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ABOUT STABILITY CONDITIONS FOR RETARDED FRACTIONAL DIFFERENTIAL SYSTEMS WITH DISTRIBUTED DELAYS

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ABSTRACT: The aim of the present work is to introduce a new approach which allows to establish explicit conditions for global asymptotic stability of incommensurate and commensurate retarded linear fractional differential system with distributed delays. The derivatives in the system can be in Riemann-Liouville or Caputo type. The established conditions are simply to verify - it must be studied either the distribution of the roots of a polynomial, or the distribution of the eigenvalues for a constant matrix, which are explicitly determined from the systems parameters. Some results for the commensurate case are given too.

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1. INTRODUCTION

The fractional calculus has several applications in science fields as rheology, viscoelasticity, electrochemistry, electromagnetism, etc. A good and deep understanding in this matter can be received from the monographs of Kilbas et al. [3], Kiryakova [4] and

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Das [1] and the references therein. We note the significant works of Deng, Li, Lu [2] and Qian, Li, Agarwal, Wong [7] for fractional system with constant delays and the survey about stability analysis of fractional differential equations from Li, Zhang [5]. The first detailed study of the linear delay differential equations and system with distributed delay (fundamental theory, stability, oscillation behavior, etc.) was done by A.D. Myshkis in his fundamental monograph [6]. Fractional system with distributed delays was studied in [8], [10] and [11].

The aim of this work is to introduce a new approach which allows to establish sufficient explicit conditions for global asymptotic stability of linear fractional differential system with distributed delays. The approach allows to replace the difficult for practical checking theoretical condition ”if all roots of an analogue of the characteristic equation have negative real parts, then the considered system is asymptotically stable”, with significantly simple task - to study the distribution of the roots of a polynomial or the distribution of the eigenvalues for a constant matrix, which are explicitly determined from the systems parameters. Note that in the work the both cases of Riemann-Liouville and Caputo type derivatives are considered and some results for commensurate case are given too.

2. PRELIMINARIES

As is known, there are many different definitions of the fractional derivative, all of which generalize on the usual integer order derivative. Below we recall only the definitions of Riemann-Liouville and Caputo fractional derivatives. For their properties and details we refer to [3] and [4].

Let $\alpha > 0$ is arbitrary number and $m = [\alpha] + 1$. Let denote by $L_1^{loc}(\mathbb{R}, \mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$. Then for each $a \in \mathbb{R}$, $\alpha \in \mathbb{R}_+ = (0, \infty)$ and $f \in L_1^{loc}(\mathbb{R}, \mathbb{R})$ the left-sided fractional integral operator of order α is defined by

$$
\begin{aligned}
(D_{a+}^{-\alpha} f)(t) &= (I_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \\
(D_{0+}^{0} f)(t) &= (I_{0+}^{0} f)(t) = f(t),
\end{aligned}
$$

the corresponding left side Riemann-Liouville fractional derivative by

$$
{}_{RL}D_{a+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^m (I_{a+}^{m-\alpha} f)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s) ds,
$$

and the Caputo fractional left side derivative ${}_{C}D^{\alpha}_{a+}$ is defined by

$${}_{C}D^{\alpha}_{a+}f(t) = {}_{RL}D^{\alpha}_{a+}\left(f(s) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(s-a)^k\right)(t),$$

where $t > a, m = [\alpha] + 1, \alpha \notin \mathbb{N}$ and $m = \alpha, \alpha \in \mathbb{N}$ (see Kalibas et al. [3], p. 91).

3. FRACTIONAL LINEAR SYSTEMS

Consider the following autonomous linear systems with distributed delay

$$D^{\alpha_k}_{0+}x_k(t) = \sum_{j=1}^{n}\int_{-\sigma}^{0} x_j(t+\theta)du^j_k(\theta), \quad k = 1, 2, \dots, n, \tag{3.1}$$

where $D^{\alpha_k}_{0+}$ denotes either ${}_{RL}D^{\alpha_k}_{0+}$ (Riemann-Liouville fractional derivative) or ${}_{C}D^{\alpha_k}_{0+}$ (Caputo fractional derivative), $\alpha_k \in (0, 1)$, $\sigma \in \bar{\mathbb{R}}_+ = [0, \infty)$ and $n \geq 2$.

