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EXISTENCE AND UNIQUENESS RESULTS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT: This paper deals with the existence and uniqueness of solutions for implicit fractional differential equations involving the Caputo fractional derivative in a Banach space by using the Banach contraction principle and Schauder's fixed point theorem.

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1. INTRODUCTION

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4, 5, 12, 13, 16, 18, 19, 22]). There has been significant development in the study of fractional differential equations in recent years; see the monographs of Abbas *et al.* [1, 2] , Kilbas *et al.* [14], Lakshmikantham *et al.* [15], Miller and Ross [17], Podlubny [19], and Samko *et al.* [20]. For some recent contributions on fractional differential equations, see [3, 6, 7, 8, 9, 10, 11, 23] and the references therein.

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Recently, considerable attention has been given to the existence of solutions of boundary value problems for implicit fractional differential equations and integral equations with Caputo fractional derivative (see [9, 10, 24]).

The Green function for linear boundary value problems for ordinary differential equations with sufficiently smooth coefficients have been investigated in detail in several studies [10, 21]. In this work, analogously with boundary value problems for differential equations of integer order, we first derive the corresponding Green's function-named by fractional Green's function.

Motivated by the above cited works, the purpose of this paper, is to establish existence and uniqueness results to the following class of implicit fractional differential equations:

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), \quad \text{for each} \quad t \in J = [0, b], \ 0 < \alpha < 1, \tag{1.1}$$

$$y(0) + \lambda \int_0^b y(t)dt = y(b), \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a given function, and $\lambda \in (0, +\infty)$.

In this paper we present two results for the problem (1.1)-(1.2). The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem. Finally, we present two illustrative examples.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J, \mathbb{R})$ we denote the space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} = \{\sup |y(t)| \quad t \in J\}.$$

Definition 2.1. ([14]) The fractional order integral of order $\alpha \in \mathbb{R}_+$ of the function $\varphi \in L^1([0, b])$, is defined by

$$(I^{\alpha}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}\varphi(s)ds,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^{+\infty} t^{\xi - 1}e^{-t}dt, \quad \xi > 0.$$

Definition 2.2. ([14]) For a function φ given on the interval $[0, b]$, the Caputo fractional-order derivative of order α of φ is defined by

$$({}^cD^\alpha\varphi)(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{n-\alpha-1}\varphi^{(n)}(s)ds,$$

where $\Gamma(\cdot)$ is the Gamma function and $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.3. *([23]) Let $\alpha > 0$; then the differential equation ${}^cD^\alpha h(t) = 0$ has a solution*

$$h(t) = c_0 + c_1t + c_2t^2 + \ldots + c_{m-1}t^{m-1},\ c_i \in \mathbb{R},\ i = 0, 1, \ldots, m-1, \quad m = [\alpha]+1.$$

Lemma 2.4. *([23]) Let $\alpha > 0$; then*

$$I^\alpha({}^cD^\alpha h(t)) = h(t) + c_0 + c_1t + c_2t^2 + \ldots + c_{m-1}t^{m-1},$$

for arbitrary $c_i \in \mathbb{R},\ i = 0, 1, \ldots, m-1, \quad m = [\alpha]+1$.

3. EXISTENCE OF SOLUTIONS

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.2).

Definition 3.1. A function $y \in C^1(J, \mathbb{R})$ is said to be a solution of (1.1)-(1.2) if y satisfies
the equation ${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t))$ on J, and the condition (1.2).

For the existence results for the problem (1.1)-(1.2) we need the following auxiliary lemmas.

Lemma 3.2. *Let $0 < \alpha < 1$ and let $h \in C(J, \mathbb{R})$ be a given function. Then the boundary value problem*

$$ {}^cD^\alpha y(t) = h(t), \quad t \in J, \tag{3.1}$$

$$ y(0) + \lambda\int_0^b y(t)dt = y(b), \tag{3.2}$$

has a unique solution given by

$$y(t) = \int_0^b G(t,s)h(s)ds$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s)=\begin{cases}\dfrac{1}{b\Gamma(\alpha)}\left(\dfrac{(b-s)^{\alpha-1}}{\lambda}+\dfrac{\alpha b(t-s)^{\alpha-1}-(b-s)^{\alpha}}{\alpha}\right) & \text{if } \ 0\le s<t\\ \dfrac{1}{b\Gamma(\alpha)}\left(\dfrac{(b-s)^{\alpha-1}}{\lambda}-\dfrac{(b-s)^{\alpha}}{\alpha}\right) & \text{if } \ t\le s<b.\end{cases} \tag{3.3}$$

Proof. By Lemma 2.4 we have

$$\begin{aligned} y(t) &= I^{\alpha}({}^{c}D^{\alpha}y(t))\\ &= I^{\alpha}h(t)-c_0 \qquad \text{for some constant } c_0\in\mathbb{R}.\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds-c_0.\end{aligned}$$

