GENERALIZATION OF SOME HADAMARD PRODUCT

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ABSTRACT: In this paper, we introduce a new operator in order to derive some new symmetric properties of Hadamard product.

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1. NOTATIONS AND MAIN RESULTS

Let $k$ and $n$ be two positive integer and \{\textit{x}_{1}, \textit{x}_{2}, ..., \textit{x}_{n}\} are set of given variables, recall [6] that the $k$-th elementary symmetric function $e_{k}(\textit{x}_{1}, \textit{x}_{2}, ..., \textit{x}_{n})$ and the $k$-th complete homogeneous symmetric function $h_{k}(\textit{x}_{1}, \textit{x}_{2}, ..., \textit{x}_{n})$ are defined respectively by

$$
e_{k}(\textit{x}_{1}, \textit{x}_{2}, ..., \textit{x}_{n}) = \sum_{i_{1}+i_{2}+...+i_{n}=k} \textit{x}_{1}^{i_{1}}\textit{x}_{2}^{i_{2}}...\textit{x}_{n}^{i_{n}}, \quad 0 \leq k \leq n,$$

with $i_{1}, i_{2}, ..., i_{n} = 0$ or $1$.

$$
h_{k}(\textit{x}_{1}, \textit{x}_{2}, ..., \textit{x}_{n}) = \sum_{i_{1}+i_{2}+...+i_{n}=k} \textit{x}_{1}^{i_{1}}\textit{x}_{2}^{i_{2}}...\textit{x}_{n}^{i_{n}}, \quad 0 \leq k \leq n,$$

with $i_{1}, i_{2}, ..., i_{n} \geq 0$.
First, we set \(e_0(x_1, x_2, \ldots, x_n) = 1\) and \(h_0(x_1, x_2, \ldots, x_n) = 1\) (by convention). For \(k > n\) or \(k < 0\), we set \(e_k(x_1, x_2, \ldots, x_n) = 0\) and \(h_k(x_1, x_2, \ldots, x_n) = 0\).

**Remark 1.** Let \(B = \{b_1, b_2, \ldots, b_n\}\) an alphabet, we have

\[h_k(b_1, b_2, \ldots, b_n) = S_k(b_1 + b_2 + \ldots + b_n)\]

Define the Hadamard product of two entire series or two functions analytic at the origin, a and b, as their termwise product,

\[a(z) \odot b(z) = \sum_{n \geq 0} a_n b_n z^n, \text{ if } a(z) = \sum_{n \geq 0} a_n z^n, \text{ b}(z) = \sum_{n \geq 0} b_n z^n.\]

**Definition 1.** [1] Let \(A\) and \(E\) be any two alphabets, then we give \(S_n(A - E)\) by the following form:

\[
\frac{\Pi_{e\in E}(1 - ze)}{\Pi_{a\in A}(1 - za)} = \sum_{j=0}^{\infty} S_n(A - E)z^j, \tag{1.1}
\]

with the condition \(S_n(A - E) = 0\) for \(n < 0\).

**Definition 2.** [2] Given a function \(f\) on \(\mathbb{R}^n\), the divided difference operator is defined as follows:

\[
\partial_{x_ix_i+1}(f) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2} \ldots x_n)}{x_i - x_{i+1}}.
\]

**Definition 3.** [4] The symmetrizing operator \(\delta_{e_1e_2}^k\) is defined by

\[
\delta_{e_1e_2}^k(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } k \in \mathbb{N}. \tag{1.2}
\]

**Proposition 1.** [3] Let \(E = \{e_1, e_2\}\) an alphabet, we define the operator \(\delta_{e_1e_2}^k\) as follows:

\[
\delta_{e_1e_2}^k f(e_1) = h_{k-1}^{(2)}(e_1, e_2) f(e_1) + e_2^k \partial_{e_1e_2} f(e_1), \text{ for all } k \in \mathbb{N}.
\]

In our main result, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following theorem.

