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A RELOOK AT QUEUEING-INVENTORY SYSTEM WITH RESERVATION, CANCELLATION AND COMMON LIFE TIME

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ABSTRACT: In this paper, we study an M/M/1 queue with a storage system having capacity S which have a common life time (CLT), exponentially distributed. On realization of common life time or the first time inventory level drops to zero in a cycle whichever occurs first, a replenishment order is placed so as to bring the inventory level back to S (zero lead time). Customers arrive to the system according to a Poisson process and their service time is exponentially distributed. Reservation of items and cancellation of sold items is permitted before the realization of common life time. Cancellation takes place according to an exponential distribution. In this paper we assume that the time required to cancel the reservation is negligible. When the inventory level becomes zero through service completion or CLT realization, a replenishment order is placed which is realized instantly. We first derive the stationary joint distribution of the queue length and the on-hand inventory in product form. Using the joint distribution, we introduce long-run performance measures and a revenue function. The case of positive lead time is also investigated. Numerical illustrations are provided.

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1. INTRODUCTION

In this paper we consider a single server queueing-inventory system with reservation of inventoried item, cancellation of the reserved item and common life time (CLT) of items in the inventory. Advance reservation/ purchase of an item for future use and, sometimes purchased item may be returned, could be seen in several day to day life scenario. Each item on hand may have an expiry date which in some cases is common to all. Typical example are flight/train/bus seats for travel, medicines with same expiry date, manufactured items in a batch and so on. The seats are considered as inventory. In this context once the flight/train/bus departs, the one holding inventory, but not using it, will loss the inventory as well.

Queueing inventory with reservation, cancellation, common life time and retrial is introduced by Krishnamoorthy et al. [8]. They assumed that a customer, on arrival to an idle server with at least one item in inventory, is immediately taken for service or else he joins a buffer of varying size depending on the number of items in the inventory. If there is no item in the inventory the arriving customer first queues up in a finite waiting space of capacity K. When it overflows an arrival goes to an orbit of infinite capacity with probability p or is lost forever with probability $1-p$. From the orbit he retries for service. However they fail to produce a product form solution for want of an appropriate blocking set.

For the model discussed in Krishnamoorthy et al. [7], the buffer, waiting space and orbit are dispensed and a single queue is considered. They assumed that arrival follows a Poisson process, service time exponentially distributed and inventoried items have a common life time which is also exponentially distributed. Reservation (purchase) and cancellation of purchased item is allowed. On realization of common life time items in stock are discarded and an order is placed to bring the inventory level back to S. Lead time follows exponential distribution. Under the crucial assumption that no customer joins the system when inventory level is zero, they established the stochastic decomposition property of the system state.

In the present paper we follow the same assumptions as in Krishnamoorthy et al. [7] and further assume that on realization of CLT, or if at a service completion epoch the storage system turns out to be empty, whichever occurs first then the inventory is brought back to the maximum level S instantaneously through a replenishment. We also consider the case of positive lead time which is exponentially distributed and assume that no customer joins the system when inventory is zero. In both cases we produce product form solutions.

Before proceeding further, we provide a brief survey on queueing-inventory models and some definitions. Queueing-inventory models (inventory with positive service time) is introduced by Sigman and Levi [20]. This was followed by contributions from Berman et al. [2], Berman and Kim [3], Berman and Sapna [4], Arivarignan et al. [1], Krishnamoorthy et al. [11] and by several other researchers. A survey of work in this area is given in Krishnamoorthy et al. [9]. Schwarz et al. [18] stand out as the first significant contribution providing stochastic decomposition of the system state of a queueing-inventory problem. Krishnamoorthy and Viswanath [10] brings in production of items for inventory replenishment, thereby subsuming Schwarz et al. [18]. The latter is also subsumed by Saffari et al. [16] in that the lead time is arbitrarily distributed. Further contribution with stochastic decomposition results could be found in Schwarz et al. [19], Schwarz and Daduna [17], Krenzler and Daduna ([5], [6]), Otten et al. [15].