Introduce the following notations:

$$X(t) = (x_1(t), ..., x_n(t))^T, \qquad |X(t)| = \sum_{k=1}^{n}|x_k(t)|,$$

$$\alpha_m = \min(\alpha_1, ..., \alpha_n), \qquad \alpha_M = \max(\alpha_1, ..., \alpha_n),$$

$$\mathbb{C}_+ = \{p \in \mathbb{C} | \mathrm{Re}\, p > 0\}, \qquad \bar{\mathbb{C}}_+ = \{p \in \mathbb{C} | \mathrm{Re}\, p \geq 0\},$$

$$\mathbb{C}_- = \mathbb{C} \setminus \bar{\mathbb{C}}_+,$$

$${}_{RL}D^{\alpha}_{0+}X(t) = \left({}_{RL}D^{\alpha_1}_{0+}x_1(t), \dots, {}_{RL}D^{\alpha_n}_{0+}x_n(t)\right)^T,$$

$${}_{C}D^{\alpha}_{0+}X(t) = \left({}_{C}D^{\alpha_1}_{0+}x_1(t), \dots, {}_{C}D^{\alpha_n}_{0+}x_n(t)\right)^T,$$

for each $t \in \mathbb{R}_+$. For $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}, W(\theta) = \left\{w^i_j(\theta)\right\}^n_{i,j=1}$ we denote $|W(\theta)| = \sum_{k,j=1}^{n} |w^j_k(\theta)|$ and by $BV[a, b]$ the linear space of functions with bounded variation in θ on $[a, b] \subset \mathbb{R}, a \leq b$, where

$$Var_{[a,b]}W(\cdot) = \sum_{k,j=1}^{n} Var_{[a,b]}w^j_k(\cdot).$$

As usual with $\mathfrak{C}$ we denote the Banach space of initial vector functions

$$\mathfrak{C} = \left\{\Phi : [-\sigma, 0] \to \mathbb{R}^n | \Phi(t) = (\phi_1(t), \dots, \phi_n(t))^T, \phi_k \in C([-\sigma, 0], \mathbb{R}),\ 1 \leq k \leq n\right\},$$

with norm $||\Phi|| = \sup\limits_{t\in[-\sigma,0]} \sum_{k=1}^{n} |\phi_k(t)| = \sup\limits_{t\in[-\sigma,0]} |\Phi(t)|$.

We say that for the kernel $U : \mathbb{R} \to \mathbb{R}^{n\times n}$ the conditions (S) are fulfilled if the following conditions hold:

(S1) The function $U \in BV[\sigma, 0]$ and is normalized so that $U(0) = 0, U(\theta) = 0$ for $\theta \geq 0$, $U(\theta) = U(-\sigma)$ for $\theta \leq -\sigma$.

(S2) The kernel $U(\theta)$ is continuous from the left in θ on $(\sigma, 0)$.

(S3) The Lebesgue decomposition for the kernel $U(\theta)$ for $\theta \in [-\sigma, 0]$ has the form:

$$U(\theta) = \aleph(\theta) + \int_{-\sigma}^{\theta} B(s)ds + \Im(\theta),$$

where

$$\aleph(\theta) = \left\{a_k^j H(\theta + \sigma_k^j)\right\}_{k,j=1}^n, \ \sigma_k^j \in [0, \sigma], \ 1 \leq j, \ k \leq n, \quad A = \left\{a_k^j\right\}_{k,j=1}^n \in \mathbb{R}^{n\times n}$$

and $H(t)$ is the Heaviside function;

$$\int_{-\sigma}^{\theta} B(s)ds = \left\{\int_{-\sigma}^{\theta} b_k^j(s)ds\right\}_{k,j=1}^n \in AC([-h, 0], \mathbb{R}^{n\times n})$$

and

$$\Im(\theta) = \left\{g_k^j(\theta)\right\}_{k,j=1}^n \in C([-h, 0], \mathbb{R}^{n\times n}).$$

