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APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR THE SOLUTION OF CUBIC BOUSSINESQ EQUATION AND BOUSSINESQ-BURGER EQUATION

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ABSTRACT: In this paper, homotopy analysis method is applied to compute the numerical solution of cubic Boussinesq equation and Boussinesq-Burger equation and compared the obtained results with the results obtained by various analytic methods like homotopy perturbation method, Laplace adomian decomposition method, Optimal homotopy asymptotic method and with exact solution. Comparison between our solutions and the exact solution shows that although all the four methods are effective and accurate in solving nonlinear problems but homotopy analysis method is more accurate with less number of iterations as compared to OHAM and other methods.

AMS Subject Classification: 35K15, 35L15

1. INTRODUCTION

The soliton equation is one of the most prominent subjects in the field of nonlinear science. In the past several decades, a great number of efforts have been made to study various nonlinear soliton equations.The dynamics of shallow water waves are seen in various places like sea beaches, lakes and rivers are governed by the Boussinesq Equation (BE). The Korteweg-de Vries (KdV) equation that models shallow water waves is definitely very well known. However, the Boussinesq equation gives

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a much better approximation to such waves. There are two forms of the Boussinesq equation that will be addressed in this paper, and both are with cubic nonlinearity [1]. The soliton solutions will be obtained for these equations. These solutions will be extremely useful in carrying out further analysis in the context of shallow water waves that arises in the context of oceanography. The traditional methods of solving nonlinear wave equations include inverse scattering theory [2,3], Backlund transformation [45], Darboux transformation [6] and Painlev expansion method [7], etc. With the rapid development of nonlinear science, some new powerful solving methods have been developed, such as homogeneous balance method [8], Jacobi elliptic function method [9], method of bifurcation [10], F-expansion method [11, 12] and some approximate method such as HPM and ADM method [13] and method of auxiliary equation [14], etc. The Homotopy Analysis Method (HAM) is an approximate analytical method, which can be adapted to solve nonlinear ordinary and partial differential equations.The homotopy analysis method (HAM) is a general analytic approach to get series solutions of various types of non- linear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them. More importantly, different from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series, and therefore, the HAM is valid even for strongly non-linear problems. The HAM was successfully applied to solve many nonlinear problems such as nonlinear Riccati differential equation with fractional order [15], nonlinear Vakhnenko equation [16], the Glauert-jet problem [17], fractional KdV-Burgers-Kuramoto Equation [18], a generalized Hirota-Satsuma coupled KdV equation [19], nonlinear heat transfer [20], projectile motion with the quadratic law [21], boundary layer flow of nanofluid past a stretching sheet [22],the Poisson-Boltzmann equation of semiconductor devices [23], solitary solution of discrete MKdV equation [24], the system of Fractional differential equations [25],the Oldroyd 6- constant fluid with magnetic field [26], MHD-flow of an Oldroyd 8-constant fluid [27], to the nonlinear flows with slip boundary condition [28] and so on.

In this paper, we have studied two types of nonlinear partial differential equations,one of which is a system of nonlinear partial differential equations namely, Boussinesq equation and generalized Boussinesq-Burgers equations. We employed the HomotopyAnalysis Method to obtain the numerical solutions to both these systems.

Here we first considered a well-known model of nonlinear dispersive waves i.e.

Cubic Boussinesq equation, which was proposed by Boussinesq, is given by

$$u_{tt} - u_{xx} - u_{xxx} + 2\left(u^3\right)_{xx} = 0. \tag{1}$$

Secondly considered the Generalized BoussinesqBurger equation given by

$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \tag{2}$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \leq x \leq 1. \tag{3}$$

Here x and t represents the normalized space and time respectively,$u(x;t)$ is the horizontal velocity and $v(x;t)$ represent the height of water surfaceabove a horizontal level at the bottom.

2. MATERIALS AND METHODS

2.1. BASIC IDEA OF HOMOTOPY ANALYSIS METHOD

In order to show the basic idea of HAM, consider the following differential equation

$$N\left[u(x,t)\right] = 0, \tag{4}$$

where N is a nonlinear operator, x and t denotes the independent variables and u is an unknown function. For simplicity, we ignore all the boundary or initial conditions,which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-q)\, L\left[\phi\left(x,t;q\right) - u_0\left(x,t\right)\right] = q\tilde{h}\, H\left(x,t\right) N\left[\phi\left(x,t;q\right)\right]. \tag{5}$$

Here $q[0,1]$ is the embedding parameter, $h0$ is an auxiliary parameter, L is an auxiliary linear operator, is an unknown function, is an initial guess and denotes a nonzero auxiliary function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, then equation (5) becomes

$$\phi\left(x,t;0\right) = u_0\left(x,t\right), \phi\left(x,t;1\right) = u\left(x,t\right).$$

Thus as q increases from 0 to 1, the solution varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi\left(x,t;q\right) = u_0\left(x,t\right) + \sum_{m=1}^{\infty} u_m\left(x,t\right) q^m, \tag{6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

The convergence of the series (6) depends upon the auxiliary parameter. If it is convergent at$q = 1$,

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{7}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [15]-[16].

