Communications in Applied Analysis **20** (2016), 367-378

ON HYERS-ULAM STABILITY OF ALMOST-PERIODIC SOLUTIONS FOR CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS IN LEAKAGE TERMS ON TIME SCALES

LIUBIN HUA[1] AND YONGJIN LI[2]

[1]Guangzhou Sontan Polytechnic College
Guangzhou, 511370, P.R. CHINA

[2]Department of Mathematics
Sun Yat-Sen University
Guangzhou, 510275, P.R. CHINA

ABSTRACT: We investigate the Hyers-Ulam stability of cellular neural networks with time-varying delays in leakage terms on time scales by using Banach fixed point method.

AMS Subject Classification: 39A10, 39A12

1. INTRODUCTION AND PRELIMINARIES

1.1. THE HYERS-ULAM STABILITY

In 1940, S.M. Ulam gave a wide range of talks at the Mathematics Club of the University of Wisconsin, in which discussed a number of important unsolved problems. He [34] posed the following question concerning the stability of group homomorphisms before a Mathematical Colloquium: *When can we assert that the solutions of an inequality are close to one of the exact solutions of the corresponding equation?*

Received: October 2, 2016

Within the next two years, Hyers [14] brilliantly gave a partial answer to this question for the case when and are assumed to be Banach spaces by using direct method. For further details and discussions, the reader is referred to the book by Jung [19].

To the best of our knowledge, the first one who pay attention to the stability of differential equations is M. Obłoza [29, 30]. Thereafter, C. Alsina and R. Ger [1] proved that the stability holds true for differential equation $y'(x) = y(x)$. Then, a generalized result was given by S.-E. Takahasi, T. Miura and S. Miyajima [33], in which they investigated the stability of the Banach space valued linear differential equation of first order (see also [26, 28]). A more general result on the linear differential equations of first order of the form $y'(t) + \alpha(t)y(t) + \beta(t) = 0$ was given by S.-M. Jung [17] and the stability of linear differential equations of second order was established by Y. Li et al. (see [22, 24, 11, 21]).

In the near past many research papers have been published about the Ulam-Hyers stability of functional, differential and difference equations. The main tool used by the authors for obtaining stability results was the direct method. Recently Rus developed a unified approach based on Gronwall type inequalities and Picard operators. This approach can be applied to a wide range of problems.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ be an operator. The fixed point equation $Tx = x$ is said to be Ulam-Hyers stable if there exists a real number $C_T > 0$ such that: for each real number $\varepsilon > 0$ and each solution y^* of the equation $d(y, Ty) < \varepsilon$, there exists a solution x^* of the $Tx = x$ such that $d(y^*, x^*) < C_T \cdot \varepsilon$.

1.2. TIME SCALE ANALYSIS

Stefan Hilger in his doctoral dissertation, that resulted in his seminal paper [13] in 1990, initiated the study of time scales in order to unify continuous and discrete analysis. In recent years, the theory of dynamic equations on time scales, which provides powerful new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. We refer the reader to the book by Bohner and Peterson [5] and to the papers cited therein. The time scales calculus has a tremendous potential for applications in mathematical models of real processes, for instance, in biotechnology, chemical technology, economic, neural networks, physics, social sciences and so on, see the monographs of Aulbach and Hilger [4], Bohner and Perterson [5] and the references therein.

For convenience, we will provide without proof several foundational definitions and results from the calculus on time scales so that the paper is self-contained.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 1.2. Fix $t \in \mathbb{T}_k$. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$. The nabla derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in U$. Define $f^{\nabla^n}(t)$ to be the nablaa derivative of $f^{\nabla^{n-1}}(t)$; i.e., $f^{\nabla^n}(t) = (f^{\nabla^{n-1}}(t))^{\nabla}$.

Definition 1.3. If f is ld-continuous, then there is a function F such that $F^{\nabla}(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

The function p is v-regressive if $1 - v(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$. Define the v-regressive class of functions on $\mathbb{T}_k$ to be $R_v = \{p \ : \ T \to R \mid p \ \text{ is ld-continuous and } \ v\text{-regressive}\}$.