We have by integration using Fubini's integral theorem

$$\begin{aligned}\int_0^b y(s)ds &= \int_0^b\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)d\tau-c_0\right)ds\\ &= \int_0^b\left(\frac{1}{\Gamma(\alpha)}\int_\tau^b(s-\tau)^{\alpha-1}ds\right)h(\tau)d\tau-c_0b\\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^b(b-\tau)^{\alpha}h(\tau)d\tau-c_0b.\end{aligned}$$

Applying the boundary condition (3.2), we have $y(0)=-c_0$

$$y(b)=\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}h(s)ds-c_0$$

that is

$$c_0=\frac{1}{b\Gamma(\alpha)}\int_0^b\left(\frac{(b-s)^{\alpha}}{\alpha}-\frac{(b-s)^{\alpha-1}}{\lambda}\right)h(s)ds.$$

Therefore, the unique solution of (3.1)-(3.2) is

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t(t-s)^{\alpha-1}h(s)ds+\frac{1}{b}\int_0^b\left(\frac{(b-s)^{\alpha-1}}{\lambda}-\frac{(b-s)^{\alpha}}{\alpha}\right)h(s)ds\right]\\ &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t(t-s)^{\alpha-1}+\frac{1}{b}\left(\frac{(b-s)^{\alpha-1}}{\lambda}-\frac{(b-s)^{\alpha}}{\alpha}\right)\right]h(s)ds\\ &+ \frac{1}{b\Gamma(\alpha)}\int_t^b\left(\frac{(b-s)^{\alpha-1}}{\lambda}-\frac{(b-s)^{\alpha}}{\alpha}\right)h(s)ds\\ &= \int_0^b G(t,s)h(s)ds.\end{aligned}$$

□

Remark 3.3. The function $t\in J\longmapsto\int_0^b G(t,s)ds$ is continuous on J, and hence is bounded. Let

$$G^*=\sup_{t\in J}\left\{\int_0^b|G(t,s)|ds\right\}.$$

Lemma 3.4. *A function $y \in C^1(J,\mathbb{R})$ is a solution of the problem (1.1)-(1.2) if and only if $y \in C(J,\mathbb{R})$ is a solution of the integral equation*

$$y(t) = \int_0^b G(t,s)\varphi(s)ds, \tag{3.4}$$

where $G(t,s)$ is the Green's function given by (3.3) and $\varphi \in C(J,\mathbb{R})$ satisfies the implicit functional equation

$$\varphi(s) = f(s, y(s), \varphi(s)).$$

Theorem 3.5. *Assume that*
*(**H1**) $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.*
*(**H2**) There exist constants $0 < l < 1$ and $0 < k < \dfrac{1-l}{bG^*}$ such that*

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq k|x-\overline{x}| + l|y-\overline{y}|$$

for any $x,\ y,\ \overline{x},\ \overline{y} \in \mathbb{R}$, and $t \in J$.
Then there exists a unique solution for the problem (1.1)-(1.2).

Proof. We transform the problem (1.1)-(1.2) into fixed point problem. Consider the operator $A : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by

$$A(y)(t) = \int_0^b G(t,s)\varphi(s)ds, \tag{3.5}$$

where $G(t,s)$ is the Green's function given by (3.3) and $\varphi \in C(J,\mathbb{R})$ satisfies the implicit functional equation

$$\varphi(s) = f(s, y(s), \varphi(s)).$$

Clearly, from Lemmas 3.2 and 3.4 , the fixed points of A are solutions to the problem (1.1)-(1.2). We shall show that A is a contraction.
Let $u,\ v \in C(J,\mathbb{R})$. Then, for each $t \in J$, we have

$$(Au)(t) - (Av)(t) = \int_0^b G(t,s)\big(\varphi(s) - \psi(s)\big)ds,$$

where

$$\varphi(s) = f(s, u(s), \varphi(s)\big),$$

$$\psi(s) = f(s, v(s), \psi(s)),$$

and

$$|\varphi(s) - \psi(s)| \leq k|u(s) - v(s)| + l|\varphi(s) - \psi(s)|.$$

Thus,

$$|\varphi(s)-\psi(s)| \leq \frac{k}{1-l}|u(s)-v(s)|.$$

Then,

$$\begin{aligned}
\Big|(Au)(t)-(Av)(t)\Big| &\leq \int_0^b \Big|G(t,s)\big(\varphi(s)-\psi(s)\big)\Big|ds \\
&\leq \frac{k}{1-l}\int_0^b |G(t,s)||u(s)-v(s)|ds \\
&\leq \frac{bkG^*}{1-l}\|u-v\|_\infty.
\end{aligned}$$

Thus

$$\|Au-Av\|_\infty \leq \frac{bkG^*}{1-l}\|u-v\|_\infty.$$

Since $\frac{bkG^*}{1-l} < 1$, the operator A is a contraction.
Then by Banach's fixed point theorem, the problem (1.1)-(1.2) has a unique solution. □

Now we give an existence result based on Schauder's fixed point theorem.

