x_n \rangle_{n=1}^{\infty}$ *is bounded, where* $x_n = fx_{n-1}$ *for all* $n \ge 1$.

Proof. If possible, suppose that $O_f(x_0)$ is unbounded. Then there exists a positive integer n such that

$$G(x_1, x_n, x_n) > \mu \max\{G(x_1, x_r, x_r) : 0 \le r \le n-1\}, \tag{16}$$

where

$$\mu = \max\left\{\tfrac{2a+2b+2e}{1-c-d-2e}, \tfrac{a+2c+2d+2e}{1-c-d-2e}\right\}. \tag{17}$$

Writing $x = x_0$, $y = z = x_{n-1}$ in (11) and then using (G5) and (2), we get

$$
\begin{aligned}
G(x_1, x_n, x_n) &= G(fx_0, fx_{n-1}, fx_{n-1}) \\
&\le aG(x_0, x_{n-1}, x_{n-1}) + bG(x_0, fx_0, fx_0) + cG(x_{n-1}, fx_{n-1}, fx_{n-1}) \\
&\quad + dG(x_{n-1}, fx_{n-1}, fx_{n-1}) + e\max\{G(x_0, fx_{n-1}, fx_{n-1}), \\
&\quad G(x_{n-1}, fx_0, fx_0), G(x_{n-1}, fx_{n-1}, fx_{n-1}), G(x_{n-1}, fx_{n-1}, fx_{n-1}), \\
&\quad G(x_{n-1}, fx_0, fx_0), G(x_0, fx_{n-1}, fx_{n-1})\} \\
&\le aG(x_0, x_{n-1}, x_{n-1}) + bG(x_0, x_1, x_1) + cG(x_{n-1}, x_n, x_n) \\
&\quad + dG(x_{n-1}, x_n, x_n) + e\max\{G(x_0, x_n, x_n), G(x_{n-1}, x_1, x_1), \\
&\qquad G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_1, x_1), G(x_0, x_n, x_n)\} \\
&\le a[G(x_0, x_1, x_1) + G(x_1, x_{n-1}, x_{n-1})] + 2bG(x_1, x_0, x_0) \\
&\qquad + (c+d)[G(x_{n-1}, x_1, x_1) + G(x_1, x_n, x_n)] \\
&\qquad + e\max\{G(x_0, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_1, x_1)\} \\
&\le a[2G(x_1, x_0, x_0) + G(x_1, x_{n-1}, x_{n-1})] + 2bG(x_1, x_0, x_0) \\
&\quad + 2(c+d)[G(x_1, x_{n-1}, x_{n-1})] + (c+d)G(x_1, x_n, x_n)] \\
&\qquad + e\max\{[G(x_0, x_1, x_1) + G(x_1, x_n, x_n)], G(x_{n-1}, x_1, x_1), \\
&\qquad\qquad [G(x_{n-1}, x_1, x_1) + G(x_1, x_n, x_n)]\} \\
&\le (2a+2b)G(x_1, x_0, x_0) + (a+2c+2d)G(x_{n-1}, x_1, x_1) \\
&\qquad + (c+d)G(x_1, x_n, x_n) + e[2G(x_1, x_0, x_0) + G(x_1, x_n, x_n) \\
&\qquad\qquad + 2G(x_1, x_{n-1}, x_{n-1}) + G(x_1, x_n, x_n)] \\
&\le (2a+2b+2e)G(x_1, x_0, x_0) + (a+2c+2d+2e)G(x_1, x_{n-1}, x_{n-1}) \\
&\qquad\qquad + (c+d+2e)G(x_1, x_n, x_n)
\end{aligned}
$$

or

$$
\begin{aligned}
(1-c-d-2e)G(x_1, x_n, x_n) \le (2a+2b+2e)&G(x_1, x_0, x_0) + \\
&(a+2c+2d+2e)G(x_1, x_{n-1}, x_{n-1})
\end{aligned}
$$

so that

$$
\begin{aligned}
G(x_1, x_n, x_n) \le &\left(\tfrac{2a+2b+2e}{1-c-d-2e}\right) G(x_1, x_0, x_0) \\
&+ \left(\tfrac{a+2c+2d+2e}{1-c-d-2e}\right) G(x_1, x_{n-1}, x_{n-1}),
\end{aligned}
$$

which in view of (16) and (17) gives is a contradiction that

$$G(x_1, x_n, x_n) \le \mu \max\{G(x_1, x_r, x_r) : 0 \le r \le n-1\} < G(x_1, x_n, x_n).$$