Define the vectors

$$u_n = (u_0(x,t), u_1(x,t), ..., u_n(x,t)). \tag{8}$$

Differentiate the zeroth-order deformation equation (4) m-times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we get the following m^{th}- order deformation equation:

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = \tilde{h} R_m(u_{m-1}), \tag{9}$$

where

$$R_m(u_{m-1}) = \frac{1}{m!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $u_m(x,t)$ for $m1$ is governed by the linear equation (7) with linear boundary conditions that comes form the original problem, which can be solved by the symbolic computation software such as Mathematica or Maple.the convergence of the above method was studied by Liao[ ]. If equation (4) admits unique solution, then this method will produce the unique solution. If equation (4) does not posses a unique solution, the HAM will give a solution among many other possible solutions.

3. APPLICATION OF HAM TO CUBIC BOUSSINESQ EQUATION

Consider a generalCubic Boussinesq equation(1) with initial condition

$$u_0(x,t) = \frac{1}{2}\tanh\left(\frac{x}{2}\right) \tag{10}$$

The exact solution of equation (1) is given by [29]:

$$u(x,t) = \frac{1}{2}\tanh\left(\frac{x}{2} + \frac{t}{2\sqrt{2}}\right) \tag{11}$$

The linear operator of equation(1) can be written as

$$L\left(\phi\left(x,t;q\right)\right)=\frac{\partial^{2}\left(\phi\left(x,t;q\right)\right)}{\partial t^{2}} \tag{12}$$

with the property where $L_C$is integral constant. Moreover, the non-linear operator of equation(1) can be defined as

$$N\left(\phi\left(x,t;q\right)\right)=\frac{\partial^{2}\left(\phi\left(x,t;q\right)\right)}{\partial t^{2}}-\frac{\partial^{2}\left(\phi\left(x,t;q\right)\right)}{\partial x^{2}}-\frac{\partial^{3}\left(\phi\left(x,t;q\right)\right)}{\partial x^{3}} + 2\frac{\partial^{2}\left(\phi\left(x,t;q\right)\right)^{3}}{\partial x^{2}}. \tag{13}$$

By using the procedure of HomotopyAnalysis Method, the zeroth-order deformation equations for (1) can be written as

$$(1-q)\left[\phi\left(x,t;q\right)-u_{0}\left(x,t\right)\right]=qc_{0}N\left[\phi\left(x,t;q\right)\right] \tag{14}$$

For $q=0$ and $q=1$,it can be written as

$$\phi\left(x,t;0\right)=u_{0}\left(x,t\right),\phi\left(x,t;1\right)=u\left(x,t\right), \tag{15}$$

and the m_{th} order deformation equations of equation (1) can be written as

$$L\left(u_{m}\left(x,t\right)-\chi_{m}u_{m-1}\left(x,t\right)\right)=c_{0}R_{m}\left(u_{m-1}\right), \tag{16}$$

where

$$R_{m}\left(u_{m-1}\right)=\left(u_{m-1}\right)_{tt}-\left(u_{m-1}\right)_{xx}-\left(u_{m-1}\right)_{xxx}+2\left(\left(u_{m-1}\right)^{3}\right)_{xx} \tag{17}$$

The approximate solution of equation (1) can be written as

$$u\left(x,t\right)=u_{0}\left(x,t\right)+\sum_{m=1}^{\infty}u_{m}\left(x,t\right), \tag{18}$$

where

$$u_{m}\left(x,t\right)=\chi_{m}u_{m-1}\left(x,t\right)+c_{0}\int_{0}^{t}\int_{0}^{t}R_{m}\left(u_{m-1}\right)dtdt+c. \tag{19}$$