Let assume that the derivatives in the system (3.1) are in Riemann-Liouville sense and consider the Cauchy problem for (3.1) under the initial conditions

$$D_{0+}^{\alpha_k - 1} x_k(t) = \phi_k(t), \quad t \in [-h, 0], \ \Phi \in \mathfrak{C}, \ 1 \leq k \leq n. \tag{3.2}$$

In the case when the derivatives in the system (3.1) are in Caputo sense then we consider the Cauchy problem for (3.1) under the initial conditions

$$X(t) = \Phi(t), \quad t \in [-h, 0], \ \Phi \in \mathfrak{C}. \tag{3.3}$$

All questions connected with existing, uniqueness and continuation on $\mathbb{R}_+$ of the solution of the Cauchy problems for the linear systems in incommensurate case with distributed delay, when the derivatives are in Riemann-Liouville or Caputo sense are considered in [8], [10] and [11]. The possibility to apply correct the Laplace transform to the system (3.1) is considered in [11] too. Note that all complex computations below are done in the branch of the principle value of argument.

Definition 1. (see [8]) For the system (3.1) we call the matrix valued function $G(p)$

$$G(p) = \begin{pmatrix} p^{\alpha_1} - \int\limits_{-\sigma}^{0} e^{p\theta} du_1^1(\theta) & -\int\limits_{-\sigma}^{0} e^{p\theta} du_1^2(\theta) & \dots & -\int\limits_{-\sigma}^{0} e^{p\theta} du_1^n(\theta) \\ -\int\limits_{-\sigma}^{0} e^{p\theta} du_2^1(\theta) & p^{\alpha_2} - \int\limits_{-\sigma}^{0} e^{p\theta} du_2^2(\theta) & \dots & -\int\limits_{-\sigma}^{0} e^{p\theta} du_2^n(\theta) \\ \dots & & & \\ -\int\limits_{-\sigma}^{0} e^{p\theta} du_n^1(\theta) & -\int\limits_{-\sigma}^{0} e^{p\theta} du_n^2(\theta) & \dots & p^{\alpha_n} - \int\limits_{-\sigma}^{0} e^{p\theta} du_n^n(\theta) \end{pmatrix}, \tag{3.4}$$

characteristic matrix and the equation

$$\det(G(p)) = 0 \tag{3.5}$$

characteristic equation.

The next lemma which gives an answer of the important question when the roots of the characteristic equation (3.5) in $\bar{\mathbb{C}}_+$ are uniformly bounded and will be used in our considerations below.

Lemma 2. (see [11], Lemmas 5 and 6 *Let the following conditions be fulfilled:*

1. *The conditions S hold.*
2. $Var_{[-h,0]}U(\cdot) > 0$.

Then all roots of the characteristic equation (3.5) *with nonnegative real parts belongs to the rectangle*

$$\Omega = \{p = \gamma + i\omega \in \mathbb{C} | 0 \leq \gamma \leq s^*, 0 \leq |\omega| \leq s^*\},$$

where $s^* \in \mathbb{R}$ *is the unique root in* $\mathbb{R}_+$ *of the equation*

$$h(s) = \min(s^{\alpha_m}, s^{\alpha_M}) - Var_{[-h,0]}U(\cdot) = 0. \tag{3.6}$$

4. STABILITY ANALYSIS

Definition 3. The system (3.1) is said to be:

(a) Stable iff for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $t_* \in \bar{\mathbb{R}}_+$, such that for every initial function $\Phi \in \mathfrak{C}$ with $||\Phi|| < \delta$ the corresponding solution $X(t)$ satisfies for $t \geq t_*$ the inequality $|X(t)| \leq \varepsilon$.

(b) Locally asymptotically stable (LAS) if there is a $\Delta > 0$ such that for every initial function $\Phi \in \mathfrak{C}$ with $||\Phi|| < \Delta$ for the corresponding solution $X(t)$ we have that $\lim_{t\to\infty} |X(t)| = 0$.

(c) Globally asymptotically stable (GAS) if for every initial function $\Phi \in \mathfrak{C}$ for the corresponding solution $X(t)$ we have that $\lim_{t\to\infty} |X(t)| = 0$.