Hence $O_f(x_0)$ is bounded. □

Proof of Theorem 3.1: We prove the results in the next sections.

STEP 1 – EXISTENCE OF THE INFIMUM

Define $S = \{G(x, fx, fx) : x \in X\}$. Note that S is a nonempty set of nonnegative numbers which is bounded below. Hence by the well-known infimum property of real numbers, $\inf S = a$ exists.

STEP 2 – VANISHING INFIMUM

We establish that $a = 0$. If possible, suppose that $a > 0$. Now $G(fx, f^2x, f^2x) \in S$. Writing $y = z = fx$ in (11), we have

$$\begin{aligned} G(fx, f^2x, f^2x) &\leq aG(x, fx, fx) + bG(x, fx, fx) \\ &\quad + cG(fx, f^2x, f^2x) + dG(fx, f^2x, f^2x) \\ &\quad + e\max\big\{G(x, f^2x, f^2x), G(fx, fx, fx), G(fx, f^2x, f^2x), \\ &\qquad G(fx, f^2x, f^2x), G(fx, fx, fx), G(x, f^2x, f^2x)\big\} \\ &\leq (a + b)G(x, fx, fx) + (c + d)G(fx, f^2x, f^2x) \\ &\quad + e\max\{G(x, fx, fx), G(fx, f^2x, f^2x)\} \\ &\leq (a + b)G(x, fx, fx) + (c + d)G(fx, f^2x, f^2x) \\ &\quad + e[G(x, fx, fx) + G(fx, f^2x, f^2x)] \\ &= (a + b + e)G(x, fx, fx) + (c + d + e)G(fx, f^2x, f^2x) \end{aligned}$$

or

$$(1 - c - d - e)G(fx, f^2x, f^2x) \leq (a + b + e)G(x, fx, fx)8$$

so that

$$G(fx, f^2x, f^2x) \leq \left(\tfrac{a+b+e}{1-c-d-e}\right) G(x, fx, fx). \tag{18}$$

Since $\frac{a+b+e}{1-(c+d+e)} < 1$, it would follow from (18), that $G(fx, f^2x, f^2x) < a$ where $G(fx, f^2x, f^2x) \in S$. In other words, a cannot be a lower bound of S, which is a contradiction. Therefore, $a = \inf S = 0$.

STEP 3 – EXISTENCE OF A SEQUENCE

By an elementary theorem on limits related to the infimum, we can choose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$G(x_n, fx_n, fx_n) \in S \text{ for all } n = 1, 2, 3, \ldots \text{ and } \lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{19}$$

STEP 4 – $\langle x_n \rangle_{n=1}^{\infty}$ IS G-CAUCHY

In fact, by the rectangle inequality of G and (2), we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ &\leq G(x_n, fx_n, fx_n) + [G(fx_n, fx_m, fx_m) + G(fx_m, x_m, x_m)] \\ &\leq G(x_n, fx_n, fx_n) \\ &\qquad + G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m). \end{aligned} \tag{20}$$

