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GENERALIZATION OF SOME HADAMARD PRODUCT

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ABSTRACT: In this paper, we introduce a new operator in order to derive some new symmetric properties of Hadamard product.

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1. NOTATIONS AND MAIN RESULTS

Let k and n be two positive integer and $\{x_1, x_2, ..., x_n\}$ are set of given variables, recall [6] that the k-th elementary symmetric function $e_k(x_1, x_2, ..., x_n)$ and the k-th complete homogeneous symmetric function $h_k(x_1, x_2, ..., x_n)$ are defined respectively by

$$e_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n = 0$ or 1.

$$h_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n \geq 0$.

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First, we set $e_0(x_1, x_2, ..., x_n) = 1$ and $h_0(x_1, x_2, ..., x_n) = 1$ (by convention). For $k > n$ or $k < 0$, we set $e_k(x_1, x_2, ..., x_n) = 0$ and $h_k(x_1, x_2, ..., x_n) = 0$.

Remark 1. Let $B = \{b_1, b_2, ..., b_n\}$ an alphabet, we have

$$h_k(b_1, b_2, ..., b_n) = S_k(b_1 + b_2 + ... + b_n)$$

Define the Hadamard product of two entire series or two functions analytic at the origin, a and b, as their termwise product,

$$a(z) \odot b(z) = \sum_{n\geq 0} a_n b_n z^n, \text{ if } a(z) = \sum_{n\geq 0} a_n z^n, \ b(z) = \sum_{n\geq 0} b_n z^n.$$

Definition 1. [1] Let A and E be any two alphabets, then we give $S_n(A - E)$ by the following form:

$$\frac{\Pi_{e\epsilon E}(1 - ze)}{\Pi_{a\epsilon A}(1 - za)} = \sum_{j=0}^{\infty} S_n(A - E)z^j, \tag{1.1}$$

with the condition $S_n(A - E) = 0$ for $n < 0$.

Definition 2. [2] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_i x_{i+1}}(f) = \frac{f(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - f(x_1, \cdots x_{i-1}, x_{i+1,} x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}.$$

Definition 3. [4] The symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{e_1 e_2}(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } k \in \mathbb{N}. \tag{1.2}$$

Proposition 1. *[3] Let $E = \{e_1, e_2\}$ an alphabet, we define the operator $\delta^k_{e_1 e_2}$ as follows:*

$$\delta^k_{e_1 e_2} f\ (e_1) = h^{(2)}_{k-1}(e_1, e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f\ (e_1), \ \textit{for all } k \in \mathbb{N}.$$

In our main result, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following theorem.

Theorem 1. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, ..., a_n\}$ then*

$$\sum_{n=0}^{\infty} h^{(n)}_n(a_1, a_2, ..., a_n) h^{(2)}_{k+n-1}(e_1, e_2) z^n$$
$$= \frac{\sum\limits_{n=0}^{k-1} S_n(-A) e_1^n e_2^n h^{(2)}_{k-n-1}(e_1, e_2) z^n - (e_1 e_2 z)^k \sum\limits_{n=0}^{\infty} S_{n+k+1}(-A) h^{(2)}_n(e_1, e_2) z^{n+1}}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)(e_1 z)^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)(e_2 z)^n\right)}. \tag{1.3}$$

Proof. Let $\sum_{n=0}^{\infty} S_n(A)z^n$ and $\sum_{n=0}^{\infty} S_n(-A)z^n$ be two sequences as $\sum\limits_{n=0}^{\infty} S_n(A)z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)z^n = 1$.

On one hand, since $f(e_1) = \sum\limits_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, ..., a_n)e_1^n z^n$, we have

$$\begin{aligned}
\delta_{e_1 e_2}^k f\ (e_1) &= \delta_{e_1 e_2}^k \left(\sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, ..., a_n)e_1^n z^n \right) \\
&= \sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, ..., a_n)\ h_{k+n-1}^{(2)}(e_1, e_2)z^n,
\end{aligned}$$

which is the left hand side of (1.3). On the other hand, since

$$f(e_1) = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n},$$

we have that

$$\begin{aligned}
\partial_{e_1 e_2} f(e_1) &= \frac{1}{e_1 - e_2} \left(\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n} - \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n} \right) \\
&= \frac{1}{e_1 - e_2} \left(\frac{\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n - \sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)} \right) \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-A)\frac{e_2^n - e_1^n}{e_1 - e_2} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)} \\
&= -\frac{\sum\limits_{n=0}^{\infty} b_n h_{n-1}^{(2)}(e_1, e_2)z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}.
\end{aligned}$$

By Proposition 1, it follows that

$$\begin{aligned}
\delta_{e_1 e_2}^k f\ (e_1) &= h_{k-1}^{(2)}(e_1, e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f\ (e_1) \\
&= \frac{h_{k-1}^{(2)}(e_1, e_2)}{\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n} - e_2^k \frac{\sum\limits_{n=0}^{\infty} S_n(-A)h_{n-1}^{(2)}(e_1, e_2)z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-A)\left[e_2^n h_{k-1}^{(2)}(e_1, e_2) - e_2^k h_{n-1}^{(2)}(e_1, e_2)\right] z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}.
\end{aligned}$$