A discrete time (s, S) inventory model in which the stored items have a random common life time (CLT) with a discrete phase type distribution, where demands arrive in batches following a discrete phase type renewal process is considered by Lian et al.[13].

Definition 1.1. A ***cycle*** *is the time starting from the maximum inventory S in stock at an epoch, until the next epoch of replenishment, that is, duration between two consecutive S to S transitions.*

The end of a cycle and hence the beginning of the next cycle can be either due to CLT realization or by a service completion when there was just one item left in the inventory (the customer completing service, walks away with this item), whichever occurs first.

We define two types of events that causes the beginning of a new cycle. We call these two events A and B, respectively.

Definition 1.2. ***A Event*** A event is the one, occurrence of which causes the end of a cycle in the following way: suppose a service is going on with just one item of inventory left. Assume that neither CLT realization nor a cancellation takes place before this service is completed. Thus at the end of the present service the customer walks away with the single item left in the inventory.

If this happens for the first time starting from the moment the inventory is replenished most recently, we refer to it as **A event**. This means that we don't allow cancellation once the inventory level goes down to zero.

Definition 1.3. ***B Event*** When a cycle ends with (and so the new cycle begins with) occurrence of CLT we say that a B event has occurred resulting in the cycle completion.

The significance of the model rests in the fact that a replenishment is triggered by either realization of CLT or by a demand when there is only one item left in the inventory, whichever occurs first. This means that the cycle time (length of a cycle) is given by min(exp(α), time until inventory level drops to zero from S (consequent to replenishment)). The distribution of this, which is phase type, will be derived at a later stage in this paper.

The article is organized as follows. In Section 2, the model under study is described. Section 3 provides the steady state analysis of the model. Some important performance measures, including the probability that next cycle starts with service completion/realization of common life time, are derived in this section. In Section 4, we consider the case of positive lead time. Numerical examples and cost analysis are discussed in Section 5.

In the sequel we need the following notations and abbreviations:

- S^* : Inventory level on realization of common life time (after the replenishment. This is same as S; however, just to distinguish the beginning of the next cycle we use it as a purely temporary notation).
- $\mathbf{e}$ = Column vector of $1'$s of appropriate order.
- I = Identity matrix of appropriate order.
- $CTMC$: Continuous time Markov chain.
- $LIQBD$: Level independent quasi-birth and death process.
- CLT : Common life time

2. MATHEMATICAL FORMULATION

We have a single server queueing-inventory system with a storage system having S items at the beginning of a cycle. Customers arrive according to a Poisson process of rate λ, each demanding exactly one unit of the item. To deliver one unit of the item to the customer in service, it requires an exponentially distributed amount of time with parameter μ. The inventoried items have a common life time (CLT) which means that they all perish together on realization of this time. We assume that this common life time is exponentially distributed with parameter α. On realization of CLT or the first time inventory level hits zero for the first time in the cycle, whichever occurs first, the inventory reaches its maximum level S (denoted by S^* for identification purpose) through an instantaneous replenishment for the next cycle. In addition the possibility of cancellation of purchased item (return of the item with a penalty) is introduced here. Inter cancellation time follows exponential distribution with parameter $i\beta$ when there are $(S-i)$ items present in the inventory.

3. STEADY STATE ANALYSIS

In this section we analyze the queueing-inventory model described in Section 2 in steady state. Let $N(t)$ and $I(t)$ denote, respectively, the number of customers in the system and the number of items in the inventory. The process $\Omega = \{(N(t), I(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by

$$\{(n,i), n \geq 0, 1 \leq i \leq S\} \bigcup \{(n,S^*), n \geq 0\}.$$

The transition rates are:

(a) Transitions due to arrival:

$$\begin{aligned} &(n,i) \to (n+1,i): && \text{rate } \lambda \quad \text{for } n \geq 0, \quad 1 \leq i \leq S, \\ &(n,S^*) \to (n+1,S^*): && \text{rate } \lambda \quad \text{for } n \geq 0. \end{aligned}$$