Remark 4. Note that the case $t_* = 0$ is possible only when the derivatives are in Caputo sense. As usual in our consideration below the expressions "the system (3.1) is stable" and "the zero solution of system (3.1) is stable" are synonymous.

Consider first the partial case when in condition (S3) we have that $\int\limits_{-\sigma}^{\theta} B(s)ds \equiv 0, \Im(\theta) \equiv 0$ for $\theta \in [-\sigma, 0]$. Then from (3.4) it follows that the characteristic matrix has the form

$$G_J(p) = \begin{pmatrix} p^{\alpha_1} - a_1^1 e^{-p\sigma_1^1} & -a_1^2 e^{-p\sigma_1^2} & \dots & -a_1^n e^{-p\sigma_1^n} \\ a_2^1 e^{-p\sigma_2^1} & p^{\alpha_2} - a_2^2 e^{-p\sigma_2^2} & \dots & -a_2^n e^{-p\sigma_2^n} \\ \dots & & & \\ a_n^1 e^{-p\sigma_n^1} & -a_n^2 e^{-p\sigma_n^2} & \dots & p^{\alpha_n} - a_n^n e^{-p\sigma_n^n} \end{pmatrix}, \tag{4.1}$$

i.e. $G_J(p) = \left\{c_k^j(p)\right\}_{k,j=1}^n$, where $c_k^j(p) = p^{\alpha_k} - a_k^k e^{-p\sigma_k^k}, j = k$, $c_k^j(p) = -a_k^j e^{-p\sigma_k^j}, j \neq k$.

If in additional we have that $\sigma_k^j = 0$, $1 \leq j, k \leq n$, then the characteristic matrix is $G_0(p) = \left\{\tilde{c}_k^j(p)\right\}_{k,j=1}^n$, $\tilde{c}_k^j(p) = p^{\alpha_k} - a_k^k, j = k, \tilde{c}_k^j(p) = -a_k^j, j \neq k$, where the matrix $A = \left\{a_k^j\right\}_{k,j=1}^n$ is the matrix of coefficients in the jump part in the Lebesgue decomposition of the kernel $U(\theta)$.

From (3.5) it follows that in this cases the characteristic equations becomes the forms

$$\det G_J(p) = 0 \tag{4.2}$$

and

$$\det G_0(p) = 0, \tag{4.3}$$

respectively.

The basic idea of our approach is as follows:

The first step is a simple task - check the matrix of coefficients in the jump part $A = \left\{a_k^j\right\}_{k,j=1}^n$ for stability (one possibility is to calculate some logarithmic norm of A). If $A = \left\{a_k^j\right\}_{k,j=1}^n$ is stable this implies that the system (3.1) in the partial case when

$$\int\limits_{-\sigma}^{\theta} B(s)ds \equiv 0,\ \Im(\theta) \equiv 0 \text{ and } \sigma_k^j = 0,\ 1 \leq j,\ k \leq n$$

(i.e. the system (3.1) without delay) is GAS.

Second step: We consider the more general case when

$$\int\limits_{-\sigma}^{\theta} B(s)ds \equiv 0,\ \Im(\theta) \equiv 0 \text{ but } \sigma_k^j \geq 0,\ 1 \leq j,\ k \leq n$$

and introduce a natural additional condition (see Condition 3 of Theorem 6). Then we can prove that the delayed system is GAS.

Third step: If in additional a simple inequality holds for $p \in \Omega$ (see Condition 2 of Corollary (9)), we can prove that the general system (with distributed delays) is GAS.

Note that if all roots of (3.5) have negative real parts, the system (3.1) with derivatives in Riemann-Liouville (Caputo) sense is globally asymptotically stable. (see [11] Theorem 4 (Theorem 5)).