We define the set R_v^+ of all positively v-regressive elements by $R_v^+ = R_v^+(\mathbb{T}, \mathbb{R}) = \{p \in R_v \mid 1 - v(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}$.

If $p \in R_v$, then we define the nabla exponential function by

$$\hat{e}_p(t, s) = exp\left\{\int_s^t \hat{\xi}_{v(\tau)}(p(\tau)) \bigtriangledown \tau\right\}$$

for $s, t \in \mathbb{T}$, where the v-cylinder transformation

$$\xi_h(z) = \begin{cases} -\frac{\log(1-hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Definition 1.4. If $p, q \in R_v$, then we define a circle plus addition by

$$(p \oplus_v q)(t) := p(t) + q(t) - p(t)q(t)v(t)$$

for all $t \in \mathbb{T}_k$. For $p \in R_v$, define a circle minus p by

$$\ominus_v p := -\frac{p}{1 - vp}.$$

Theorem 1.5. *If $p, q \in R_v$ and $s, t, r \in \mathbb{T}$ then*

(1) $\hat{e}_0(t, s) \equiv 1$ *and* $\hat{e}_p(t, t) \equiv 1$;

(2) $\hat{e}_p(\sigma(t), s) = (1 - v(t)p(t))\hat{e}_p(t, s)$;

(3) $\hat{e}_p(t, s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus_v p}(s, t)$;

(4) $\hat{e}_p(t, s)\hat{e}_p(s, r) = \hat{e}_p(t, r)$;

(5) $(\hat{e}_p(t, s))^{\nabla} = p(t)\hat{e}_p(t, s)$.

Theorem 1.6. *Let f, g be nabla differentiable functions on $\mathbb{T}$. Then*

(1) $(v_1 f + v_2 g)^{\nabla} = v_1 f^{\nabla} + v_2 g^{\nabla}$, *for any constants v_1 and v_2;*

(2) $(fg)^{\nabla} = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\rho(t))$;

(3) *If f and f^{∇} are continuous, then* $(\int_a^t f(t, s) \nabla s)^{\nabla} = f(\rho(t), t) + \int_a^t f^{\nabla}(t, s) \nabla s$.

Theorem 1.7. *Assume $p \in R_v$ and $t_0 \in T$. If $1 - v(t)p(t) > 0$ for $t \in \mathbb{T}$, then $\hat{e}_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.*

For more details about calculus on time scales, one can see [4], [5].

2. HYERS-ULAM STABILITY OF CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS IN LEAKAGE TERMS ON TIME SCALES

S. András, A.R. Mészáros discussed the Ulam-Hyers stability of dynamic equations on time scales via Picard operators [3]. D.R. Anderson, B. Gates and D. Heuer studied the Hyers-Ulam stability of second-order linear dynamic equations on time scales [2]. In the study of dynamic equations on time scales, most often the analysis turns to that of a related integral equation on time scales.

Cellular neural networks were first introduced by Chua and Yang [7, 8]. They have been paid much attention in the past decades due to their applications in many fields such as image processing, pattern recognition and associative memories. In [32, 35] the phenomenon of stochastic and coherence resonance on Hodgkin-Huxley neuronal networks was studied with delay. Compared with periodicity, almost-periodicity occurs more frequently and reflects the nature more accurately. Hence, the existence and stability of almost-periodic solutions of neural networks have been widely investigated. In 2013, Zhang and Shao [36] discussed the cellular neural network with time-varying delays in leakage terms.

To the best of our knowledge, up to now, there are very few papers in the literature on the Hyers-Ulam stability of almost-periodic solutions of cellular neural networks on time scales. The Hyers-Ulam stability of almost-periodic solutions is important topics in the study of cellular neural networks on time scales.

In this paper, we consider a cellular neural network with time-varying delays in leakage terms on time scale of the following form

$$x_i^{\nabla}(t) = -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{g}_j(x_j(t-\tau_{ij}(t)))$$

$$+\sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(x_j(t-u))\nabla u + I_i(t). \qquad (2.1)$$

Where $i = 1, 2, \cdots, n$. $t \in \mathbb{T}$. $\mathbb{T}$ is an almost-periodic time scale.