**Theorem 1.** Given two alphabets \(E = \{e_1, e_2\}\) and \(A = \{a_1, a_2, \ldots, a_n\}\) then

\[
\sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, \ldots, a_n) h_{k+n-1}^{(2)}(e_1, e_2) z^n = \sum_{n=0}^{k-1} S_n(-A) e_1^n e_2^n h_{k+n-1}^{(2)}(e_1, e_2) z^n - (e_1 e_2 z)^k \sum_{n=0}^{\infty} S_{n+k+1}(-A) h_{n}^{(2)}(e_1, e_2) z^{n+1} \left(\sum_{n=0}^{\infty} S_n(-A)(e_1 z)^n\right) \left(\sum_{n=0}^{\infty} S_n(-A)(e_2 z)^n\right). \tag{1.3}
\]
Proof. Let $\sum_{n=0}^{\infty} S_n(A) z^n$ and $\sum_{n=0}^{\infty} S_n(-A) z^n$ be two sequences as $\sum_{n=0}^{\infty} S_n(A) z^n \times \sum_{n=0}^{\infty} S_n(-A) z^n = 1$.

On one hand, since $f(e_1) = \sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, \ldots, a_n) e_1^n z^n$, we have

$$
\delta_{e_1 e_2}^k f(e_1) = \delta_{e_1 e_2}^k \left( \sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, \ldots, a_n) e_1^n z^n \right) = \sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, \ldots, a_n) h_{k+n-1}^{(2)}(e_1, e_2) z^n,
$$

which is the left hand side of (1.3). On the other hand, since

$$
f(e_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n},
$$

we have that

$$
\delta_{e_1 e_2} f(e_1) = \frac{1}{e_1 - e_2} \left( \frac{1}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n} \frac{1}{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n} \right) \sum_{n=0}^{\infty} S_n(-A) \frac{e_2^n - e_1^n}{e_1 - e_2} z^n

= \frac{1}{e_1 - e_2} \left( \frac{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n} \frac{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n} \right) \sum_{n=0}^{\infty} S_n(-A) \left( e_2^n h_{n-1}^{(2)}(e_1, e_2) \right) z^n

= \frac{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n} \frac{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n} \sum_{n=0}^{\infty} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1, e_2) - e_2^n h_{n-1}^{(2)}(e_1, e_2) \right] z^n

= \frac{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n} \frac{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n}{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n} \sum_{n=0}^{\infty} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1, e_2) - e_2^n h_{n-1}^{(2)}(e_1, e_2) \right] z^n.
$$

By Proposition 1, it follows that

$$
\delta_{e_1 e_2}^k f(e_1) = h_{k-1}^{(2)}(e_1, e_2) f(e_1) + e_2^k \delta_{e_1 e_2} f(e_1)

= h_{k-1}^{(2)}(e_1, e_2) - e_2^k \sum_{n=0}^{\infty} S_n(-A) h_{n-1}^{(2)}(e_1, e_2) z^n

= \sum_{n=0}^{\infty} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1, e_2) - e_2^n h_{n-1}^{(2)}(e_1, e_2) \right] z^n

= \left( \sum_{n=0}^{\infty} S_n(-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-A) e_2^n z^n \right) \sum_{n=0}^{\infty} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1, e_2) - e_2^n h_{n-1}^{(2)}(e_1, e_2) \right] z^n
$$
Hence, we have that

\[ \delta_{e_1 e_2}^k f (e_1) = \sum_{n=0}^{k-1} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1 + e_2) - e_2^k h_{n-1}^{(2)}(e_1, e_2) \right] z^n \]

\[ + \sum_{n=k+1}^{\infty} S_n(-A) \left[ e_2^n h_{k-1}^{(2)}(e_1 + e_2) - e_2^k h_{n-1}^{(2)}(e_1, e_2) \right] z^n \]

\[ = \sum_{n=0}^{k-1} S_n(-A) e_1^n z^n + \sum_{n=k+1}^{\infty} S_n(-A) e_2^n z^n \]

This completes the proof.

If \( k = 1 \) and \( A = \{a_1, a_2\} \), the following Lemma holds:

**Lemma 1.** Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1, a_2\} \), then

\[ \sum_{n=0}^{\infty} h_n^{(2)}(a_1, a_2) h_n^{(2)}(e_1, e_2) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left( \sum_{n=0}^{\infty} S_n(-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-A) e_2^n z^n \right)}. \] (1.4)

**Definition 4.** Let \( A = \left\{ 1, 1, \ldots, 1 \right\} \), we have

\[ S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n + k - 1}{k} \] (1.5)

**2. THE HADAMARD PRODUCT**

In this section, we show the efficiency of the proposed method by determining the Hadamard product. In fact, by taking \( E = 0 \) in (1.1), we obtain

\[ \sum_{n=0}^{\infty} S_n(A) z^n = \frac{1}{\prod_{a \in A} (1 - a z)}. \] (2.1)

For the special case where \( a_1 = a_2 = 1 \) in (2.1), we have

\[ \sum_{n=0}^{\infty} (n + 1) z^n = \frac{1}{(1 - z)^2}, \] (2.2)
which is found in [5].