Hence, we have that

$$\delta^k_{e_1e_2} f\ (e_1) = \frac{\sum_{n=0}^{k-1} S_n(-A)\left[e_2^n h^{(2)}_{k-1}(e_1+e_2) - e_2^k S_{n-1}(e_1,e_2)\right] z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}$$

$$+ \frac{\sum_{n=k+1}^{\infty} S_n(-A)\left[e_2^n h^{(2)}_{k-1}(e_1+e_2) - e_2^k h^{(2)}_{n-1}(e_1,e_2)\right] z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}$$

$$= \frac{\sum_{n=0}^{k-1} S_n(-A)e_1^n e_2^n) h^{(2)}_{k-n-1}(e_1,e_2)z^n - (e_1e_2z)^k \sum_{n=0}^{\infty} S_{n+k+1}(-A)h^{(2)}_n(e_1,e_2)z^{n+1}}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}.$$

This completes the proof. □

If $k = 1$ and $A = \{a_1, a_2\}$, the following Lemma holds:

Lemma 1. *Given two alphabets* $E = \{e_1, e_2\}$ *and* $A = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{\infty} h^{(2)}_n(a_1,a_2)h^{(2)}_n(e_1,e_2)z^n = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}. \tag{1.4}$$

Definition 4. Let $A = \left\{\underbrace{1, 1, ..., 1}_{n}\right\}$, we have

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n+k-1}{k} \tag{1.5}$$

2. THE HADAMARD PRODUCT

In this section, we show the efficiency of the proposed method by determining the Hadamard product. In fact, by taking $E = 0$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} S_n(A)z^n = \frac{1}{\prod_{a\in A}(1-az)}. \tag{2.1}$$

For the special case where $a_1 = a_2 = 1$ in (2.1), we have

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2}, \tag{2.2}$$

which is found in [5].

By replacing z by $e_1 z$ in (2.2), we get

$$\sum_{n=0}^{\infty}(n+1)e_1^n z^n = \frac{1}{(1-e_1 z)^2}. \tag{2.3}$$

Using Theorem 1 with the action of the operator $\delta_{e_1 e_2}$ on both sides of the identity (2.3) one can obtain

$$\sum_{n=0}^{\infty}(n+1)h_n^{(2)}(e_1,e_2)z^n = \frac{1-e_1e_2z^2}{(1-e_1z)^2(1-e_2z)^2}. \tag{2.4}$$

By taking $e_1 = 1$ and $e_2 = 1$, we have

$$\sum_{n=0}^{\infty}(n^2+2n+1)z^n = \frac{1+z}{(1-z)^3}. \tag{2.5}$$

which is found also in [5].

On the other hand, using formula (1.4) with the action of the operator $\delta_{e_1e_2}$ on both sides of (2.5), where replacing z by $e_1 z$ leads to

$$\sum_{n=0}^{\infty}(n+1)^2 h_n^{(2)}(e_1,e_2)z^n = \delta_{e_1e_2}\frac{1}{(1-e_1z)^3} + z\times\delta_{e_1e_2}\frac{e_1}{(1-e_1z)^3}. \tag{2.6}$$

By using formulas (1.2), (1.4) and (1.6),it follows that

$$\delta_{e_1e_2}\frac{1}{(1-e_1z)^3} = \frac{1-e_1e_2z^2\sum\limits_{n=0}^{1}(-1)^{n+2}\,C_3^{n+2}h_n^{(2)}(e_1,e_2)z^n}{(1-e_1z)^3(1-e_2z)^3}, \tag{2.7}$$

$$\delta_{e_1e_2}\frac{e_1}{(1-e_1z)^3} = \frac{\left[\sum\limits_{n=0}^{1}(-1)^n\,C_3^n e_1^n e_2^n h_{1-n}^{(2)}(e_1,e_2)z^n - e_1^2e_2^2z^3\sum\limits_{n=0}^{1}(-1)^{n+3}\,C_3^{n+2}h_n^{(2)}(e_1,e_2)z^n\right]}{(1-e_1z)^3(1-e_2z)^3}. \tag{2.8}$$

Notice that, for $e_1 = 1$ and $e_2 = 1$, we have

$$\sum_{n=0}^{\infty}(n^3+3n^2+3n+1)z^n = \frac{1+4z+z^2}{(1-z)^4}, \tag{6.9}$$

Using the same procedure, we obtain the following new generatings functions:

$$\sum_{n=0}^{\infty}(n^2+2n+1)^2 z^n = \frac{1+11z+11z^2+z^3}{(1-z)^5},$$

$$\sum_{n=0}^{\infty}(n^2+2n+1)^2(n+1)z^n = \frac{1+26z+66z^2+26z^3+z^4}{(1-z)^6},$$

$$\sum_{n=0}^{\infty}(n^3+3n^2+3n+1)^2z^n = \frac{1+57z+302z^2+302z^3+57z^4+z^5}{(1-z)^7},$$

$$\sum_{n=0}^{\infty}(n^3+3n^2+3n+1)^2(n+1)z^n$$
$$= \frac{1+120z+1191z^2+2416z^3+1191z^4+120z^{5+}z^6}{(1-z)^8},$$

$$\sum_{n=0}^{\infty}(n^3+3n^2+3n+1)^2(n+1)^2z^n$$
$$= \frac{1+247z+4293z^2+15619z^3+15619z^4+4293z^5+247z^6+z^7}{(1-z)^9}.$$

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