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BIFURCATION ANALYSIS OF AN *SIR* MODEL

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ABSTRACT: This paper is devoted to study a three dimensional Susceptible-Infected-Recovered (SIR) epidemic model. The stability of the equilibrium points in dynamical models is one of the most important issues. Here so we will study the stability of equilibrium points of the epidemic model by using bifurcation theory. For this purpose, we investigate transcritical, pitchfork and saddle-node bifurcation points.

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1. INTRODUCTION

Knowing the stability of the rest points is an important part in the analysis of a dynamical system, so it can be used in various sciences. That's why scientists are very interested in this issue. In this paper, by using the basic idea of the theory of bifurcations, various bifurcations of a epidemic model will be studied.

An SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations

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relating the number of susceptible people $S(t)$, number of people infected $I(t)$, and number of people who have recovered $R(t)$. One of the simplest SIR models is the Kermack-McKendrick model.

In 1927 they formulate the behavior of epidemic outbreaks in [1].They introduced the following nonlinear system of ordinary differential equations [1]

$$\begin{cases} \dot{x}(t) = -\beta x(t)y(t), \\ \dot{y}(t) = \beta x(t)y(t) - \gamma y(t), \\ \dot{z}(t) = \gamma y(t). \end{cases} \tag{1.1}$$

By initial conditions $x(0) = N_1 \geqslant 0$, $y(0) = N_2 \geqslant 0$ and $z(0) = N_3 \geqslant 0$, $N_i \in R$, $i = 1, 2, 3$, and where the parameter β and γ are the infection rate and the recovery rate.

If we consider a death rate μ and a birth rate equal to the death rate,then this epidemic model can be given by

$$\begin{cases} \dot{x}(t) = -\beta x(t)y(t) + \mu(N - x), \\ \dot{y}(t) = \beta x(t)y(t) - \gamma y(t) - (\gamma + \mu)y, \\ \dot{z}(t) = \gamma y(t) - \mu z(t). \end{cases} \tag{1.2}$$

See[2]-[5]. To study bifurcations of the model we organize this paper as follows: Section 2 gives some mathematical concepts which are going to be used in other sections. In Section 3 we study the equilibrium points and stable condition. Section 4 is devoted to analyse transcritical, pitchfork and saddle-node bifurcations.

2. PRELIMINARIES

In this section we shall state some mathematical concepts and basic theorems.

In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour. In general, at a bifurcation point, the local stability properties of equilibria, periodic orbits or other invariant sets change. A saddle-node bifurcation or tangent bifurcation is a collision and disappearance of two branches of equilibria. One of these branches is stable, the other is unstable. In a transcritical bifurcation, two families of fixed points collide and exchange their stability properties. The family that was stable before the bifurcation is unstable after it. The other family of fixed points goes from being unstable to being stable. But, in pitchfork bifurcation one family of fixed points transfers its stability properties to two families after or before the bifurcation point. If this occurs after the bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial fixed points occur for values of the parameter lower than the bifurcation value, see [6], [7], [8].

Consider the following system

$$\dot{x} = f(x, \mu),$$

with $x \in R^n$, $\mu \in R$, and smooth function f . Assume that at $\mu = \mu_0$, $x = x_0$, the above system has an equilibrium for which there exists a simple zero eigenvalue. The following theorem states the sufficient conditions for existence of saddle-node, transcritical and pitchfork bifurcations.

Theorem 2.1. *Let $\dot{x} = f(x, \mu)$ be a system of differential equations in R^n depending on the single parameterμ. When $\mu = \mu_0$, assume that there is an equilibrium p for which the following hypotheses are satisfied:*

1. *$D_x f_{\mu_0}(p)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w. $D_x f_{\mu_0}(p)$ has k eigenvalues with negative real parts and $(n-k-1)$ eigenvalues with positive real parts (counting multiplicity).*
2. $w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) \neq 0,$
3. $w^T[D_x^2(p, \mu_0)(v, v)] \neq 0.$

Then the system $\dot{x} = f(x, \mu)$ experiences a saddle-node bifurcation at the equilibrium point p as the parameter μ varies through the bifurcation value $\mu = \mu_0$. Moreover, if the condition (2) are changed to:

(a) $w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) = 0, \qquad w^T(\frac{\partial^2 f}{\partial \mu \partial x}(p, \mu_0)v) \neq 0.$

the system has a transcritical bifurcation at the equilibrium point p. And if the conditions (2), (3) are changed to

(a')$w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) = 0, \qquad w^T(\frac{\partial^2 f}{\partial \mu \partial x}(p, \mu_0)(v) \neq 0,$

(b')$w^T[D_x^2(p, \mu_0)(v, v)] = 0, \qquad w^T[D_x^3(p, \mu_0)(v, v, v)] \neq 0,$

Then the system has a pitchfork bifurcation at p, see [7], [8].