Let $\overline{c}_i = \sup_{t\in\mathbb{T}} |c_i(t)|, \underline{c}_i = \inf_{t\in\mathbb{T}} |c_i(t)|, \overline{\eta}_i = \sup_{t\in\mathbb{T}} |\eta_i(t)|, \overline{a}_{ij} = \sup_{t\in\mathbb{T}} |a_{ij}(t)|$, $\overline{b}_{ij} = \sup_{t\in\mathbb{T}} |b_{ij}(t)|, \overline{I}_i = \sup_{t\in\mathbb{T}} |I_i(t)|, i,j = 1, 2, 3, \cdots, n$, and $(-\infty, 0]_{\mathbb{T}} = \{t \mid t \in (-\infty, 0] \bigcap \mathbb{T}\}$. The initial condition is

$$x_i(s) = \varphi_i(s), s \in (-\infty, 0]_{\mathbb{T}}, i = 1, 2, 3, \cdots, n,$$

where $\varphi_i(s) \in C^1((-\infty, 0]_{\mathbb{T}}, \mathbb{R})$.

Throughout this paper, the following conditions are hold:

$(H_1) K_{ij} : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is ld-continuous and $|K_{ij}(u)|$ is ∇-integrable on $[0, \infty)_{\mathbb{T}}$, $g_j(x), \widetilde{g}_j(x) : \mathbb{R} \to \mathbb{R}, c_i(t), \eta_i(t), \tau_{ij}(t) : \mathbb{T} \to (0, \infty), a_{ij}(t), b_{ij}(t), I_i(t) : \mathbb{T} \to \mathbb{R}$ are all almost-periodic functions, and $c_i(t) \in \mathbb{R}_v^+$ and $\underline{c}_i > 0, i, j = 1, 2, 3, \cdots, n$.

(H_2) there exist positive constants M_j, G_j, R_j, such that

$$|\widetilde{g}_j(x) \le M_j,$$

$$|\widetilde{g}_j(u_1) - \widetilde{g}_j(u_2)| \le G_j|u_1 - u_2|,$$

$$|g_j(u_1) - g_j(u_2)| \le \mathbb{R}_j|u_1 - u_2|, \text{ for any } x, u_1, u_2 \in R, j = 1, 2, 3, \cdots, n.$$

(H_3) there exists a positive constant r_0, such that

$$\max_{1\le i\le n}\left\{\frac{Q_i}{\underline{c}_i}, \left(1+\frac{\overline{c}_i}{\underline{c}_i}\right)Q_i\right\} \le r_0, \max_{1\le i\le n}\left\{\frac{N_i}{\underline{c}_i}, \left(1+\frac{\overline{c}_i}{\underline{c}_i}\right)N_i\right\} \le \alpha < 1,$$

$$Q_i = \overline{c}_i r_0 \overline{\eta}_i + \sum_{j=1}^{n} \overline{a}_{ij}M_j + \sum_{j=1}^{n} \overline{b}_{ij}(|g_j(0)| + R_j r_0)\int_0^{\infty} |K_{ij}(u)|\nabla u + \overline{I}_i,$$

$$N_i = \overline{c}_i\overline{\eta}_i + \sum_{j=1}^{n} \overline{a}_{ij}G_j + \sum_{j=1}^{n} \overline{b}_{ij}R_j\int_0^{\infty} |K_{ij}(u)| \bigtriangledown u.$$

for $i = 1, 2, 3, \cdots, n$.

Let $X = \{\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)^T \mid \varphi_i \in C^1(\mathbb{T}, \mathbb{R})$ is an almost-periodic function, $i = 1, 2, 3, \cdots, n\}$, then X is a linear space. Define the norm

$$\|\varphi\| = \sup_{t\in\mathbb{T}} \|\varphi(t)\| \text{ for } \|\varphi(t)\| = \max_{1\le i\le n}\{|\varphi_i(t)|, |\varphi_i^{\nabla}(t)|\},$$

then it is easy to prove that $(X, \|\cdot\|)$ is Banach space.