Now, with $x = x_n$ and $y = z = x_m$, (11) gives,

$$\begin{aligned} G(fx_n, fx_m, fx_m) &\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n) + cG(x_m, fx_m, fx_m) \\ &\quad + dG(x_m, fx_m, fx_m) + e\max\big\{G(x_n, fx_m, fx_m), \\ &\quad G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m), G(x_m, fx_m, fx_m), \\ &\qquad G(x_m, fx_n, fx_n), G(x_n, fx_m, fx_m)\big\} \\ &\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n) + (c+d)G(x_m, fx_m, fx_m) \\ &\quad + e\max\{G(x_n, fx_m, fx_m), G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m)\} \\ &\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n) + (c+d)G(x_m, fx_m, fx_m) \\ &\quad + e\max\{[G(x_n, x_m, x_m) + G(x_m, fx_m, fx_m)], [G(x_m, x_n, x_n) \\ &\qquad + G(x_n, fx_n, fx_n)], G(x_m, fx_m, fx_m)\} \\ &\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n) + (c+d)G(x_m, fx_m, fx_m) \\ &\quad + e\max\{[G(x_n, x_m, x_m) + G(x_m, fx_m, fx_m)], [2G(x_n, x_m, x_m) \\ &\qquad\qquad + G(x_n, fx_n, fx_n)]\} \\ &\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n) + (c+d)G(x_m, fx_m, fx_m) \\ &\quad + e[3G(x_n, x_m, x_m) + G(x_m, fx_m, fx_m) + G(x_n, fx_n, fx_n)] \\ &= (a+3e)G(x_n, x_m, x_m) + (b+e)G(x_n, fx_n, fx_n) \\ &\qquad + (c+d+e)G(x_m, fx_m, fx_m). \end{aligned}$$

Inserting this in (20), we get

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + (a+3e)G(x_n, x_m, x_m) \\ &\quad + (b+e)G(x_n, fx_n, fx_n) + (c+d+e)G(x_m, fx_m, fx_m) \\ &\qquad + 2G(x_m, fx_m, fx_m) \end{aligned}$$

or

$$(1-a-3e)G(x_n, x_m, x_m) \leq (1+b+e)G(x_n, fx_n, fx_n)$$

$$+ (2+c+d+e)G(x_m, fx_m, fx_m)$$

so that

$$G(x_n, x_m, x_m) \le \left(\tfrac{1+b+e}{1-a-3e}\right) G(x_n, fx_n, fx_n) + \left(\tfrac{2+c+d+e}{1-a-3e}\right) G(x_m, fx_m, fx_m).$$

Applying the limit as $m, n \to \infty$ in this and using (19), we obtain that $\langle x_n \rangle_{n=1}^{\infty}$ is a G-Cauchy sequence in X.

STEP 5 – *G*-CONVERGENCE

Since, X is G-complete, we find the point p in X such that

$$\lim_{n\to\infty} x_n = p. \tag{21}$$

STEP 6 – *G*-CONVERGENT LIMIT IS A FIXED POINT

Again repeatedly using (G5),

$$\begin{aligned} G(p, fp, fp) &\le G(p, fx_n, fx_n) + G(fx_n, fp, fp). \\ &\le [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp). \end{aligned} \tag{22}$$

Now, from (11) with $x = x_n$ and $y = z = p$, it follows that

$$\begin{aligned} G(fx_n, fp, fp) &\le aG(x_n, p, p) + bG(x_n, fx_n, fx_n) + cG(p, fp, fp) \\ &\quad + dG(p, fp, fp) + e \max\big\{G(x_n, fp, fp), G(p, fx_n, fx_n), \\ &\quad G(p, fp, fp), G(p, fp, fp), G(p, fx_n, fx_n), G(x_n, fp, fp)\big\} \\ &\le aG(x_n, p, p) + bG(x_n, fx_n, fx_n) + (c+d)G(p, fp, fp) \\ &\quad + e \max\big\{G(x_n, fp, fp), G(p, fp, fp), G(p, fx_n, fx_n)\big\} \\ &\le aG(x_n, p, p) + bG(x_n, fx_n, fx_n) + (c+d)G(p, fp, fp) \\ &\quad + e \max\big\{[G(x_n, p, p) + G(p, fp, fp)], G(p, fp, fp), \\ &\qquad [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)]\big\} \\ &\le aG(x_n, p, p) + bG(x_n, fx_n, fx_n) + (c+d)G(p, fp, fp) \\ &\quad + e \max\big\{[G(x_n, p, p) + G(p, fp, fp)], \\ &\qquad [2G(x_n, p, p) + G(x_n, fx_n, fx_n)]\big\} \\ &\le aG(x_n, p, p) + bG(x_n, fx_n, fx_n) + (c+d)G(p, fp, fp) \\ &\quad + e[3G(x_n, p, p) + G(p, fp, fp) + G(x_n, fx_n, fx_n)], \end{aligned}$$