(b) Transitions due to service completions:

$$\begin{aligned} &(n,i) \to (n-1,i-1): && \text{rate } \mu \quad \text{for } n \geq 1, \quad 2 \leq i \leq S, \\ &(n,1) \to (n-1,S^*): && \text{rate } \mu \quad \text{for } n \geq 1, \\ &(n,S^*) \to (n-1,S-1): && \text{rate } \mu \quad \text{for } n \geq 1. \end{aligned}$$

(c) Transitions due to common life time realization:

$$(n,i) \to (n,S^*): \quad \text{rate } \alpha \quad \text{for } n \geq 0, 1 \leq i \leq S.$$

(d) Transition due to cancellation:

$$(n,i) \to (n,i+1): \quad \text{rate } (S-i)\beta \quad \text{for } n \geq 0, 1 \leq i \leq S-1.$$

Other transitions have rate zero.

Thus the infinitesimal generator of Ω is of the form

$$\mathcal{Q} = \begin{pmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.1}$$

Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $S+1$.

Entries of A_0 are given in (a); that of A_2 are given in (b) and those in $A_{0,0}$ and A_1 correspond to transition rates given by (c) and (d). In addition diagonal entries in A_{00} and A_1 are non-positive, having numerical value equal to but with negative sign the sum of other elements of the same row found in A_{00}, A_0, A_1 and A_2. All other transitions have rate zero.

3.1. STABILITY CONDITION

Let $\boldsymbol{\pi}$ be the steady state probability vector of $A = A_0 + A_1 + A_2$, where

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \left(\begin{array}{cccccccc} 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \\ b_{S-1} & a_{S-1} & & & & & & \alpha+\mu \\ \mu & b_{S-2} & a_{S-2} & & & & & \alpha \\ & \mu & b_{S-3} & a_{S-3} & & & & \alpha \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & \mu & b_2 & a_2 & & \alpha \\ & & & & \mu & b_1 & a_1 & \alpha \\ & & & & & \mu & b_0 & \alpha \\ & & & & & \mu & & -\mu \end{array} \right),$$

with $b_i = -(\mu + \alpha + i\beta), 0 \leq i \leq S-1$ and $a_j = j\beta, 1 \leq j \leq S-1$. That is, $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} A = \mathbf{0}, \quad \boldsymbol{\pi}\mathbf{e} = 1. \tag{3.2}$$

In the sequel, let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_S, \pi_{S^*})$.

The following theorem establishes the stability condition of the queueing-inventory system under study.

Theorem 3.1. *The queueing-inventory system under study is stable if and only if* $\lambda < \mu$.

Proof. The queueing-inventory system under study with the *LIQBD* type generator given in (3.1) is stable if and only if (see Neuts [14])

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{3.3}$$

Note that from the transition rates (a) (which give the elements of A_0, and (b) (which give the form of A_2), we get

$$\boldsymbol{\pi} A_0 \mathbf{e} = \lambda(\pi_1 + ... + \pi_S + \pi_{S^*}) \text{ and } \boldsymbol{\pi} A_2 \mathbf{e} = \mu(\pi_1 + ... + \pi_S + \pi_{S^*}). \tag{3.4}$$

From the normalizing condition we have $\pi_1 + ... + \pi_S + \pi_{S^*} = 1$.