Now by using equation (16) and (17), the following results can be obtained in the following form

$$R_{1}\left(u_{0}\right)=t\left(\frac{1}{8}\text{sech}^{4}\left(\frac{x}{2}\right)-\frac{1}{4}\tanh^{2}\left(\frac{x}{2}\right)\text{sech}^{2}\left(\frac{x}{2}\right)+\frac{1}{4}\tanh\left(\frac{x}{2}\right)\text{sech}^{2}\left(\frac{x}{2}\right)\right) + 2\left(\frac{3}{16}\tanh\left(\frac{x}{2}\right)\text{sech}^{4}\left(\frac{x}{2}\right)-\frac{3}{16}tanh^{3}\left(\frac{x}{2}\right)\text{sech}^{2}\left(\frac{x}{2}\right)\right), \tag{20}$$

and the first approximation is

$$u_1(x,t) = c_0\left(\frac{1}{16}t^2\text{sech}^4\left(\frac{x}{2}\right) + \frac{3}{16}t^2\tanh\left(\frac{x}{2}\right)\text{sech}^4\left(\frac{x}{2}\right)\right.$$
$$\left.-\frac{3}{16}t^2 tanh^3\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\right)$$
$$-\frac{1}{8}\left(t^2\tanh^2\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right) + t^2\tanh\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\right). \quad (21)$$

Hence the solution of equation (1) can be written as

$$u(x,t) = \frac{1}{2}\tanh\left(\frac{x}{2}\right) + \left\{\begin{array}{l} c_0\left(\begin{array}{l}\frac{1}{16}t^2\text{sech}^4\left(\frac{x}{2}\right) + \frac{3}{16}t^2\tanh\left(\frac{x}{2}\right)\text{sech}^4\left(\frac{x}{2}\right)\\ -\frac{3}{16}t^2 tanh^3\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\end{array}\right)\\ -\frac{1}{8}\left(t^2\tanh^2\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right) + t^2\tanh\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\right)\end{array}\right\}. \quad (22)$$

Which represents the approximate solution for cubic boussinesq equation.

4. APPLICATION OF HAM TO GENERALIZED BOUSSINESQ-BURGER EQUATION

Consider the generalized Boussinesq-Burger equation(2) and (3) with initial condition

$$u(x,0) = \frac{ck}{2} + \frac{ck}{2}\left(\frac{-kx-\ln b}{2}\right), \quad (23)$$

$$v(x,0) = \frac{-k^2}{8}\text{sech}^2\left(\frac{kx+\ln b}{2}\right). \quad (24)$$

The exact solution of equations (2) and (3) (are given by [29]):

$$u(x,t) = \frac{ck}{2} + \frac{ck}{2}\left(\frac{ck^2t-kx-\ln b}{2}\right), \quad (25)$$

$$v(x,t) = \frac{-k^2}{8}\text{sech}^2\left(\frac{kx+\ln b-ck^2t}{2}\right). \quad (26)$$

The initial approximation of equation (2) and (3) can be represented as

$$u_0(x,t) = \frac{ck}{2} + \frac{ck}{2}\left(\frac{-kx-\ln b}{2}\right), \quad (27)$$

$$v_0(x,t) = \frac{-k^2}{8}\text{sech}^2\left(\frac{kx+\ln b}{2}\right). \quad (28)$$

Now we can define the linear operator of (2) as

$$L_i\left(\phi_i(x,t;q)\right) = \frac{\partial\phi_i(x,t;q)}{\partial t}, \quad i=0,1, \quad (29)$$

with the property where L_C is integral constant.

Moreover, the non-linear operator of equations (2) and (3) as

$$N_1\left(\phi_i\left(x,t;q\right)\right) = \frac{\partial\phi_1\left(x,t;q\right)}{\partial t} - \frac{1}{2}\frac{\partial\phi_2\left(x,t;q\right)}{\partial x} + 2\phi_1\left(x,t;q\right)\frac{\partial\phi_1\left(x,t;q\right)}{\partial x}, \quad (30)$$

$$N_2\left(\phi_i\left(x,t;q\right)\right) = \frac{\partial\phi_2\left(x,t;q\right)}{\partial t} - \frac{1}{2}\frac{\partial^3\phi_1\left(x,t;q\right)}{\partial x^3} + 2\frac{\partial\left(\phi_1\left(x,t;q\right)\phi_2\left(x,t;q\right)\right)}{\partial x}. \quad (31)$$