$$= (a+3e)G(x_n,p,p) + (b+e)G(x_n,fx_n,fx_n) + (c+d+e)G(p,fp,fp). \quad (23)$$

Substituting (23) in (22) and then using (G5),

$$\begin{aligned} G(p,fp,fp) &\le [G(p,x_n,x_n) + G(x_n,fx_n,fx_n)] + (a+3e)G(x_n,p,p) \\ &\quad + (b+e)G(x_n,fx_n,fx_n) + (c+d+e)G(p,fp,fp) \\ &\le (2+a+3e)G(x_n,p,p) + (1+b+e)G(x_n,fx_n,fx_n) \\ &\quad + (c+d+e)G(p,fp,fp) \end{aligned}$$

or

$$(1-c-d-e)G(p,fp,fp) \le (2+a+3e)G(x_n,p,p) + (1+b+e)G(x_n,fx_n,fx_n)$$

or

$$G(p,fp,fp) \le \left(\tfrac{2+a+3e}{1-c-d-e}\right) G(x_n,p,p) + \left(\tfrac{1+b+e}{1-c-d-e}\right) G(x_n,fx_n,fx_n)$$

In the limiting case as $n \to \infty$, this in view of (19), (21) and Lemma 2.1, implies $G(p,fp,fp) = 0$ or $fp = p$. Thus p is a fixed point.

STEP 7 – UNIQUENESS OF THE FIXED POINT

Let q be another fixed point of f so that $fq = q$. Then writing $x = p$ and $y = z = q$ in (11),

$$\begin{aligned} G(p,q,q) &= G(fp,fq,fq) \\ &\le aG(p,q,q) + bG(p,fp,fp) + cG(q,fq,fq) + dG(q,fq,fq) \\ &\quad + e\max\{G(p,fq,fq), G(q,fp,fp), G(q,fq,fq), \\ &\qquad G(q,fq,fq), G(q,fp,fp), G(p,fq,fq)\} \\ &\le aG(p,q,q) + 2eG(p,q,q) \\ &= (a+2e)G(p,q,q) \end{aligned}$$

or $(1-a-2e)G(p,q,q) \le 0$ so that $p = q$. Thus p is a unique fixed point of f. □

Employing this technique, the following theorems can be proved in a similar way. So, the proofs are omitted here. It may be noted that the unique fixed point becomes a G-contractive fixed point in each case.

Theorem 3.2 (Mohanta [3], Theorem 3.4). *Let (X,G) be a complete G-metric space and f, a self-map on X satisfying*

$$G(fx,fy,fz) \le a[G(x,fy,fy) + G(y,fx,fx)] + b[G(y,fz,fz) + G(z,fy,fy)]$$

$$+ c[G(x, fz, fz) + G(z, fx, fx)] + dG(x, y, z)$$
$$+ e \max \{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)$$
$$G(x, fz, fz)\} \text{ for all } x, y, z \in X, \tag{24}$$

where $a.b, c, d, e \geq 0$, not all zero, with $2a+2b+2c+d+2e < 1$. Then f has a unique fixed point.

With $d = e = 0$, Theorem 3.2 reduces to

Theorem 3.3 (Mustafa and Sims [7], Theorem 2.4). *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max \{[G(x, fy, fy) + G(y, fx, fx)], [G(y, fz, fz) + G(z, fy, fy)],$$
$$[G(x, fz, fz) + G(z, fx, fx)]\} \text{ for all } x, y, z \in X, \tag{25}$$

where $0 < k < 1.2\cdot$ Then f will have a unique fixed point.