Definition 2.1. A time scale $\mathbb{T}$ is called an almost-periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} \mid t \pm \tau \in \mathbb{T}, \text{for any } t \in \mathrm{T}\} \neq \{0\}$$

Definition 2.2. Let $\mathbb{T}$ be an almost-periodic time scale and $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$. A function $f \in C(\mathbb{T}, \mathbb{E}^n)$ is called an almost-periodic function if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\tau \in \Pi \mid |f(t+\tau) - f(t)| < \varepsilon, \text{for any } t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \ \text{for any } t \in \mathbb{T}.$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

In [10], Gao establish sufficient conditions on the existence and uniqueness of almost-periodic solutions. Next, we present our result on the Hyers-Ulam stability of (2.1).

Theorem 2.3. *Suppose that $(H_1) - (H_3)$ hold, then (2.1) has Hyers-Ulam stability.*

Proof. (1) It is easy to see (2.1) can be rewrote in the form

$$x_i^{\nabla}(t) = -c_i(t)x_i(t) + c_i(t)\int_{t-\eta_i(t)}^{t} x_i^{\nabla}(u) \bigtriangledown u + \sum_{j=1}^{n} a_{ij}(t)\tilde{g}_j(x_j(t - \tau_{ij}(t)))$$

$$+\sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(x_j(t-u)) \bigtriangledown u + I_i(t).$$

For any given $\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))$, we consider the following almost-periodic system

$$x_i^{\nabla}(t) = -c_i(t)x_i(t) + c_i(t)\int_{t-\eta_i(t)}^{t} \varphi_i^{\nabla}(u) \bigtriangledown u + \sum_{j=1}^{n} a_{ij}(t)\tilde{g}_j(\varphi_j(t - \tau_{ij}(t)))$$

$$+\sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(\varphi_j(t-u)) \bigtriangledown u + I_i(t).$$

From (H_1) and Lemma 2.5 in [10], it follows that the linear system $x_i^{\nabla}(t) = -c_i(t)x_i(t)$ admits an exponential dichotomy. Thus, by Lemma 2.4 in [10], we see that (3.1) has a unique almost-periodic solution, which can be expressed as

$$x_{\varphi_i}(t) = \int_{-\infty}^{t} \hat{e}_{-c_i}(t, \rho(s))(c_i(s)\int_{s-\eta_i(s)}^{s} \varphi_i^{\nabla}(u)\nabla u + \sum_{j=1}^{n} a_{ij}(s)\tilde{g}_j(\varphi_j(s - \tau_{ij}(s)))$$

$$+\sum_{j=1}^{n} b_{ij}(s)\int_0^{\infty} K_{ij}(u)g_j(\varphi_j(s-u))\nabla u + I_i(s))\nabla S,\ i=1,2,\cdots.$$

(2) Define an operator T by $T(\varphi)(t) = x_\varphi(t)$ for any $\varphi \in \mathbb{B}$, where $x_\varphi(t) = (x_{\varphi_1}(t), x_{\varphi_2}(t), \cdots, x_{\varphi_n}(t))^T$. We will show that $T(E) \subseteq E$.

$$F_i = c_i(s)\int_{s-\eta_i(s)}^{s} \varphi_i^{\nabla}(u)\nabla u + \sum_{j=1}^{n} a_{ij}(s)\tilde{g}_j(\varphi_j(s-\tau_{ij}(s)))$$

$$+\sum_{j=1}^{n} b_{ij}(s)\int_0^{\infty} K_{ij}(u)g_j(\varphi_j(s-u)) \bigtriangledown u + I_i(s),\ i=1,2,\cdots,n.$$

By (H_2), we have

$$|g_j(\varphi_j(s-u))| - |g_j(0)| \le |g_j(\varphi_j(s-u)) - g_j(0)| \le R_j|\varphi_j(s-u)|.$$

Follows from (H_1) and (H_2), we get

$$|F_i| \le \overline{c}_i r_0 \overline{\eta}_i + \sum_{j=1}^{n} \overline{a}_{ij} M_j + \sum_{j=1}^{n} \overline{b}_{ij}\int_0^{\infty} |K_{ij}(u)|(R_j|\varphi_j(s-u)| + |g_j(0)|)\nabla u + \overline{I}_i$$

$$\le \overline{c}_i r_0 \overline{\eta}_i + \sum_{j=1}^{n} \overline{a}_{ij} M_j + \sum_{j=1}^{n} \overline{b}_{ij}\int_0^{\infty} |K_{ij}(u)|(R_j r_0 + |g_j(0)|) \bigtriangledown u + \overline{I}_i$$

$$\le \overline{c}_i r_0 \overline{\eta}_i + \sum_{j=1}^{n} \overline{a}_{ij} M_j + \sum_{j=1}^{n} \overline{b}_{ij}|(R_j r_0 + |g_j(0)|)\int_0^{\infty} |K_{ij}(u)| \bigtriangledown u + \overline{I}_i = Q_i.$$