For proof and more information see [6].

3. SINGULAR POINTS OF THE SUSCEPTIBLE-INFECTED-RECOVERED (SIR) EPIDEMIC MODEL

Considering the system (1.2) to describe the type of equilibrium points. The actual equilibrium point of system (1.2) can be studied. There are two biologically feasible equilibria for the system (1.2), namely,

(i) $E_1 = (N, 0, 0)$ is the trivial steady state. and

(ii) $E^* = (X^*, Y^*, Z^*)$ is endemic equilibrium state, where

$$X^* = \frac{\mu+\gamma}{\beta}, \quad Y^* = \frac{\mu(N\beta-\mu-\gamma)}{\beta(\mu+\gamma)}, \quad Z^* = \frac{\gamma(N\beta-\mu-\gamma)}{\beta(\mu+\gamma)}.$$

consider the system (1.2), with the following Jacobian matrix at an equilibrium point

$$J = \begin{bmatrix} -\beta y - \mu & -\beta x & 0 \\ \beta y & \beta x - \mu - \gamma & 0 \\ 0 & 0 & -\mu \end{bmatrix}.$$

The equilibrium point $E_1 = (N, 0, 0)$ is a fixed point. Consider the system (1.2) with the following Jacobian matrix at equilibrium point E^*

$$J^* = \begin{bmatrix} -\beta y^* - \mu & -\beta x^* & 0 \\ \beta y^* & \beta x^* - \mu - \gamma & 0 \\ 0 & 0 & -\mu \end{bmatrix}.$$

Therefore

$$J^* = \begin{bmatrix} a_{11} & -\mu-\gamma & 0 \\ a_{21} & 0 & 0 \\ 0 & \gamma & -\mu \end{bmatrix}.$$

Here

$$a_{11} = -\frac{\mu N\beta}{\mu+\gamma}, \quad a_{21} = \frac{\mu(N\beta-\mu-\gamma)}{\mu+\gamma}$$

4. TRANSCRITICAL, PITCHFORK AND SADDLE-NODE BIFURCATIONS OF SYSTEM (??)

4.1. PITCHFORK BIFURCATIONS

The eigenvalues of (3) at equilibrium $E_1 = (N, 0, 0)$ are:

$$\lambda_1 = \beta - h, \qquad \lambda_{2,3} = -\mu, \quad \text{where } h = \frac{\mu + \gamma}{N}.$$

Choosing β as the bifurcation parameter. Suppose that all the other parameters are fixed and positive. Then we state the following theorem.

Theorem 4.1. *Consider the dynamical system (1.2):*

(i) If $\beta^2 N\mu \neq 0$, then $\beta = h$ is a transcritical bifurcation value for system (1.2) at the equilibrium point (N,0,0).

(ii) Pitchfork bifurcation for system (1.2) at the equilibrium point $(N, 0, 0)$ *does not happen.*

Proof. If $\beta = h$, then the corresponding right and left eigenvectors of $\lambda_1 = 0$ are $v = (-\frac{\beta N}{\gamma}, \frac{\mu}{\gamma}, 1)^T$ and $w = (0, 1, 0)^T$ respectively. Now we check the conditions of Theorem 2.1:

1)
$$w^T(\frac{\partial f}{\partial \beta} \mid_{(N,0,0,h)}) = 0,$$

2)
$$w^T(\frac{\partial^2 f}{\partial x \partial \beta} \mid_{(N,0,0,h)} .v) = \frac{N\mu}{\gamma} \neq 0.$$

