Communications in Applied Analysis **20** (2016), 457-465

COMPLETE SYMMETRIC FUNCTIONS AND k-FIBONACCI NUMBERS

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ABSTRACT: In this paper, we introduce a operator in order to derive some new symmetric properties of k-Fibonacci numbers and Tchebychev polynomials of second kind. By making use of the operator defined in this paper, we give some new generating functions for k- Fibonacci numbers and Tchebychev polynomials of second kinds.

AMS Subject Classification: 05E05, 11B39

1. INTRODUCTION AND NOTATIONS

In mathematics, orthogonal polynomials consist of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebychev polynomials, and the Legendre polynomials cf. [1], [5].

Let k and n be two positive integer and $\{b_1, b_2, \ldots, b_n\}$ are set of given variables, recall [11] that the k-th elementary symmetric function $e_k(b_1, b_2, \ldots, b_n)$ and the k-th complete homogeneous symmetric function $h_k(b_1, b_2, \ldots, b_n)$ are defined respectively by

$$e_k(b_1, b_2, \ldots, b_n) = \sum_{i_1+i_2+\cdots+i_n=k} b_1^{i_1} b_2^{i_2} \cdots {}_n^{i_n} \ , \ 0 \le k \le n,$$

with $i_1, i_2, \ldots, i_n = 0$ or 1.

$$h_k(b_1, b_2, \ldots, b_n) = \sum_{i_1+i_2+\cdots+i_n=k} b_1^{i_1} b_2^{i_2} \cdots {}_n^{i_n} \ , \ 0 \le k \le n,$$

Received: September 12, 2016

with $i_1, i_2, \ldots, i_n \geq 0$.

First, we set $e_0(b_1, b_2, \ldots, b_n) = 1$ and $h_0(b_1, b_2, \ldots, b_n) = 1$ (by convention). For $k > n$ or $k < 0$, we set $e_k(b_1, b_2, \ldots, b_n) = 0$ and $h_k(b_1, b_2, \ldots, b_n) = 0$.

Definition 1. *[3] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{p_ip_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}. \qquad (1.2)$$

Definition 2. *[3] The symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{p_1p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathbb{N}. \qquad (1.3)$$

Proposition 1. *[2] Let $P = \{p_1, p_2\}$ an alphabet, we define the operator $\delta^k_{p_1p_2}$ as follows:*

$$\delta^k_{p_1p_2} g(p_1) = h_{k-1}(p_1, p_2) g(p_1) + p_2^k \partial_{p_1p_2} g\ (p_1), \text{ for all } k \in \mathbb{N}.$$

2. THE *K*-FIBONACCI NUMBERS AND PROPERTIES

The k-Fibonacci numbers have been defined in [9] for any number k as follows.

Definition 3. *[8] For any positive real number k, the k-Fibonacci numbers, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1, \qquad (2.1)$$

with initial conditions $F_{k,0} = 1; F_{k,1} = 1$.

Note that if k is a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1 \text{ if } n = 0 \\ x \text{ if } n = 1 \\ xF_n(x) + F_{n-1}(x) \text{ if } n > 1 \end{cases}.$$

Particular cases of the k-Fibonacci numbers are

• If $k = 1$, the classical Fibonacci numbers is obtained:
$F_0 = 1, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$:
$\{F_n\}_{n\in\mathbb{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$

• If $k = 2$, the Pell numbers appears:
$P_0 = 0, P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$:

$\{P_n\}_{n\in\mathbb{N}} = \{0, 1, 2, 5, 12, 29, 70, \dots\}$.

• If $k = 3$, the following sequence appears:
$H_0 = 0, H_1 = 1$, and $H_{n+1} = 3H_n + H_{n-1}$ for $n \geq 1$:
$\{H_n\}_{n\in\mathbb{N}} = \{0, 1, 3, 10, 33, 109, \dots\}$.

The well-known Binet's formula in the *Fibonacci numbers theory* [9, 10] allows us to express the k-Fibonacci number in function of the roots r_1 and r_2 of the characteristic equation, associated to the recurrence relation (2.1):

$$r^2 = kr + 1. \tag{2.2}$$

Proposition 2. *(Binet's formula) The nth k-Fibonacci number is given by*

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where $r_1,\ r_2$ are the roots of the characteristic equation (2.2) and $r_1 > r_2$.

Proof. The roots of the characteristic equation (2.2) are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$.

Note that, since $k > 0$, the
$r_2 < 0 < r_1$ and $|r_2| < |r_1|$,
$r_1 + r_2 = k$ and $r_1.r_2 = -1$,
$r_1 - r_2 = \sqrt{k^2 + 4}$. □

If σ denotes the positive root of the characteristic equation, the general term may be written in the form [9]

$$F_{k,n} = \frac{\sigma^n - \sigma^{-n}}{\sigma + \sigma^{-1}},$$

and the limit of the quotient of two terms is

$$\lim_{n\to\infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma^r.$$

In addition, the general term of the k-Fibonacci numbers may be obtained by the formula [8]:

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-2i-1}(k^2+4)^i,$$

or, equivalently, by

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i}.$$

3. ON THE GENERATING FUNCTIONS OF SOME NUMBERS

In this part, we are now in a position to provide Theorem 1. Also we derive the new generating functions of the products of some known numbers.