Thus, for $i = 1, 2, \cdots, n$, we get

$$|T(\varphi)_i(t)| = \left|\int_{-\infty}^{t} \hat{e}_{-c_i}(t,\rho(s))F_i\nabla s\right| \le \int_{-\infty}^{t} \hat{e}_{-c_i}(t,\rho(s))|F_i|\nabla s$$

$$\le \int_{-\infty}^{t} \hat{e}_{-c_i}(t,\rho(s))Q_i\nabla s \le \frac{Q_i}{\underline{c}_i}.$$

On the other hand, for $i = 1, 2, \cdots, n$, we have

$$|T(\varphi)_i^{\nabla}(t)| = \left|\left(\int_{-\infty}^{t} \hat{e}_{-c_i}(t,\rho(s))F_i\nabla s\right)_t^{\nabla}\right|$$

$$= \left|F_i - c_i(t)\int_{-\infty}^{t} \hat{e}_{-c_i}(t,\rho(s))F_i\nabla s\right|$$

$$\le |F_i| + |c_i(t)||T(\varphi)_i(t)|$$

$$\le Q_i + \overline{c}_i\frac{Q_i}{\underline{c}_i} = \left(1 + \frac{\overline{c}_i}{\underline{c}_i}\right)Q_i.$$

In view of (H_3), we obtain

$$\|T\varphi\| \le \max_{1\le i\le n}\left\{\frac{Q_i}{\underline{c}_i}, \left(1+\frac{\overline{c}_i}{\underline{c}_i}\right)\right\} \le r_0,$$

Thus $T\varphi \in E$ for any $\varphi \in E$.

(3) Next we will show that T is a contraction.

For any $\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T, \psi = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T \in E$, we denote

$$J_i = c_i(s)\int_{s-\eta_i(s)}^{s}(\varphi_i^\nabla(u)-\psi_i^\nabla(u))\nabla u+\sum_{j=1}^{n} a_{ij}(s)(\tilde{g}_j(\varphi_j(s-\tau_{ij}(s)))-\tilde{g}_j(\psi_j(s-\tau_{ij}(s))))$$

$$+\sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)(g_j(\varphi_j(s-u))-g_j(\psi_j(s-u)))\nabla u,\ i=1,2,\cdots,n.$$

By (H_1) and (H_2), we have

$$|J_i| \leq \left(\overline{c}_i\overline{\eta}_i + \sum_{j=1}^{n}\overline{a}_{ij}G_j + \sum_{j=1}^{n}\overline{b}_{ij}\int_0^\infty |K_{ij}(u)|R_j\nabla u\right)\|\varphi-\psi\| = N_i\|\varphi-\psi\|.$$

Then, for $i = 1, 2, \cdots, n$, we get

$$|T(\varphi)_i(t) - T(\psi)_i(t)| = \left|\int_{-\infty}^{t}\hat{e}_{-c_i}(t,\rho(s))J_i\nabla s\right| \leq \int_{-\infty}^{t}\hat{e}_{-c_i}(t,\rho(s))|J_i|\nabla s$$

$$\int_{-\infty}^{t}\hat{e}_{-c_i}(t,\rho(s))N_i\nabla s\|\varphi-\psi\| \leq \frac{N_i}{\underline{c}_i}\|\varphi-\psi\|.$$