Since

$$D_x^2 f(N,0,0,h)(v,v) = \begin{pmatrix} \sum_{m,n=1}^{3} \frac{\partial^2 f_1(N,0,0,h)}{\partial x_m \partial x_n} v_m v_n \\ \sum_{m,n=1}^{3} \frac{\partial^2 f_2(N,0,0,h)}{\partial x_m \partial x_n} v_m v_n \\ \sum_{m,n=1}^{3} \frac{\partial^2 f_3(N,0,0,h)}{\partial x_m \partial x_n} v_m v_n \end{pmatrix} = \begin{pmatrix} \frac{2\beta^2 N\mu}{\gamma^2} \\ -\frac{2\beta^2 N\mu}{\gamma^2} \\ 0 \end{pmatrix},$$

where $x = (x_1, x_2, x_3) =$ (x,y,z) and $v = (v_1, v_2, v_3)$. Hence:

3) $w^T[D_x^2 f(N,0,0,h)(v,v)] = -\frac{2\beta^2 N\mu}{\gamma^2}.$

Hence if $\beta^2 N\mu \neq 0$, then by Theorem 2.1, the point $(N, 0, 0)$ is a transcritical bifurcation point which completes the proof (2). (ii) Wince we have

$$D_x^2 f(N,0,0,h)(v,v) = \begin{pmatrix} \sum_{m,n,k=1}^{3} \frac{\partial^3 f_1(N,0,0,h)}{\partial x_m \partial x_n \partial x_k} v_m v_n v_k \\ \sum_{m,n,k=1}^{3} \frac{\partial^3 f_2(N,0,0,h)}{\partial x_m \partial x_n \partial x_k} v_m v_n v_k \\ \sum_{m,n,k=1}^{3} \frac{\partial^3 f_3(N,0,0,h)}{\partial x_m \partial x_n \partial x_k} v_m v_n v_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

In which $x = (x_1, x_2, x_3) =$ (x,y,z) and $v = (v_1, v_2, v_3)$, thus

$$w^T[D_x^3(N,0,0,h)(v,v,v)] = 0.$$

Therefore, by Theorem 2.1 for the system (1.2) at the equilibrium point (X^*, Y^*, Z^*) pitchfork bifurcation does not occur.

4.2. SADDLE-NODE BIFURCATION

Here we study saddle-node bifurcation for system (1.2) analytically and show that there is a bridge between profitability and bifurcation theory.

Let $\bar{E} = (X, Y, Z)$ be an equilibrium point for system (1.2) and A be the Jacobian matrix of that at $\bar{E}$. In addition, suppose that for $\beta = \beta_0$, $\lambda = 0$ is a simple root of the characteristic polynomial of A. Hence $\bar{\omega} = (1, Q, 0)^T$ is the left eigenvector corresponding to $\lambda = 0$ and $\bar{v} = (1, D, \frac{\gamma}{\mu} D)^T$ is its right eigenvector where

$$Q = \frac{\beta Y + \mu}{\beta Y}, \qquad D = -\frac{\beta Y}{\beta X - \gamma - Y}.$$

Furthermore, we have:

1) $\bar{w}^T(\frac{\partial f}{\partial \beta} \mid_{(X,Y,Z,\beta_0)}) = XY(Q-1)$.

2) $\bar{w} D^2 f(X, Y, Z, \beta_0)(\bar{v}, \bar{v}) = \bar{w}^T = \beta D(Q-1)$.

Now we state the sufficient conditions to happen the saddle-node bifurcation for (1.2) in the following theorem.

Theorem 4.2. *Suppose that $\bar{E} = (X, Y, Z)$ is an equilibrium point for system (1.2). If $\frac{X\mu}{\beta} > 0$ and $\beta\mu \neq 0$ then $\beta = \beta_0$ is a saddle-node bifurcation value for (1.2) at $\bar{E}$.*

Proof. By above discussion and Theorem 2.1, $\beta = \beta_0$ is a saddle-node bifurcation value for (1.2) at $\bar{E}$ if $\frac{X\mu}{\beta} \neq 0$ and $-\frac{\beta\mu}{\beta X - \gamma - Y} \neq 0$. Since $X, \mu > 0$ it means that $\frac{X\mu}{\beta} > 0$ and $\beta\mu \neq 0$.

5. CONCLUSION

In this paper we study the three dimensional system (1.2). Where it is susceptible-Infected-Recovered (SIR) epidemic model with equal death and birth rates. Moreover, we analysed system (1.2) by means of bifurcation theory and found transcritical, pitchfork and saddle-node bifurcation points. These methods were used to check on the quality of solutions of the system.

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