Theorem 3.4 (Mustafa, [4]). *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max \{[G(z, fx, fx) + G(y, fx, fx)], [G(y, fz, fz) + G(x, fz, fz)],$$
$$[G(x, fy, fy) + G(z, fy, fy)]\} \text{ for all } x, y, z \in X, \tag{26}$$

where $0 < k < 1/3\cdot$ Then f will have a unique fixed point.

Theorem 3.5 (Mohanta [3], Theorem 3.9). *Let (X, G) be a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max \{[G(x, fy, fy) + G(y, fx, fx) + G(z, fz, fz)],$$
$$[G(y, fz, fz) + G(z, fy, fy) + G(x, fx, fx)],$$
$$[G(z, fx, fx) + G(x, fz, fz) + G(y, fy, fy)]\}$$
$$\text{for all } x, y, z \in X, \tag{27}$$

where $0 \leq k < 1/3\cdot$ Then f has a unique fixed point.

Theorem 3.6 (Mohanta [3], Theorem 3.7). *Let (X, G) be a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max \{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), G(x, fy, fy),$$
$$G(y, fz, fz), G(z, fx, fx), G(x, fz, fz), G(y, fx, fx),$$
$$G(z, fy, fy), G(x, fy, fz), G(y, fz, fx), G(z, fx, fy),$$
$$G(x, y, fz), G(y, z, fx), G(z, x, fy), G(x, y, z)\}$$

$$for\ all\ x, y, z \in X, \tag{28}$$

where $0 \leq k < 1/3\cdot$ *Then* f *has a unique fixed point.*

Theorem 3.7 (Vats et al, [13]). *Suppose that* (X, G) *is a complete G-metric space and f, a self-map on* X *satisfying the condition*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{&G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz), \\ &G(x, fy, fy) + G(y, fx, fx) + G(z, fy, fy), \\ &G(x, fz, fz) + G(y, fz, fz) + G(z, fx, fx)\big\} \\ &for\ all\ x, y, z \in X, \end{aligned} \tag{29}$$

where $0 \leq k < 1/4\cdot$ *Then f will have a unique fixed point.*

Theorem 3.8 (Vats et al, [13]). *Suppose that* (X, G) *is a complete G-metric space and f, a self-map on* X *satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{&G(x, fx, fx), G(x, fy, fy), \\ &G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), \\ &G(z, fy, fy)\big\}\ for\ all\ x, y, z \in X, \end{aligned} \tag{30}$$

where $0 \leq k < 1/2\cdot$ *Then f will have a unique fixed point.*

Theorem 3.9 (Mustafa and Sims [7], Theorem 2.1). *Suppose that* (X, G) *is a complete G-metric space and f, a self-map on* X *satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{&G(x, y, z), G(x, fx, fx), G(x, fy, fy), G(z, fx, fx), \\ &G(y, fy, fy), G(z, fz, fz), G(y, fz, fz)\big\}\ for\ all\ x, y, z \in X, \end{aligned} \tag{31}$$

where $0 < k < 1/2\cdot$ *Then* f *will have a unique fixed point.*

Theorem 3.10 (Mustafa et al, [5]). *Let* (X, G) *be a complete G-metric space and* $f : X \to X$ *satisfying one of the following conditions:*

$$\begin{aligned} G(fx, fy, fz) \leq &k \max\{G(x, fy, fy), G(x, fz, fz), G(y, fx, fx) \\ &G(y, fz, fz), G(z, fx, fx), G(z, fy, fy)\} \end{aligned} \tag{32}$$

or

$$\begin{aligned} G(fx, fy, fz) \leq &k \max\{G(x, x, fy), G(x, x, fz), G(y, y, fx) \\ &G(y, y, fz), G(z, z, fx), G(z, z, fy)\} \end{aligned} \tag{33}$$

for all $x, y, z \in X$, *where* $0 \leq k < 1$. *Then* f *has a unique fixed point.*

4. CONCLUSION

The method of using the greatest lower bound property is more analytical and does not depend on the choice of the initial iterations of f. It is so simple that even a student of analysis in undergraduate level can understand without tears. It is always interesting to carry out an extensive research on G-contractive fixed points.

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