On the other hand, for $i = 1, 2, \cdots, n$, we have

$$|(T(\varphi)_i - T(\psi)_i)^\nabla(t)| = \left|\left(\int_{-\infty}^{t}\hat{e}_{-c_i}(t,(s))J_i\nabla s\right)_t^\nabla\right|$$

$$= \left|J_i - c_i(t)\int_{-\infty}^{t}\hat{e}_{-c_i}(t,\rho(s))J_i\nabla s\right| \leq |J_i| + |c_i(t)||T(\varphi)_i(t) - T(\psi)_i(t)|$$

$$\leq \left(N_i + \overline{c}_i\frac{N_i}{\underline{c}_i}\right)\|\varphi-\psi\| = \left(1+\frac{\overline{c}_i}{\underline{c}_i}\right)\|\varphi-\psi\|.$$

so

$$\|T\varphi - T\psi\| \leq \max_{1\leq i\leq n}\left\{\frac{N_i}{\underline{c}_i}, \left(1+\frac{\overline{c}_i}{\underline{c}_i}\right)N_i\right\}\|\varphi-\psi\| \leq \alpha\|\varphi-\psi\|.$$

Hence, by $(H_3), T$ is a contraction mapping.

(4) Since T is a contraction mapping, by Banach fixed point theorem, T has a fixed point in E. That is, (2.1) has a unique almost-periodic solution in E.

(5) Finally, we will show that (2.1) has Hyers-Ulam stability.

For any $\varepsilon > 0$, since E is complete linear metric space with the metric $d(x,y) = \|x-y\|$, if $\|Tu-u\| \leq \varepsilon$, by $Tv^* = v^*$, we have

$$\begin{aligned}\|u-v^*\| &= \|u-Tu+Tu-Tv^*\| \\ &\leq \|Tu-u\| + \|Tu-Tv^*\| \\ &\leq \varepsilon + \alpha\|u-v^*\|.\end{aligned}$$

Thus

$$\|u - v^*\| \leq \frac{1}{(1-\alpha)} \cdot \varepsilon$$

Hence, (2.1) has Hyers-Ulam stability.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (11571378).

REFERENCES

[1] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function, J. Inequal. Appl. 2 (1998), 373–380, **doi:** 10.1155/S102558349800023X

[2] D.R. Anderson, B. Gates, D. Heuer, Hyers-Ulam stability of second-order linear dynamic equations on time scales. Commun. Appl. Anal. 16 (2012), no. 3, 281–291.

[3] S. András, A.R. Mészáros, Ulam-Hyers stability of dynamic equations on time scales via Picard operators. Appl. Math. Comput. 219 (2013), no. 9, 4853–4864, **doi:** 10.1016/j.amc.2012.10.115

[4] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scales, in Nonlinear Dynamics and Quantum Dynamical Systems. Akademie Verlage, Berlin, 1990.

[5] M. Bohner, A. Peterson, Dynamic equations on time scales: An introduction with applications, Birkhäuser, Boston, (2001), **doi:** 10.1007/978-1-4612-0201-1

[6] E. A. Bohner, M. Bohner and F. Akin, Pachpatte Inequalities on time scale, J. Inequal. Pure. Appl. Math. 6 (2005), no. 1, Art 6.

[7] L.O. Chua, L. Yang, Cellular neural networks: theory, IEEE Trans. Circuits Syst. 35 (1988), 1257–1272, **doi:** 10.1109/31.7600

[8] L.O. Chua, L. Yang, Cellular neural networks: applications, IEEE Trans. Circuits Syst. 35 (1988), 1273–1290, **doi:** 10.1109/31.7601

[9] J.B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305–309, **doi:** 10.1090/S0002-9904-1968-11933-0

[10] J. Gao, Q.R. Wang, L.W. Zhang, Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales. Appl. Math. Comput. 237 (2014), 639-649, **doi:** 10.1016/j.amc.2014.03.051

[11] P. Găvruţă, S.-M Jung and Y. Li, Hyers-Ulam stability for second-order linear differential equations with boundary conditions, Electron. J. Differential Equations. 2011 (2011), no. 80, 1–5.

[12] O. Hatori, K. Kobayasi, T. Miura, H. Takagi, S. E. Takahasi, On the best constant of Hyers-Ulam stability, J. Nonlinear Convex Anal. 5 (2004), 387-393.

[13] S. Hilger, Analysis on Measure chain-A unified approch to continuous and discrete calculus, Results Math. 18 (1990), 18–56.