The approximate solution of the system (2) and (3) can be defined as

$$u\left(x,t\right) = u_0\left(x,t\right) + \sum_{m=1}^{\infty} u_m\left(x,t\right), \quad (32)$$

and

$$v\left(x,t\right) = v_0\left(x,t\right) + \sum_{m=1}^{\infty} v_m\left(x,t\right), \quad (33)$$

where

$$u_m\left(x,t\right) = \chi_m u_{m-1}\left(x,t\right) + c_0\int_0^t R_m\left(u_{m-1}\right)dt + c, \quad (34)$$

$$v_m\left(x,t\right) = \chi_m v_{m-1}\left(x,t\right) + c_1\int_0^t R_m\left(v_{m-1}\right)dt + c. \quad (35)$$

The first approximation gives

$$\begin{aligned} u_1\left(x,t\right) = c_0 t\left(-\frac{1}{2}ck^2\text{sech}^2\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right)\right) \\ \times\left(\frac{1}{2}ck\tanh\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right)+\frac{ck}{2}\right) \\ -\frac{1}{16}k^3\tanh\left(\frac{1}{2}\left(\log\left(b\right)+kx\right)\right)\text{sech}^2\left(\frac{1}{2}\left(\log\left(b\right)+kx\right)\right), \end{aligned} \quad (36)$$

$$\begin{aligned} v_1\left(x,t\right) = c_1 t\left(\frac{1}{2}\left(\begin{array}{c} \frac{1}{4}ck^4 tan\,h^2\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right)\text{sech}^2\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right) \\ -\frac{1}{8}ck^4 sec\,h^4\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right) \end{array}\right)\right) \\ +2\left(\frac{1}{32}ck^4 sec\,h^4\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right)\right)\text{sech}^2\left(\frac{1}{2}\left(\log\left(b\right)+kx\right)\right) \\ +\frac{1}{8}k^3\tanh\left(\frac{1}{2}\left(\log\left(b\right)+kx\right)\right)\text{sech}^2\left(\frac{1}{2}\left(\log\left(b\right)+kx\right)\right) \\ \times\left(\frac{1}{2}ck\tanh\left(\frac{1}{2}\left(-\log\left(b\right)-kx\right)\right)+\frac{ck}{2}\right). \end{aligned} \quad (37)$$

Hence the approximate solution of generalized Boussinesq-Burger equation(2) and (3) can be written as

$$u(x,t) = \frac{ck}{2} + \frac{ck}{2}\left(\frac{-kx-\ln b}{2}\right)$$
$$+\left\{\begin{array}{l} c_0 t\left(-\frac{1}{2}ck^2\operatorname{sech}^2\left(\frac{1}{2}\left(-\log(b)-kx\right)\right)\right)\left(\frac{1}{2}ck\tanh\left(\frac{1}{2}\left(-\log(b)-kx\right)\right)+\frac{ck}{2}\right) \\ -\frac{1}{16}k^3\tanh\left(\frac{1}{2}\left(\log(b)+kx\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\left(\log(b)+kx\right)\right) \end{array}\right\}, \tag{38}$$

$$v(x,t) = \frac{-k^2}{8}\operatorname{sech}^2\left(\frac{kx+\ln b}{2}\right)$$
$$+c_1 t\left(\frac{1}{2}\left(\begin{array}{l} \frac{1}{4}ck^4 tan\,h^2\left(\frac{1}{2}\left(-\log(b)-kx\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\left(-\log(b)-kx\right)\right) \\ -\frac{1}{8}ck^4 sec\,h^4\left(\frac{1}{2}\left(-\log(b)-kx\right)\right) \end{array}\right)\right)$$
$$+\ 2\left(\frac{1}{32}ck^4 sec\,h^4\left(\frac{1}{2}\left(-\log(b)-kx\right)\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\left(\log(b)+kx\right)\right)+$$
$$\frac{1}{8}k^3\tanh\left(\frac{1}{2}\left(\log(b)+kx\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\left(\log(b)+kx\right)\right)$$
$$\times\left(\frac{1}{2}ck\tanh\left(\frac{1}{2}\left(-\log(b)-kx\right)\right)+\frac{ck}{2}\right). \tag{39}$$

5. RESULTS AND DISCUSSION

The following Table-1 shows the comparisons of the absolute errors of Cubic Boussinesq equation obtained by using two terms approximation for HPM and HAM at different values of x and t for fixed C_0 . To show the effectiveness and accuracy of proposed schemes, we compared our results with the results obtained by HPM and obtained the L_2 and L_∞ error norm which is presented in Table- 2.

Similarly the tables -3 and 4 shows the comparisons of the absolute errors of Boussinesq-Burger equation obtained by using two terms approximation for HPM, LADM, OHAM, and HAM at different values of x and t for fixed C_0 and C_1 . To show the effectiveness and accuracy of proposed schemes, we compared our results with the results obtained by HPM, LADM, and OHAM and obtained the L_2 and L_∞ error norms which is presented in Tables 5, 6,7 and 8.