Introduce the fractional degree quasi-polynomial

$$E(p) = \sum_{k=0}^{n} e_k^0 p^{\alpha_k} + \sum_{i=1}^{m} (\sum_{k=0}^{n} e_k^i p^{\alpha_k}) e^{h_i p^r}, \tag{4.4}$$

where $\alpha_0 = 0$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $r \in (0, 1]$, $h_1 \geq 0$, $h_1 < h_2 < \cdots < h_m$, $e_k^i \in \mathbb{R}$ and $p \in \mathbb{C}$. In our consideration below we need the following statement

Theorem 5. (see [9]) *The fractional characteristic quasi-polynomial* (4.4) *(of commensurate or non-commensurate degree) is stable if and only if*

$$\Delta_{\omega \in \mathbb{R}} arg \Psi(i\omega) = \Delta_{\omega \in \mathbb{R}} arg \frac{E(i\omega)}{w(i\omega)} = 0, \tag{4.5}$$

where $w(p)$ is arbitrary reference polynomial or quasi-polynomial, which must be stable (i.e. all roots of the equations $w(p) = 0$ lie in $\mathbb{C}_-$) and have the same fractional degree as the quasi-polynomial (4.4).

The next theorem clears when is possible to replace the more complicated task of localization of the roots of equation (4.2) with a significantly simple task to localize the roots of equation (4.3).

Theorem 6. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 2 hold.*
2. $\det G_0(0) > 0$ *and* $\det G_0(p) \neq 0$ *for every $p \in \Omega$, where Ω is the rectangle defined in Lemma 2.*
3. $\min_{|\omega| \leq s^*} |\det G_J(i\omega)| > 0$,*where s^* is the unique root of the equation* (3.6).
4. *In the Lebesgue decomposition we have that* $\int_{-\sigma}^{\theta} B(s)ds \equiv 0$, $\Im(\theta) \equiv 0$ *for* $\theta \in [-\sigma, 0]$.

Then the system (3.1) *with derivatives in Riemann-Liouville (or Caputo) sense is GAS.*

Proof. For $p \in \bar{\mathbb{C}}_+$ introduce the generalized modified Mikhailov plot $\Psi(p) = \frac{\det G_J(p)}{\det G_0(p)}$, where for the reference quasi-polynomial we have chosen $w(p) = \det G_0(p)$ which under conditions 1 and 2 of Theorem 6 have no roots for $p \in \bar{\mathbb{C}}_+$. According Theorem 5 we must prove that for $\Psi(i\omega), p = i\omega, \omega \in \mathbb{R}$ the condition (4.5) of Theorem 5 holds. It is not difficult to see that the condition (4.5) of Theorem 5 holds if and only if, when the generalized modified Mikhailov plot $\Psi(i\omega)$ does not cross the origin of the complex plane and does not encircle the origin, when ω runs from $-\infty$ to ∞. Since $\lim_{\omega \to \pm\infty} \Psi(i\omega) = 1$ and $\Psi(0) = 1$, then from the condition 3 of Theorem 6 it follows that the generalized modified Mikhailov plot $\Psi(i\omega)$ cannot cross the imaginary axis. Therefore it is no possible for $\Psi(i\omega)$ to cross the origin, or encircle the origin when ω runs from $-\infty$ to ∞. □

Let all $\alpha_k \in (0,1) \cap \mathbb{Q}$ in (3.1), i.e. $\alpha_k = \frac{l_k}{r_k} \in \mathbb{Q}$, $l_k, r_k \in \mathbb{N}$, $1 \le k \le n$, and denote by m the lowest common multiple of the denominators of all α_k. Then substitute $p = \lambda^m$ in (3.4) we obtain that the characteristic matrix obtains the form

$$G_{\mathbb{Q}}(p) = G_{\mathbb{Q}}(\lambda^m) = \left\{ \lambda^{m\alpha_k} - \int_{-\sigma}^{0} e^{\theta\lambda^m} du_k^j(\theta) \right\}_{k,j=1}^{n} \tag{4.6}$$

and the characteristic equation (3.5) obtains the form

$$\det G_{\mathbb{Q}}(\lambda^m) = 0 \tag{4.7}$$

In the case when in addition we have that $\sigma_k^j = 0,\ 1 \le j,\ k \le n$ the characteristic equation obtains a polynomial form

$$\det G_{\mathbb{Q}}^0(\lambda^m) = \det \left\{ \bar{c}_k^j(\lambda) \right\}_{k,j=1}^{n} = 0, \tag{4.8}$$

where $c_k^j(\lambda) = \lambda^{m\alpha_k} - a_k^k$, $j = k$, $c_k^j(\lambda) = -a_k^j$, $j \neq k$.