By replacing \(z\) by \(e_1z\) in (2.2), we get

\[
\sum_{n=0}^{\infty} (n + 1)e_1^n z^n = \frac{1}{(1 - e_1z)^2}.
\] (2.3)

Using Theorem 1 with the action of the operator \(\delta_{e_1e_2}\) on both sides of the identity (2.3) one can obtain

\[
\sum_{n=0}^{\infty} (n + 1)h^{(2)}_n(e_1, e_2)z^n = \frac{1 - e_1e_2z^2}{(1 - e_1z)^2(1 - e_2z)^2}.
\] (2.4)

By taking \(e_1 = 1\) and \(e_2 = 1\), we have

\[
\sum_{n=0}^{\infty} (n^2 + 2n + 1)z^n = \frac{1 + z}{(1 - z)^3}.
\] (2.5)

On the other hand, using formula (1.4) with the action of the operator \(\delta_{e_1e_2}\) on both sides of (2.5), where replacing \(z\) by \(e_1z\) leads to

\[
\sum_{n=0}^{\infty} (n + 1)^2h^{(2)}_n(e_1, e_2)z^n = \delta_{e_1e_2} \frac{1}{(1 - e_1z)^3} + z \times \delta_{e_1e_2} \frac{e_1}{(1 - e_1z)^3}.
\] (2.6)

By using formulas (1.2), (1.4) and (1.6), it follows that

\[
\delta_{e_1e_2} \frac{e_1}{(1 - e_1z)^3} = \frac{1 - e_1e_2z^2}{(1 - e_1z)^3(1 - e_2z)^3} \sum_{n=0}^{\infty} \left( -1 \right)^{n+2} C_3^{n+2} h^{(2)}_n(e_1, e_2)z^n.
\] (2.7)

\[
\delta_{e_1e_2} \frac{e_1}{(1 - e_1z)^3} = \frac{1 - e_1e_2z^2}{(1 - e_1z)^3(1 - e_2z)^3} \sum_{n=0}^{\infty} \left( -1 \right)^{n+3} C_3^{n+3} h^{(2)}_n(e_1, e_2)z^n.
\] (2.8)

Notice that, for \(e_1 = 1\) and \(e_2 = 1\), we have

\[
\sum_{n=0}^{\infty} (n^3 + 3n^2 + 3n + 1)z^n = \frac{1 + 4z + z^2}{(1 - z)^4}.
\] (6.9)

Using the same procedure, we obtain the following new generating functions:

\[
\sum_{n=0}^{\infty} (n^2 + 2n + 1)^2z^n = \frac{1 + 11z + 11z^2 + z^3}{(1 - z)^5},
\]
\[
\sum_{n=0}^{\infty} (n^2 + 2n + 1)^2 (n+1)z^n = \frac{1 + 26z + 66z^2 + 26z^3 + z^4}{(1 - z)^6},
\]
\[
\sum_{n=0}^{\infty} (n^3 + 3n^2 + 3n + 1)^2 z^n = \frac{1 + 57z + 302z^2 + 302z^3 + 57z^4 + z^5}{(1 - z)^7},
\]
\[
\sum_{n=0}^{\infty} (n^3 + 3n^2 + 3n + 1)^2 (n+1)z^n = \frac{1 + 120z + 1191z^2 + 2416z^3 + 1191z^4 + 120z^5 + z^6}{(1 - z)^8},
\]
\[
\sum_{n=0}^{\infty} (n^3 + 3n^2 + 3n + 1)^2 (n+1)^2 z^n = \frac{1 + 247z + 4293z^2 + 15619z^3 + 15619z^4 + 4293z^5 + 247z^6 + z^7}{(1 - z)^9}.
\]

REFERENCES


