Communications in Applied Analysis **20** (2016), 307-315

FUZZY β-SUBALGEBRAIC TOPOLOGICAL SPACES

M. CHANDRAMOULEESWARAN[1], P. MURALIKRISHNA[2], and K. SUJATHA[3]

[1,3]Department of Mathematics
Saiva Bhanu Kshatriya College
Aruppukottai, 626101, Tamilnadu, INDIA
[2]PG & Research Department of Mathematics
Muthurangam Govt. Arts College (Autonomous)
Vellore, 612 001, Tamilnadu, INDIA

ABSTRACT: In this paper, we define the notion of fuzzy β-subalgebraic topological space and investigate some of their properties and results.

AMS Subject Classification: 08A72, 03E72

1. INTRODUCTION

Fuzzy set was developed by Lofti.A Zadeh [6]. After that, many researchers used the fuzzy set in many directions and applications in various areas of sciences. Chang [4] made the notion of a fuzzy set on to general topology. Recently in 2014, Annalakshmi et.al [1], introduced the concept of fuzzy topological subsystem on a TM-algebra.

The concept of β- algebra is introduced in 2002 by J.Neggers and H.S.Kim [5]. Aub Ayub Anasri et.al [2], applied the idea of fuzzy set in β- algebra and developed the new notion, called fuzzy β- subalgebras of β- algebra in 2013.

This paper dealt the idea of fuzzy β-subalgebraic topological space on β-algebras, by connecting the two concepts fuzzy β-algebras and fuzzy topological spaces and proved some of their properties.

Received: September 18, 2016

2. PRELIMINARIES

In this section some basic definitions which are required are recalled in the sequel.

Definition 2.1. A β-algebra is a non-empty set X with a constant 0 and two binary operations + and − satisfying the following axioms:

1. $x - 0 = x$,
2. $(0 - x) + x = 0$,
3. $(x - y) - z = x - (z + y)$, $\forall\ x, y, z\ \in\ X$.

Definition 2.2. Let X be a non empty set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. For each element $x \in X$, $\mu(x)$ such that $0 \leq \mu(x) \leq 1$ is called the membership value of x in X.

Definition 2.3. If μ_1 and μ_2 are two fuzzy sets of X, then the intersection of μ_1 and μ_2, denoted by, $\mu_1 \cap \mu_2$ is defined by,

$$(\mu_1 \cap \mu_2)(x) = min\left\{\mu_1(x), \mu_2(x)\right\},\ \forall\ x \in X.$$

Definition 2.4. If μ_1 and μ_2 are two fuzzy sets of X, then the union of μ_1 and μ_2, denoted by, $\mu_1 \cup \mu_2$, is defined by

$$(\mu_1 \cup \mu_2)(x) = max\left\{\mu_1(x), \mu_2(x)\right\},\ \forall\ x \in X.$$

Definition 2.5. Letμ_1 and μ_2 be two fuzzy sets of X. μ_1 is said to be a subset of μ_2, denoted by $\mu_1 \subseteq \mu_2$, if $\mu_1(x) \leq \mu_2(x)$, $\forall\ x \in X$.

Definition 2.6. Let f be a function from X to Y and μ be a fuzzy set in X. The image of μ is defined as

$$\mu(y) = \begin{cases} \sup\limits_{x \in\ f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Let λ be a fuzzy set in Y. The inverse of function, f^{-1} is defined as $\lambda_{f^{-1}}(x) = \lambda(f(x))$ for all $x \in X$

Definition 2.7. A non empty subset A of a β-algebra (X,+,-,0) is called a β-subalgebra of X, if both $x + y$ & $x - y \in A$, $\forall x, y \in A$.

Definition 2.8. Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-subalgebra of X if for all $x, y \in X$

1. $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,

2. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$.

Example 2.9. Let $(X, +, -, 0)$ be a β-algebra with the Cayley table.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	a	0
c	c	b	0	a

-	0	a	b	c
0	0	a	c	b
a	a	0	b	c
b	b	c	0	a
c	c	b	a	0

Define the fuzzy set $\mu : X \to [0,1]$ such that

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, \\ 0 & \text{if } x = b, c. \end{cases}$$

then μ is a fuzzy β-subalgebra in X.

Definition 2.10. Let X be any set. Then a fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions

1. $\overline{0}$ and $\overline{1} \in \tau$ where $\overline{0} = \mu(x) = 0$ and $\overline{1} = \mu(x) = 1$,

2. If $\mu_A, \mu_B \in \tau$ then $\mu_{(A \cap B)} \in \tau$,

3. If $\mu_{A_i} \in T$ for each $i \in I$ then $\cup_I \mu_{A_i} \in \tau$ where I is an indexing set.

Remark 2.11. If X is a set with a fuzzy topology τ then (X, τ) is called a fuzzy topological space and any element in τ is called a τ- open fuzzy set in X.