[14] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222–224, **doi:** 10.1073/pnas.27.4.222

[15] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett. 17 (2004), 1135–1140, **doi:** 10.1016/j.aml.2003.11.004

[16] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, III, J. Math. Anal. Appl. 311 (2005), 139–146, **doi:** 10.1016/j.jmaa.2005.02.025

[17] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, II, Appl. Math. Lett. 19 (2006), 854–858, **doi:** 10.1016/j.aml.2005.11.004

[18] S.-M. Jung, A fixed point approach to the stability of differential equations $y' = F(x, y)$, Bull. Malays. Math. Sci. Soc. (2) 33 (2010), no. 1, 47–56.

[19] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, vol. 48 of Springer Optimization and Its Applications, Springer, New York, NY, USA, 2011, **doi:** 10.1007/978-1-4419-9637-4

[20] T. Kulik and C. C. Tisdell, Volterra integral equations on time scales: Basic qualitative and quantitative results with applications to initial value problems on unbounded domains, Int. J. Difference Equ. 3 (2008), no 1, 103–133.

[21] Y. Li and Y. Shen, Hyers-Ulam stability of nonhomogeneous linear differential equations of second order, Internat. J. Math. Math. Sci. 2009 (2009), Article ID 576852, **doi:** 10.1155/2009/576852

[22] Y. Li, Hyers-Ulam stability of linear differential equations $y'' = \lambda^2 y$, Thai J. Math. 8 (2010), no. 2, 215–219.

[23] Y. Li, L. Hua, Hyers-Ulam stability of a polynomial equation. Banach J. Math. Anal. 3 (2009), no. 2, 86–90, **doi:** 10.15352/bjma/1261086712

[24] Y. Li and Y. Shen, Hyers-Ulam stability of linear differential equations of second order, Appl. Math. Lett. 23 (2010), 306–309, **doi:** 10.1016/j.aml.2009.09.020

[25] T. Miura, On the Hyers-Ulam stability of a differentiable map, Sci. Math. Japan. 55 (2002), 17–24.

[26] T. Miura, S.-M. Jung and S.-E. Takahasi, Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$, J. Korean Math. Soc. 41 (2004), 995–1005, **doi:** 10.4134/JKMS.2004.41.6.995

[27] T. Miura, M. Miyajima and S.-E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, Math. Nachr. 258 (2003), 90–96, **doi:** 10.1002/mana.200310088

[28] T. Miura, H. Oka, S.-E. Takahasi and N. Niwa, Hyers-Ulam stability of the first order linear differential equation for Banach space-valued holomorphic mappings, J. Math. Inequal. 3 (2007), 377–385, **doi:** 10.7153/jmi-01-32

[29] M. Obłoza, Hyers stability of the linear differential equation, Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993), 259–270.

[30] M. Obłoza, Connections between Hyers and Lyapunov stability of the ordinary differential equations, Rocznik Nauk.-Dydakt. Prace Mat. 14 (1997), 141–146.

[31] H. Rezaei, S.-M. Jung and Th.M. Rassias, Laplace transform and Hyers-Ulam stability of linear differential equations, J. Math. Anal. Appl. 403 (2013), 244–251, **doi:** 10.1016/j.jmaa.2013.02.034

[32] X.J. Sun, M. Perc, Q.S. Lu, J. Kurths, Spatial coherence resonance on diffusive and small-world networks of Hodgkin-Huxley neurons, Chaos 18 (2008), 023102. 7pp, **doi:** 10.1063/1.2900402

[33] S.-E. Takahasi, T. Miura and S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, Bull. Korean Math. Soc. 39 (2002), 309–315, **doi:** 10.4134/BKMS.2002.39.2.309

[34] S. M. Ulam, A Collection of the Mathematical Problems, Interscience, New York, 1960.

[35] Q.Y. Wang, M. Perc, Z.S. Duan, G.R. Chen, Delay-enhanced coherence of spiral waves in noisy HodgkinCHuxley neuronal networks, Phys. Lett. A 372 (2008), 5681–5687.

[36] H. Zhang, J.Y. Shao, Almost periodic solutions for cellular neural networks with time-varying delays in leakage terms, Appl. Math. Comput. 219 (2013), 11471–11482, **doi:** 10.1016/j.amc.2013.05.046