Figures 1 and 2 shows the comparison graphically between the approximate solutions obtained by HPM, HAM and the exact solutions for different values of x and for $t = 0.4$ and $t = 0.2$ which shows the reliability and efficiency of HAM as compared to HPM for the solution of Cubic Boussinesq equation. Figures 3-8 shows the comparison graphically between the approximate solutions of Boussinesq - Burger equationobtained by HPM, HAM and the exact solutions for different values of x and at $t = 0.5, t = 0.3$ and $t = 0.1$ which shows the reliability,accuracy and efficiency of HAM as compared to HPM for the solution of Boussinesq-Burger equation.

(x,t)	$\lvert u_{HPM} - u_{Exact}\rvert$	$\lvert u_{HAM} - u_{Exact}\rvert$
(-10 , 0.2)	6.4387E-6	6.4558E-6
(-10 , 0.4)	1.2974E-5	1.3025E-5
(-10 , 0.6)	1.9655E-5	1.9710E-5
(-10 , 0.8)	2.6448E-5	2.6432E-5
(-10 , 1.0)	3.3420E-5	3.3051E-5
(-5 , 0.2)	9.3572E-4	9.3834E-4
(-5 , 0.4)	1.8682E-3	1.8753E-3
(-5 , 0.6)	2.7968E-3	2.8004E-3
(-5 , 0.8)	3.7087E-3	3.6839E-3
(-5 , 1.0)	4.5780E-3	4.4780E-3
(0 , 0.2)	3.7701E-2	3.7558E-2
(0 , 0.4)	7.8710E-2	7.7635E-2
(0 , 0.6)	1.1924E-1	1.1471E-1
(0 , 0.8)	1.5319E-1	1.4074E-1
(0 , 1.0)	1.7209E-1	1.4294E-1
(5 , 0.2)	6.9448E-4	9.3334E-4
(5 , 0.4)	9.0571E-4	7.0169E-4
(5 , 0.6)	6.4019E-4	6.9778E-4
(5 , 0.8)	1.0301E-4	1.1969E-5
(5 , 1.0)	1.3319E-3	1.2124E-3
(10 , 0.2)	4.6256E-6	4.6800E-6
(10 , 0.4)	5.7393E-6	5.9570E-6
(10 , 0.6)	3.4450E-6	3.9547E-6
(10 , 0.8)	2.1679E-6	1.2911E-6
(10 , 1.0)	1.1022E-5	9.6640E-6

Table 1: The absolute errors in the solution of Cubic Boussinesq equation using three terms approximation for HPM and HAM at various points with $C_0 = -1.2$

6. CONCLUSION

In this paper, the Cubic Boussinesq equation and the Boussinesq-Burger equations has been solved by using Homotopy analysis method (HAM). The obtained results are then compared with the exact solutions as well as Homotopy perturbation method (HPM), Laplace adomian decomposition method (LADM) and Optimal homotopy asymptotic method(OHAM) . The accuracy and efficiency of HAM has been demon-

	HPM	HPM	HAM	HAM
x	L_2	L_∞	L_2	L_∞
-10	1.9787E-5	3.3420E-5	1.9734E-5	3.3051E-5
-5	2.7774E-3	4.5780E-3	2.7552E-3	4.4780E-3
0	1.1218E-1	1.7209E-1	1.0271E-1	1.4294E-1
5	7.3506E-4	1.3319E-3	7.1143E-4	1.2124E-3
10	5.3990E-6	1.1022E-5	5.1050E-6	9.6640E-6

Table 2: L_2 and L_∞ error norm for Cubic Boussinesq equation using three terms approximation for HPM and HAM at various points x

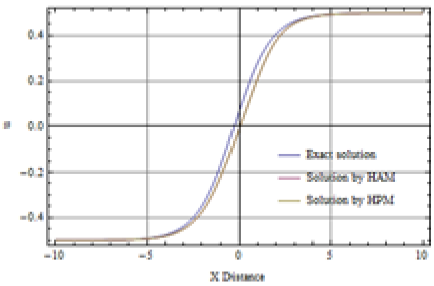

Figure 1: comparision of HPM solution and HAM solution with exact solution at $t = 0.4$