Definition 2.12. Let (X, τ) be a fuzzy topological space and $\mu \in \tau$. A fuzzy set $\nu \in \tau$ is said to a neighbourhood of μ if there exists a τ-open fuzzy set O such that $\mu \subset O \subset \nu$, i.e, $\mu(x) \leq O(x) \leq \nu(x)$, $\forall x \in X$.

Definition 2.13. Let A and B be fuzzy sets in a fuzzy topological space (X, τ) and $A \supset B$. Then B is called an interior of A if A is a neighbourhood of B. The union of all interior fuzzy sets of A is again a an interior of A and is denoted by A^0

3. FUZZY β-SUBALGEBRAIC TOPOLOGICAL SPACE

This section, introduces the notion of fuzzy β-subalgebraic topological space and discusses some related results.

Definition 3.1. Let $(X, +, -, 0)$ be a β-algebra. Then (X, τ) is said to be a fuzzy β-subalgebraic topological space on X, if there is a family τ of fuzzy β-subalgebras, which satisfies the following conditions:

1. $\overline{0}$ and $\overline{1} \in \tau$ where $\overline{0} = \mu(x) = 0$ and $\overline{1} = \mu(x) = 1$,
2. If $\mu_A, \mu_B \in \tau$ then $\mu_{(A\cap B)} \in \tau$,
3. If $\mu_{A_i} \in T$ for each $i \in I$ then $\cup_I \mu_{A_i} \in \tau$ where I is an indexing set.

Any element in τ is called a τ-open fuzzy set in β-algebra of X.

Example 3.2. Consider the above example 2.9 is β-algebra. Let the fuzzy β-subalgebras, $\mu_i : X \to [0, 1]$,i = 1,2,3,4,5 be given

$$\mu_1(x) = \begin{cases} .8 & \text{if } x = 0, \\ .4 & \text{if } x = a, b, \\ .2 & \text{if } x = c, \end{cases}$$

$$\mu_2(x) = \begin{cases} .9 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, b, \\ 0.2 & \text{if } x = c, \end{cases}$$

$$\mu_3(x) = \begin{cases} 1.0 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, b, \\ 0.2 & \text{if } x = c, \end{cases}$$

$$\mu_4(x) = \begin{cases} .7 & \text{if } x = 0, \\ .4 & \text{if } x = a, b, \\ .1 & \text{if } x = c, \end{cases}$$

$$\mu_5(x) = \begin{cases} .8 & \text{if } x = 0 \\ .4 & \text{if } x = a, b \\ .1 & \text{if } x = c \end{cases}$$

Then the collection $\tau = \left\{\overline{0}, \overline{1}, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\right\}$ is a fuzzy β-subalgebras on X. Hence (X, τ) is a fuzzy β-subalgebraic Topological space on X.

Definition 3.3. Let (X, τ) be a fuzzy β-subalgebraic topological space. Let μ be a fuzzy set in τ. A fuzzy set $\nu \in \tau$ is said to a neighbourhood of μ if there exists a τ-open fuzzy set O such that $\mu \subset O \subset \nu$, ie $\mu(x) \leq O(x) \leq \nu(x)$ for all $x \in X$

Example 3.4. Consider the fuzzy β- subalgebraic topological space as in the example 3.2. μ_3 is a fuzzy neighbourhood of a fuzzy set μ_1, for $\mu_1(x) \leq \mu_2(x) \leq \mu_3(x)$

Definition 3.5. Let A and B be fuzzy sets in a fuzzy β-subalgebraic topological space (X, τ). Let $A \supset B$ Then B is called an interior of A if A is a neighbourhood of B. The union of all interior fuzzy sets of A is again a an interior of A and is denoted by A^0

Example 3.6. Consider the above example 3.2 is a Fuzzy β-subalgebraic Topological space on X.

So, μ_2 is a fuzzy neighbourhood of a fuzzy sets μ_1, μ_4, μ_5.

That is, μ_1, μ_4, μ_5 are fuzzy interiors of μ_2:

$$\begin{aligned} \mu_2^0 &= \cup \{\mu_1, \mu_4, \mu_5\} \\ &= \max \{\mu_1(x), \mu_4(x), \mu_5(x)\} \\ &= \mu_1(x). \end{aligned}$$

Theorem 3.7. *Let (X, τ) be a fuzzy β-subalgebraic topological space on X. A fuzzy β-subalgebra of A is τ-open iff for each fuzzy set B contained in A, A is a fuzzy neighbourhood of B.*

Proof. Suppose a fuzzy β-subalgebra of A is τ-open.