Corollary 7. *Let the following conditions be fulfilled:*

1. *Conditions 1,2 and 3 of Theorem 6 hold.*
2. $\alpha_k \in (0,1) \bigcap \mathbb{Q}, 1 \le k \le n$.
3. *The arguments of all roots* $\lambda_* = p_*^{\frac{1}{m}} \in \mathbb{C}$ *of equation* (4.7) *satisfy the condition* $|arg\lambda_*| > \frac{\pi m^{-1}}{2}$, *where* m *is the lowest common multiple of the denominators of all* $\alpha_k, 1 \le k \le n$.

Then the system (3.1) *with derivatives in Riemann-Liouville (or Caputo) sense is GAS.*

Proof. To apply Theorem 6 we need only to check that its condition 3 fulfilled.

Obviously if $p_* \in \mathbb{C}$ is a root of (3.5) then $\lambda_* = p_*^{\frac{1}{m}}$ is a root of (4.7) and vice versa. Then for all roots $p_* \in \mathbb{C}$ of equation (3.5) the condition $Rep_* < 0$ is fulfilled if and only if, when for the arguments of all roots $\lambda_* = p_*^{\frac{1}{m}} \in \mathbb{C}$ of equation (4.7) the condition $|arg\lambda_*| > \frac{\pi m^{-1}}{2}$ hold. Therefore the condition 3 of the Corollary 7 implies that all roots of the equation (4.3) have negative real parts. Thus Condition 3 of Theorem 6 holds too. □

Remark 8. Theorem 6 generalizes the result obtained in Corollary 3 in [2] for the partial case when $\alpha_k = \alpha \in (0, 1)$, $1 \leq k \leq n$, under the restrictive condition "the characteristic equation has no purely imaginary roots for any $\sigma_k^j > 0, 1 \leq j, k \leq n$". This condition is essentially weakened in our work - see condition 3 of Theorem 6. In them we replace the condition "the characteristic equation has no purely imaginary roots on the whole imaginary axis" used in [2], with " the characteristic equation has no purely imaginary roots in a fixed compact interval on the imaginary axis". Note that this interval for every choice of the delays $\sigma_k^j > 0, 1 \leq j, k \leq n$ can be explicitly determined from the systems parameters. Since our proof is not based on the so called "crossing technique" used in [2] we obtain an additional advantage - if for some choice of delays $\sigma_k^j \geq 0, 1 \leq j, k \leq n$ we have that the characteristic equation has no purely imaginary roots in the compact interval mentioned above, then the homogeneous system (3.1) with derivatives in Riemann-Liouville (or Caputo) sense is GAS for this choice of the delays. Note that the statement of Corollary 7 generalize the result obtained in Corollary 5 in [2] also.

Corollary 9. *Let the following conditions be fulfilled:*

1. *Condition 1, 2 and 3 of Theorem 6 hold.*
2. $|\det G_J(p)| \leq |\det G(p)|$ *for every* $p \in \Omega$, *where* Ω *is the rectangle defined in Lemma* (2).

   Then the system (3.1) *with derivatives in Riemann-Liouville (or Caputo) sense is GAS.*

Proof. Consider as an auxiliary system the system (3.1) in the case when $\int_{-\sigma}^{\theta} B(s)ds \equiv 0, \Im(\theta) \equiv 0$ for $\theta \in [-\sigma, 0]$. Then Theorem 6 implies that $|\det G_J(p)| > 0$ for every $p \in \bar{\mathbb{C}}_+$. From condition 2 of Corollary 9 it follows that $|\det G(p)| \geq |\det G_J(p)| > 0$ for each $p \in \Omega$. Then taking into account Lemma 2, we conclude that the characteristic equation (3.5) has no roots in $\bar{\mathbb{C}}_+$. □

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