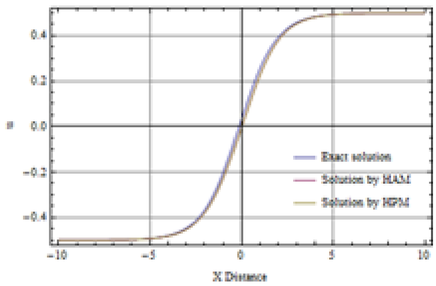

Figure 2: comparision of HPM solution and HAM solution with exact solution at $t = 0.2$

strated graphically as well as by tables to check the accuracy and efficiency of the method which is very useful for solving highly nonlinear problem as well as system of nonlinear PDE generally arising in real world problems. An advantage of HAM over perturbation methods and LADM is that it does not depends on small parameters. Similarly HAM requires less no. control parameter and it allows a good rate of convergence by suitably identifying convergence control parameters and as compared to OHAM. It explores the reliability and powerfulness of HAM and an easy way to control and adjust the convergence region for strongly nonlinear problems as compared to other analytical method.

(x,t)	$\|u_{HPM} - u_{Exact}\|$	$\|u_{LADM} - u_{Exact}\|$	$\|u_{OHAM} - u_{Exact}\|$	$\|u_{HAM} - u_{Exact}\|$
(0.1 , 0.1)	4.0376E-05	4.0376E-05	5.4357E-05	4.5588E-05
(0.1 , 0.2)	1.5774E-04	1.5774E-04	3.1723E-05	1.4119E-05
(0.1 , 0.3)	3.4620E-04	3.4620E-04	6.2001E-05	8.8306E-05
(0.1 , 0.4)	5.9952E-04	5.9952E-04	2.2059E-04	2.5567E-04
(0.1 , 0.5)	9.1119E-04	9.1119E-04	4.3752E-04	4.8137E-05
(0.2 , 0.1)	3.4524E-05	3.4524E-05	6.2743E-05	5.3739E-05
(0.2 , 0.2)	1.3393E-04	1.3393E-04	6.0619E-05	4.2611E-05
(0.2 , 0.3)	2.9167E-04	2.9167E-04	1.5265E-07	2.6859E-05
(0.2 , 0.4)	5.0096E-04	5.0096E-04	1.1186E-04	1.4787E-04
(0.2 , 0.5)	7.5475E-04	7.5475E-04	2.6837E-05	3.1338E-04
(0.3 , 0.1)	2.8060E-05	2.8060E-05	7.1361E-05	6.2159E-05
(0.3 , 0.2)	1.0766E-04	1.0766E-04	9.1179E-05	7.2774E-05
(0.3 , 0.3)	2.3177E-04	2.3177E-04	6.6498E-05	3.8891E-05
(0.3 , 0.4)	3.9310E-04	3.9310E-04	4.5809E-06	3.2228E-05
(0.3 , 0.5)	5.8424E-04	5.8424E-04	8.7131E-05	1.3314E-04
(0.4 , 0.1)	2.1056E-05	2.1056E-05	8.0069E-05	7.07071E-05
(0.4 , 0.2)	7.9349E-05	7.9349E-05	1.2290E-04	7.2774E-05
(0.4 , 0.3)	1.6743E-04	1.6743E-04	1.3594E-04	3.8891E-05
(0.4 , 0.4)	2.7779E-04	2.7779E-04	1.2679E-04	8.9328E-05
(0.5 , 0.5)	4.0253E-04	4.0253E-04	1.0359E-04	5.6239E-05
(0.5 , 0.1)	1.3644E-05	1.3644E-05	8.8711E-05	7.9238E-05
(0.5 , 0.2)	4.9424E-05	4.9424E-05	1.5522E-04	1.3627E-04
(0.5 , 0.3)	9.9848E-05	9.9848E-05	2.0722E-04	1.7880E-04
(0.5 , 0.4)	1.5694E-04	1.5694E-04	2.5248E-04	2.1459E-04
(0.5 , 0.5)	2.1296E-04	2.1296E-04	2.9882E-04	2.5143E-04

Table 3: The absolute errors in the solution of Boussinesq-Burger equation $u(x,t)$ using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points with $c = 1/2$,$k = -1$ and $b - 2$,$c_0 = -1.015$ and and $c_1 = -0.92$