Let B be any fuzzy β-subalgebra contained in A. Since A is open, and $B \subset A$. Therefore A is fuzzy neighbourhood of B.

Conversely, for each fuzzy β-subalgebra B contained in A, A is a fuzzy neighbourhood of B.

For $A \subset A$, by our assumption, A is a fuzzy neighbourhood of A.

Hence there exits an open fuzzy set O such that $A \subset O \subset A$.

Hence $A = O$ and A is τ- open on (X, τ). □

Theorem 3.8. *Let (X, τ) be a fuzzy β-subalgebraic Topological space on X. Let A be a fuzzy β-subalgebra on X.*

1. *A^0 is open and A^0 is the largest fuzzy open set contained in A.*

2. *The fuzzy β- subalgebra A is open iff $A = A^0$*

Proof. 1) Let (X, τ) be a fuzzy β-subalgebraic Topological space on X.

Let A be a fuzzy β- subalgebra in X. By definition of fuzzy interior, A^0 is again a fuzzy interior set of A.

Hence there exist an τ- fuzzy open set O such that $A^0 \subset O \subset A$. But O is an fuzzy interior set of $A, O \subset A^0$ Hence $A^0 = O$.

Thus A_0 is open and is the largest fuzzy open set contained in A.

2) Suppose the fuzzy set A is open. If A is open, then $A \subset A^0$, for A is an fuzzy interior set of A.

Hence $A = A^0$

Conversely, let us suppose $A = A^0$. By definition of fuzzy interior, The union of all fuzzy interior sets of A is again a fuzzy interior of A and is denoted by A^0.

Therefore A is a neighbourhood of A^0.

Therefore, a fuzzy set A is τ- open. □

Definition 3.9. Let (X, τ) and (Y, σ) be a fuzzy β-subalgebraic Topological spaces on the β- algebras, X and Y respectively . A function f from (X, τ) to (Y, σ) is called a $\mathcal{F}$-continuous function if the inverse of each σ-open fuzzy set of Y is τ-open fuzzy set of X.

Theorem 3.10. *Let (X, τ) and (Y, σ) be a fuzzy β-subalgebraic Topological spaces on the β- algebras, X and Y respectively. Then the function f is $\mathcal{F}$-continuous if and only if the inverse image of every closed fuzzy set is closed.*

Proof. Suppose the function f is $\mathcal{F}$-continuous, i.e. the inverse of each σ-open fuzzy set is τ-open.

Let $\sigma\ '$ be the set of closed fuzzy set in Y. Then

$$\begin{aligned}\mu_{f^{-1}(\sigma\ ')}(x) =& \mu'_{\sigma}(f(x))\\ =& \mu_{\sigma\ '}(f(x))\\ =& 1 - \mu_{\sigma}(f(x))\\ =& 1 - \mu_{f^{-1}(\sigma)}(x)\\ =& \mu'_{f^{-1}(\sigma\ ')}(x).\\ \Rightarrow f^{-1}(\sigma\ ') =& \{f^{-1}(\sigma)\}',\end{aligned}$$

for all x in X.

Since f is $\mathcal{F}$-continuous, the inverse of every closed fuzzy set is closed.

Conversely, let σ be the set of open fuzzy set in Y. Then $\mu_{f^{-1}(\sigma)}(x) = \mu_{\sigma}(f(x))$ for all x in X. Since the inverse of every closed fuzzy set is closed.

Therefore, the inverse of every open fuzzy set is open and f is $\mathcal{F}$-continuous. □

Theorem 3.11. *Let (X, τ) and (Y, σ) be a fuzzy β-subalgebraic Topological spaces on the β- algebras, X and Y respectively. Then for each fuzzy set A in X , the inverse of every neighbourhood of $f(A)$ is a neighbourhood of A if and only if for each fuzzy set A in X and each neighbourhood v of $f(A)$, there is neighbourhood w of A such that $f(w) \subset v$.*

Proof. Let $\mathcal{A}$ be the fuzzy sets of X.

Let $\mathcal{U}$, $\mathcal{I}$ be the family of neighbourhoods of fuzzy sets and their image. Let, moreover $A \in \mathcal{A}$.