Theorem 1. *Given two alphabets* $P = \{p_1, p_2\}$ *and* $B = \{b_1, b_2, b_3\}$, *then*

$$\sum_{n=0}^{\infty} h_n(b_1,b_2,b_3)h_n(p_1,p_2)t^n = \frac{e_0(b_1,b_2,b_3) - p_1p_2e_2(b_1,b_2,b_3)t^2 - p_1p_2e_3(b_1,b_2,b_3)\ h_1(p_1,p_2)t^3}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n\right)}. \quad (3.1)$$

Proof. Let $\sum_{n=0}^{\infty} h_n(b_1,b_2,b_3)p_1^nt^n$ and $\sum_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n$ be two sequences such that $\sum\limits_{n=0}^{\infty} h_n(b_1,b_2,b_3)t^n = \frac{1}{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)t^n}$, the left-hand side of the formula (3.1) can be written as:

$$\begin{aligned}\delta_{p_1p_2}g(p_1) &= \delta_{p_1p_2}\left(\sum_{n=0}^{\infty} h_n(b_1,b_2,b_3)p_1^nt^n\right)\\ &= \sum_{n=0}^{\infty} h_n(b_1,b_2,b_3)\ h_n(p_1,p_2)t^n,\end{aligned}$$

while the right-hand side can be expressed as:

$$\begin{aligned}\partial_{p_1p_2}g(p_1) &= \frac{1}{p_1-p_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n} - \frac{1}{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n}\right)\\ &= \frac{1}{p_1-p_2}\left(\frac{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n - \sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n\right)}\right)\\ &= \frac{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)\frac{p_2^n-p_1^n}{p_1-p_2}t^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n\right)}\\ &= -\frac{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)\ h_{n-1}(p_1,p_2)t^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^nt^n\right)}.\end{aligned}$$

By Proposition 1, it follows that

$$\delta_{p_1p_2}g\ (p_1) = h_0(p_1,p_2)g(p_1) + p_2\partial_{p_1p_2}g(p_1)$$

$$= \frac{h_0(p_1,p_2)}{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^n t^n} - p_2 \frac{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)h_{n-1}(p_1,p_2)t^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^n t^n\right)}$$

$$= \frac{\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)\left[p_2^n\ h_0(p_1,p_2) - p_2\ h_{n-1}(p_1,p_2)\right]t^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^n t^n\right)}.$$

Hence, we have that

$$\delta_{p_1p_2} g(p_1) = \frac{\sum\limits_{n=0}^{3} e_n(b_1,b_2,b_3)\left[p_2^n\ h_0(p_1,p_2) - p_2\ h_{n-1}(p_1,p_2)\right]t^n}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^n t^n\right)} + 0$$

$$= \frac{e_0(b_1,b_2,b_3) - p_1p_2e_2(b_1,b_2,b_3)t^2 - p_1p_2e_3(b_1,b_2,b_3)\ h_1(p_1,p_2)t^3}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1,b_2,b_3)p_2^n t^n\right)}.$$

This completes the proof. □

3.1. THE CASE $B = \{1, 0, 0\}$

Based on Theorem 1, we deduce the following Lemmas.

Lemma 1. *Given an alphabet $P = \{p_1, -p_2\}$, we have*

$$\sum_{n=0}^{\infty} h_n(p_1,[-p_2])t^n = \frac{e_0(1,0,0)}{e_0(p_1,[-p_2]) - e_1(p_1,[-p_2])t + e_2(p_1,[-p_2])t^2}. \tag{3.2}$$

Lemma 2. *Given an alphabet $P = \{p_1, -p_2\}$, we have*

$$\sum_{n=0}^{\infty} h_{n+1}(p_1,[-p_2])t^n = \frac{e_1(p_1,[-p_2]) - e_2(p_1,[-p_2])t}{e_0(p_1,[-p_2]) - e_1(p_1,[-p_2])t + e_2(p_1,[-p_2])t^2}. \tag{3.3}$$

Taking $p_1 - p_2 = 1$ and $p_1p_2 = 1$ in (3.2) and (3.3) , we obtain the generating functions given by Boussayoud et al [2] which arises

1. The generating function of the Fibonacci numbers F_n.

2. The generating function of the Lucas numbers L_n.

Taking $p_1 - p_2 = 2$ and $p_1p_2 = 1$ in (3.2) and (3.3) , we obtain the generating functions given by Boussayoud et al [2] which arises:

1. The generating function of the Pell numbers P_n.

2. The generating function of the Pell numbers Q_n.

Choosing p_1 and p_2 such that $\begin{cases} p_1p_2 = 1 \\ p_1 - p_2 = k \end{cases}$ and substituting in (3.2) and (3.3) we end up with

$$\sum_{n=0}^{\infty} h_n(p_1, [-p_2])t^n = \frac{1}{1 - kt - t^2}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} h_{n+1}(p_1, [-p_2])t^n = \frac{k + t}{1 - kt - t^2}, \tag{3.5}$$

from which we have the following theorem.