(x,t)	$\|u_{HPM} - u_{Exact}\|$	$\|u_{LADM} - u_{Exact}\|$	$\|u_{OHAM} - u_{Exact}\|$	$\|u_{HAM} - u_{Exact}\|$
(0.1, 0.1)	5.49815E-05	5.49815E-05	5.30293E-05	7.7140E-05
(0.1, 0.2)	2.24577E-04	2.24577E-04	8.55529E-06	3.9666E-05
(0.1, 0.3)	5.15476E-04	5.15476E-04	1.91443E-04	1.1911E-04
(0.1, 0.4)	9.33935E-04	9.33935E-04	5.01892E-04	4.0545E-04
(0.1, 0.5)	1.48572E-03	1.48572E-03	9.45668E-04	8.2512E-04
(0.2, 0.1)	6.17326E-05	6.17326E-05	3.12906E-05	5.2056E-05
(0.2, 0.2)	2.51026E-04	2.51026E-04	6.49797E-05	2.3450E-05
(0.2, 0.3)	5.73647E-04	5.73647E-04	2.94577E-04	2.3280E-04
(0.2, 0.4)	1.03482E-03	1.03482E-03	6.62723E-04	5.7966E-04
(0.2, 0.5)	1.63915E-03	1.63915E-03	1.17404E-03	1.0702E-03
(0.3, 0.1)	6.75811E-05	6.75811E-05	8.76219E-06	2.5804E-05
(0.3, 0.2)	2.73710E-04	2.73710E-04	1.21023E-04	8.6904E-05
(0.3, 0.3)	6.23007E-04	6.23007E-04	3.93977E-04	3.4283E-04
(0.3, 0.4)	1.11945E-03	1.11945E-03	8.14078E-04	7.4591E-04
(0.3, 0.5)	1.76633E-03	1.76633E-03	1.38462E-03	1.2994E-03
(0.4, 0.1)	7.23025E-05	7.23025E-05	1.40750E-05	1.0773E-06
(0.4, 0.2)	2.91751E-04	2.91751E-04	1.75296E-04	1.4930E-04
(0.4, 0.3)	6.61637E-04	6.61637E-04	4.86954E-04	4.4796E-04
(0.4, 0.4)	1.18453E-03	1.18453E-03	9.51621E-04	8.9963E-04
(0.4, 0.5)	1.86226E-03	1.86226E-03	1.57112E-03	1.5016E-03
(0.5, 0.1)	7.57082E-05	7.57082E-05	3.67202E-05	2.8017E-05
(0.5, 0.2)	3.04426E-04	3.04426E-04	2.26450E-04	2.0904E-04
(0.5, 0.3)	6.87978E-04	6.87978E-04	5.71013E-04	5.4490E-04
(0.5, 0.4)	1.22742E-03	1.22742E-03	1.07147E-03	1.0367E-03
(0.5, 0.5)	1.92305E-03	1.92305E-03	1.72811E-03	1.6846E-03

Table 4: The absolute errors in the solution of Boussinesq-Burger equation $v(x,t)$ using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points with $c = 1/2$,$k = -1$ and $b - 2$,$c_0 = -1.015$ and and $c_1 = -0.92$

Error in case of two terms approximation for $u(x,t)$

x	HPM	HPM	HAM	HAM
0.1	5.1693E-04	5.1693E-04	2.2266E-04	1.7702E-04
0.2	4.3008E-04	4.3008E-04	1.3575E-04	1.1689E-04
0.3	3.3525E-04	3.3525E-04	7.1331E-05	6.7838E-05
0.4	2.3407E-04	2.34.7E-04	1.1549E-04	8.5673E-05
0.5	1.2851E-04	1.2851E-04	2.1351E-04	1.7207E-04

Table 5: L_2 error norm for Boussinesq-Burger equation using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points x

x	HPM	HPM	HAM	HAM
0.1	9.1119E-04	9.1119E-04	4.3752E-04	2.5567E-04
0.2	7.5475E-04	7.5475E-04	1.1186E-04	3.1338E-04
0.3	5.8424E-04	5.8424E-04	9.1179E-05	1.3314E-04
0.4	4.0253E-04	4.02530E-04	1.3594E-04	8.9328E-05
0.5	2.1296E-04	2.1296E-04	2.9882E-04	2.5143E-04

Table 6: L_∞ error norm for Boussinesq-Burger equation using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points x. Error in case of two terms approximation for $u(x,t)$

x	HPM	HPM	HAM	HAM
0.1	6.4294E-04	6.4294E-04	4.8697E-04	2.9330E-04
0.2	9.1143E-04	9.1143E-04	6.1794E-04	3.9153E-04
0.3	9.8394E-04	9.8394E-04	7.4160E-04	5.0018E-04
0.4	1.0392E-03	1.0392E-03	8.5347E-04	6.0082E-04
0.5	1.0748E-03	1.0748E-03	9.5006E-04	7.6663E-04