Theorem 3.6. *Assume (***H1***) and (***H2***) hold. If*

$$1-l-bkG^* > 0, \tag{3.6}$$

the problem (1.1)-(1.2) has at least one solution.

Proof. Let

$$D = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leq \gamma\}$$

where

$$\gamma > \frac{bf^*G^*}{1-l-bkG^*}, \tag{3.7}$$

with $\quad f^* = \sup_{t\in J}|f(t,0,0)|.$

It is clear that D is a closed, convex subset of $C(J,\mathbb{R})$. Let the operator A be defined in (3.5). We shall show that A satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: A is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \longrightarrow u$ in $C(J,\mathbb{R})$. Then for each $t \in J$, we have

$$\varphi_n(s) = f\big(s, u_n(s), \varphi_n(s)\big)$$

and

$$\varphi(s) = f(s, u(s), \varphi(s)).$$

We have

$$|\varphi_n(s) - \varphi(s)| \leq k|u_n(s) - u(s)| + l|\varphi_n(s) - \varphi(s)|.$$

Thus,

$$|\varphi_n(s) - \varphi(s)| \leq \frac{k}{1-l}|u_n(s) - u(s)|.$$

Then,

$$\begin{aligned} \Big|(Au_n)(t) - (Au)(t)\Big| &\leq \int_0^b \Big|G(t,s)\big(\varphi_n(s) - \varphi(s)\big)\Big|ds \\ &\leq \frac{k}{1-l}\int_0^b |G(t,s)||u_n(s) - u(s)|ds. \end{aligned}$$

Since $u_n \longrightarrow u$, we get $\varphi_n \longrightarrow \varphi$, and the Lebesgue dominated convergence theorem implies that

$$\|A(u_n) - A(u)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence A is continuous.

Step 2: $A(D) \subset D$

Let $y \in D$. We will show that $Ay \in D$. For each $t \in J$, we have

$$\begin{aligned} |(Ay)(t)| &= \Big|\int_0^b G(t,s)\varphi(s)ds\Big| \\ &\leq \int_0^b |G(t,s)||\varphi(s)|ds. \end{aligned}$$

By (**H2**) we have

$$\begin{aligned} |\varphi(s)| &= |f(s, y(s), \varphi(s))| \\ &\leq |f(s, y(s), \varphi(s)) - f(s,0,0)| + |f(s,0,0)| \\ &\leq k|y(s)| + l|\varphi(s)| + |f(s,0,0)| \\ &\leq \frac{k|y(s)| + |f(s,0,0)|}{1-l} \\ &\leq \frac{k\|y\|_\infty + f^*}{1-l}. \end{aligned}$$

Then,

$$\begin{aligned} |(Ay)(t)| &\leq \frac{k\|y\|_\infty + f^*}{1-l}\int_0^b |G(t,s)|ds \\ &\leq \frac{k\|y\|_\infty + f^*}{1-l}bG^* \\ &\leq \frac{k\gamma + f^*}{1-l}bG^*. \end{aligned}$$

By (3.7) we have

$$\|Ay\|_\infty \leq \gamma,$$

so $A(D) \subset D$.

Step 3: A maps D into a equicontinuous set of $C(J, \mathbb{R})$.

Let $y \in D$, t_1, $t_2 \in J$, $t_1 < t_2$; then

$$\begin{aligned}\Big|(Ay)(t_2) - (Ay)(t_1)\Big| &= \Big|\int_0^b G(t_2, s)\varphi(s))ds - \int_0^b G(t_1, s)\varphi(s))ds\Big| \\ &\le \int_0^b \Big|G(t_2, s) - G(t_1, s)\Big||\varphi(s)|ds \\ &\le \frac{k\|y\|_\infty + f^*}{1-l}\int_0^b \Big|G(t_2, s) - G(t_1, s)\Big|ds.\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. By the Arzelà-Ascoli theorem, A is completely continuous. Therefore, we deduce that A has a fixed point y which is a solution of problem (1.1)-(1.2). $\square$

4. EXAMPLES

In this section we give two examples to illustrate the usefulness of our main results.