Theorem 2. *For $n \in \mathbb{N}$, the generating function of the k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{k,n}t^n = \frac{1}{1 - kt - t^2}.$$

Multiplying the equation (3.4) by $(2 + k^2)$ and subtract it from (3.5) by (k), we obtain

$$\sum_{n=0}^{\infty} \left[(2 + k^2)h_n(p_1, [-p_2]) - kh_{n+1}(p_1, [-p_2])\right] t^n = \frac{2 - kt}{1 - kt - t^2},$$

from which we have the following theorem.

Theorem 3. *For $n \in \mathbb{N}$, the generating function of the k-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{k,n}t^n = \frac{2 - kt}{1 - kt - t^2}.$$

3.2. THE CASE $B = \{b_1, b_2, 0\}$

Theorem 4. *Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2\}$, we have*

$$\sum_{n=0}^{\infty} h_n(b_1, b_2)h_n(p_1, p_2)t^n = \frac{1 - p_1p_2b_1b_2t^2}{\left(\sum\limits_{n=0}^{\infty} e_n(b_1, b_2)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} e_n(b_1, b_2)p_2^nt^n\right)}. \tag{3.6}$$

Case 1. Substituting $p_1 = b_1 = 1$, $p_2 = x$ and $b_2 = y$ in (3.6), we obtain the following identity of Ramanunja [3]

$$\sum_{n=0}^{\infty} [1 + x + \cdots + x^n] [1 + y + \cdots + y^n] t^n = \frac{1 - xyt^2}{[(1 - t)(1 - xt)(1 - yt)(1 - xyt)]}.$$

Case 2. Replacing p_2 by $(-p_2)$ and b_2 by $(-b_2)$ in (3.6) yields

$$\sum_{n=0}^{\infty} h_n(b_1,[-b_2])h_n(p_1,[-p_2])t^n = \frac{1-p_1p_2b_1b_2t^2}{(1-b_1p_1t)(1+b_2p_1t)(1+b_1p_2t)(1-b_2p_2t)}. \quad (3.7)$$

Firstly, the substitutions of $\begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases}$ and $\begin{cases} p_1-p_2=k \\ p_1p_2=1 \end{cases}$ in (3.7) give

$$\sum_{n=0}^{\infty} h_n(b_1,[-b_2])h_n(p_1,[-p_2])t^n = \frac{1-t^2}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4},$$

which represents a new generating function for k-Fibonacci numbers of second order.

From those applications, we deduce the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2 t^n = \frac{1-t^2}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}. \quad (3.8)$$

Put $k=2$ in the relationship (3.8) we get

$$\sum_{n=0}^{\infty} P_n^2 t^n = \frac{1-t^2}{1-4t-10t^2-4t^3+t^4},$$

which represents a new generatings functions for Pell numbers of second order.

Secondly, by making the following restrictions: $p_1-p_2=k$, $p_1p_2=1$, $4b_1b_2=-1$, and by replacing (b_1-b_2) by $2(b_1-b_2)$ in (3.7), we get a new generating function, involving the product of k-Fibonacci numbers with Tchebychev polynomial of second kind as follows:

$$\sum_{n=0}^{\infty} h_n(2b_1,[-2b_2])h_n(p_1,[-p_2])t^n = \frac{1+t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}.$$

Thus we conclude with the following theorem.

Theorem 6. *We have the following a new generating function of the product of k-Fibonacci numbers and Tchebychev polynomials of second kind as*

$$\sum_{n=0}^{\infty} F_{k,n}U_n(b_1-b_2)t^n = \frac{1+t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}. \quad (3.9)$$

Put $k = 2$ in the relationship (3.9) we get

$$\sum_{n=0}^{\infty} P_n U_n(b_1 - b_2)t^n = \frac{1 + t^2}{1 - 4(b_1 - b_2)t + (6 - 4(b_1 - b_2)^2)t^2 + 4(b_1 - b_2)t^3 + t^4},$$

which represents a new generating function, involving the product of Pell with Tchebychev polynomial of second kind.

4. CONCLUSION

In this paper, we have derived new theorems in order to determine generating functions of k-Fibonacci numbers and Tchebychev polynomials of the second kinds. The derived theorems and Lemmas are based on symmetric functions and products of these numbers.

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