Table 7: L_2 error norm for Boussinesq-Burger equation using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points x. Error in case of two terms approximation for $v(x,t)$

x	HPM	HPM	HAM	HAM
0.1	1.4857E-03	1.4857E-03	9.4567E-04	8.2512E-04
0.2	1.6592E-03	1.6592E-03	1.1740E-03	1.0702E-03
0.3	1.7665E-03	1.7665E-03	1.3846E-03	1.2994E-03
0.4	1.8625E-03	1.8625E-03	1.5711E-03	1.5061E-03
0.5	1.9250E-03	1.9250E-03	1.7281E-03	1.6846E-03

Table 8: L_∞ error norm for Boussinesq-Burger equation using two terms approximation for HPM, LADM [29], OHAM [30] and HAM at various points x. Error in case of two terms approximation for $v(x,t)$

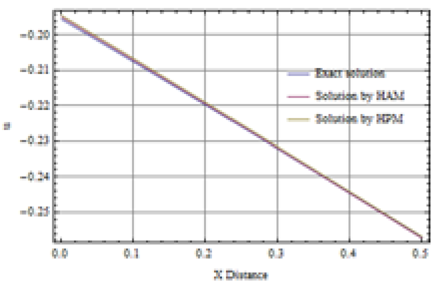

Figure 3: Comparison of HAM and HPM for solution of $u(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1 = -0.92$, $t = 0.5$

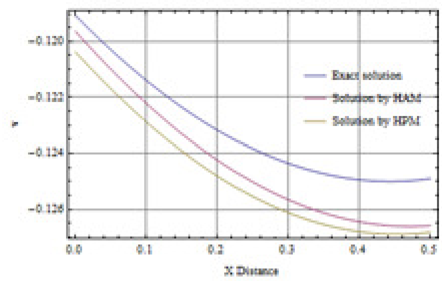

Figure 4: Comparison of HAM and HPM for solution of $v(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1 = -0.92$, $t = 0.5$

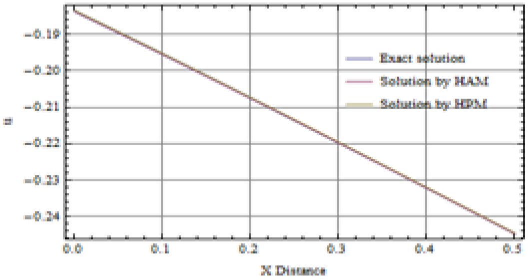

Figure 5: Comparison of HAM and HPM for solution of $u(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1=-0.92$, $t=0.3$

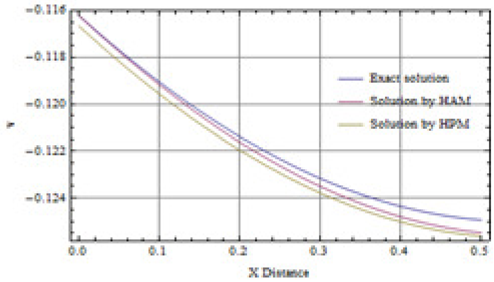

Figure 6: Comparison of HAM and HPM for solution of $v(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1=-0.92$, $t=0.3$

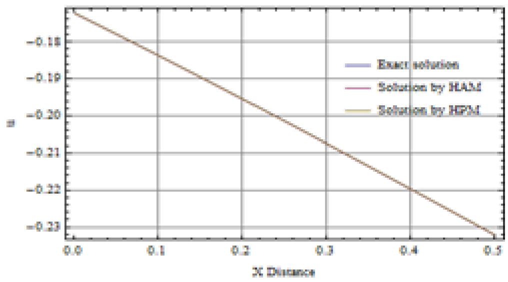

Figure 7: Comparison of HAM and HPM for solution of $u(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1=-0.92$, $t=0.1$

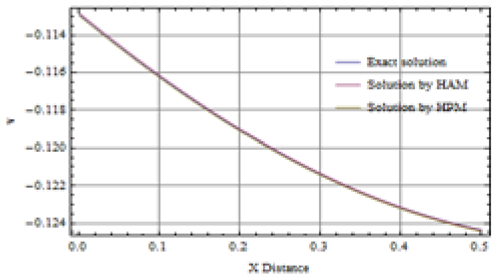

Figure 8: Comparison of HAM and HPM for solution of $v(x,t)$ for Boussinesq - Burger equation when $C_0 = -1.015$ and$C-1=-0.92$, $t=0.1$

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