Suppose $v \in \mathcal{I}$,$w \in \mathcal{U}$ is the neighbourhood of $f(A)$ and A. Since the inverse of every neighbourhood of $f(A)$ is a neighbourhood of A. Therefore

$$f(w) = f(f^{-1}(v)), \tag{1}$$

if $f^{-1}(y)$ is not empty. Moreover

$$\begin{aligned}\mu_{f(f^{-1}(v))}(y) &= \sup_{z\in f^{-1}(y)} \mu_{f^{-1}(v)}(z)\\ &= \sup_{z\in f^{-1}(y)} \{\mu_v(f(z))\}\\ &=\mu_v(y),\end{aligned}$$

for all y in Y, if $f^{-1}(y)$ is not empty $\mu_{f(f^{-1}(v))}(y) = 0$.

Therefore

$$\mu_{f(f^{-1}(v))}(y) \leq \mu_v(y),$$

for all y in Y.

Hence

$$f(f^{-1}(v)) \subset v \tag{2}$$

or from (1) and (2) we receive $f(w) \subset v$.

Conversely, let V be a neighbourhood of $f(A)$.

Since there is a neighbourhood w of A such that $f(w) \subset v$, we have

$$f^{-1}(f(w)) \subset f^{-1}(v), \tag{3}$$

$$\begin{aligned}\mu_{f^{-1}(f(w))}(x) &=\mu_{f(w)}(f(x))\\ &= \sup_{z\in f^{-1}(f(x))} \{\mu_w(z)\}\\ &\geq\mu_w(x),\end{aligned}$$

for all x in X.

Therefore

$$w \subset f^{-1}(f(w)), \tag{4}$$

or from (3) and (4) we have

$$w \subset f^{-1}(f(w)) \subset f^{-1}(v),$$

i.e. $f^{-1}(v)$ is a neighbourhood of w. □

Theorem 3.12. *Let (X, τ) and (Y, σ) be a fuzzy β-subalgebraic Topological spaces on the β- algebras, X and Y respectively. If the function f is $\mathcal{F}$-continuous then for each fuzzy set A in X, the inverse of every neighbourhood of $f(A)$ is a neighbourhood of A.*

Proof. Let $\mathcal{A}$ be the fuzzy sets of X.

Let $\mathcal{I}$ be the family of neighbourhood of fuzzy set on $\mathcal{A}$ and let $A \in \mathcal{A}$ and $v \in \mathcal{I}$, i.e. A is a fuzzy set in X and v is a neighbourhood of $f(A)$..

Then there is an open neighbourhood w of $f(A)$ such that

$$f(A) \subset w \subset v.$$

Hence

$$f^{-1}(f(A)) \subset f^{-1}(w) \subset f^{-1}(v). \tag{5}$$

Since f is $\mathcal{F}$-continuous, $f^{-1}(w)$ is open, and

$$\begin{aligned}\mu_{f^{-1}(f(A))}(x) &= \mu_{f(A)}(f(x)) \\ &= \sup_{z \in f^{-1}(f(x))} \{\mu_A(z)\} \\ &\geq \mu_A(x),\end{aligned}$$

for all $x \in X$.

So

$$A \subset f^{-1}(f(A)). \tag{6}$$

Therefore, inclusions (5) and (6) imply

$$A \subset f^{-1}(w) \subset f^{-1}(v),$$

or $f^{-1}(v)$ is a neighbourhood of A. □

4. CONCLUSION

In this paper several interesting results were discussed by joining the notions of fuzzy β-subalgebras and fuzzy topology. One can further study on L-fuzzy, intuitionistic fuzzy and intuitionistic L-fuzzy sub structures on β-algebras by connecting with L-fuzzy, intuitionistic fuzzy and intuitionistic L-fuzzy topological spaces.

REFERENCES

[1] M. Annalakshmi, M. Chandramouleeswaran, Fuzzy topological subsystem on a TM-algebra, *Intetnational Journal of Pure and Applied Mathematics*, **94**, No. 3 (2014), 439-449.

[2] M. Aub Ayub Anasri, M. Chandramouleeswaran, Fuzzy β- subalgebras of β-algebras, *International J. of Math. Sci. & Engg. Appls.*, **5**, No. 7 (2013), 239-249.

[3] A.K. Chanduraty, P. Das, Some results on fuzzy Topology on fuzzy sets, *Fuzzy Sets and Systems*, **58** (1993), 331-336.

[4] C.L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications*, **24** (1968), 182-190.

[5] J. Neggers, Kim Hee Sik, On β-algebras, *Mathematica Slovaca*,52, No. 5 (2002), 517-530.

[6] L.A. Zadeh, Fuzzy sets, *Inform. Control*, **8**, No. 3 (1965), 338-353.