Example 4.1. Consider the boundary value problem

$$^cD^{\frac{1}{2}}y(t) = \frac{|y(t)| + |^cD^{\frac{1}{2}}y(t)|}{10(1 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|)}, \quad t \in J = [0, 1], \tag{4.1}$$

$$y(0) + \int_0^1 y(t)dt = y(1). \tag{4.2}$$

Set

$$f(t, x, y) = \frac{x + y}{10(1 + x + y)} \qquad (t, x, y) \in J \times [0, \infty) \times [0, \infty).$$

It is clear that f is continuous. Let x, $y \in [0, \infty)$ and $t \in J$; then

$$\begin{aligned}|f(t, x, y) - f(t, \overline{x}, \overline{y})| &= \frac{1}{10}\left|\frac{x + y}{1 + x + y} - \frac{\overline{x} + \overline{y}}{1 + \overline{x} + \overline{y}}\right| \\ &= \frac{1}{10}\left|\frac{1}{1 + \overline{x} + \overline{y}} - \frac{1}{1 + x + y}\right| \\ &\le \frac{1}{10}\Big|x + y - \overline{x} - \overline{y}\Big| \\ &\le \frac{1}{10}\Big(|x - \overline{x}| + |y - \overline{y}|\Big).\end{aligned}$$

Then the assumption (**H2**) holds with

$$k = l = \frac{1}{10}.$$

From (3.3), G is given by

$$G(t,s)=\begin{cases}\dfrac{(1-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}+\dfrac{\frac{1}{2}(t-s)^{-\frac{1}{2}}-(1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})} & \text{if} \quad 0\le s<t\\ \dfrac{(1-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}-\dfrac{(1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})} & \text{if} \quad t\le s<1.\end{cases} \tag{4.3}$$

From (4.3) we have

$$\begin{aligned}\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds+\int_t^1 G(t,s)ds\\ &= -\frac{(1-t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}+\frac{1}{\Gamma(\frac{3}{2})}+\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\\ &\quad -\frac{1}{\Gamma(\frac{5}{2})}+\frac{(1-t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}.\end{aligned}$$

We can easily see that

$$G^*<\frac{4}{\Gamma(\frac{3}{2})}+\frac{1}{\Gamma(\frac{5}{2})}<\frac{10}{\sqrt{\pi}}.$$

Since $0<\dfrac{bkG*}{1-l}<\dfrac{10}{9\sqrt{\pi}}<1$, Theorem 3.5 implies that the problem (4.1)-(4.2) has a unique solution.

Example 4.2. Consider the boundary value problem

$${}^cD^\alpha y(t)=\frac{3+|y(t)|+|{}^cD^\alpha y(t)|}{(20+e^t)(1+|y(t)|+|{}^cD^\alpha y(t)|)} \qquad t\in J=[0,1],\ \alpha\in(0,1), \tag{4.4}$$

$$y(0)+\int_0^1 y(t)dt=y(1). \tag{4.5}$$

Set

$$f(t,x,y)=\frac{3+|x|+|y|}{(20+e^t)(1+|x|+|y|)} \qquad (t,x,y)\in J\times\mathbb{R}\times\mathbb{R}.$$

It is clear that f is continuous .

Let x, $y\in\mathbb{R}$ and $t\in J$; then

$$\begin{aligned}|f(t,x,y)-f(t,\overline{x},\overline{y})| &= \frac{1}{(20+e^t)}\left|\frac{3+|x|+|y|}{1+|x|+|y|}-\frac{3+|\overline{x}|+|\overline{y}|}{1+|\overline{x}|+|\overline{y}|}\right|\\ &= \frac{2}{(20+e^t)}\left|\frac{1}{1+|\overline{x}|+|\overline{y}|}-\frac{1}{1+|x|+|y|}\right|\\ &\le \frac{2}{(20+e^t)}\Big||x|+|y|-|\overline{x}|-|\overline{y}|\Big|\\ &\le \frac{1}{10}\Big(|x-\overline{x}|+|y-\overline{y}|\Big).\end{aligned}$$

Then the assumption (**H2**) holds with

$$k = l = \frac{1}{10}.$$

And we have

$$f^* = \sup_{t\in J} |f(t,0,0)| = \frac{1}{7}.$$

From (3.3), G is given by

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\alpha(t-s)^{\alpha-1} - (1-s)^{\alpha}}{\alpha\Gamma(\alpha)} & \text{if} \quad 0 \le s < t \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} & \text{if} \quad t \le s < 1. \end{cases} \tag{4.6}$$

From (4.6) we have

$$\begin{aligned} \int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\ &= -\frac{(1-t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad -\frac{1}{\Gamma(\alpha+2)} + \frac{(1-t)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

We can easily see that

$$G^* < \frac{4}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}$$

Condition (3.7) is satisfied for appropriate values of $\alpha \in (0,1)$. Theorem 3.6 implies that the problem (4.4)-(4.5) has a at least one solution.

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