Substituting this expression into (3.4) and using (3.3) we get the stated result. □

3.2. STEADY STATE PROBABILITY VECTOR

Let $\mathbf{x}$ be the steady state probability vector of $\mathcal{Q}$. Then $\mathbf{x}$ must satisfy the set of equations

$$\mathbf{x}\mathcal{Q} = \mathbf{0}, \quad \mathbf{xe} = 1. \tag{3.5}$$

Note that the vector $\mathbf{x}$ partitioned as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$, is such that the i^{th} component of $\mathbf{x}_n$ gives the steady state probability that there are n customers in the system and i items in the inventory. Then the above set of equations reduce to:

$$\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 = 0, \tag{3.6}$$

$$\mathbf{x}_{n-1} A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1} A_2 = 0, \quad n \geq 1. \tag{3.7}$$

For finding the steady state probability vector of the $CTMC$ Ω, we first consider the system with negligible service time. Thus the infinitesimal generator is given by

$$\tilde{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \left(\begin{array}{cccccccc} 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \\ d_{S-1} & a_{S-1} & & & & & & \alpha+\lambda \\ \lambda & d_{S-2} & a_{S-2} & & & & & \alpha \\ & \lambda & d_{S-3} & a_{S-3} & & & & \alpha \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & \lambda & d_2 & a_2 & & \alpha \\ & & & & \lambda & d_1 & a_1 & \alpha \\ & & & & & \lambda & d_0 & \alpha \\ & & & & & \lambda & & -\lambda \end{array} \right)$$

with $d_i = -(\lambda + \alpha + i\beta), 0 \leq i \leq S-1$ and $a_j = j\beta, 1 \leq j \leq S-1$.

Let $\boldsymbol{\xi} = (\xi_1, ..., \xi_S, \xi_{S^*})$ be the steady state vector of $\tilde{A}$. Then $\boldsymbol{\xi}$ satisfies the equations

$$\boldsymbol{\xi}\tilde{A} = \mathbf{0}, \qquad \boldsymbol{\xi}\mathbf{e} = 1. \tag{3.8}$$

From $\boldsymbol{\xi}\tilde{A} = 0$ we have

$$\begin{aligned} -(\lambda + \alpha + (S-1)\beta)\xi_1 + \lambda\xi_2 &= 0, \\ -(\lambda + \alpha + (S-i+1)\beta)\xi_{i-1} + (S-i)\beta\xi_i + \lambda\xi_{i+1} &= 0, \quad 2 \leq i \leq S-2 \\ 2\beta\xi_{S-2} - (\lambda + \alpha + \beta)\xi_{S-1} + \lambda\xi_S + \lambda\xi_{S^*} &= 0, \\ \beta\xi_{S-1} - (\lambda + \alpha)\xi_S &= 0, \\ \alpha(\xi_1 + ... + \xi_S) + \lambda\xi_1 - \lambda\xi_{S^*} &= 0 \end{aligned}$$

and ξ_i can be obtained as $\xi_i = \mathcal{V}_i \xi_1,\ 1 \le i \le S$ and $i = S^*$ where

$$\mathcal{V}_i = \begin{cases} 1 & i = 1, \\ \dfrac{\lambda + \alpha + (S-1)\beta}{\lambda} & i = 2, \\ \dfrac{\lambda + \alpha + (S-i+1)\beta}{\lambda}\mathcal{V}_{i-1} - \dfrac{(S-i+2)\beta}{\lambda}\mathcal{V}_{i-2} & 3 \le i \le S-1, \\ \dfrac{\beta}{\lambda + \alpha}\mathcal{V}_{S-1} & i = S \\ \dfrac{\lambda + \alpha + \beta}{\lambda}\mathcal{V}_{S-1} - \dfrac{2\beta}{\lambda}\mathcal{V}_{S-2} - \mathcal{V}_S & i = S^*. \end{cases}$$

The unknown probability ξ_1 can be found from the normalizing condition

$$\xi_1 = \left[\sum_{i=1}^{S} \mathcal{V}_i + \mathcal{V}_{S^*}\right]^{-1}.$$

Now using the vector $\boldsymbol{\xi}$ we proceed to compute the steady state probability vector of the original system. We show that

$$\mathbf{x}_n = \mathcal{K}\left(\frac{\lambda}{\mu}\right)^n \boldsymbol{\xi} \text{ for } n \ge 0 \tag{3.9}$$

where $\mathcal{K}$ is a constant to be determined is the unique solution to (3.5). From (3.6), we have

$$\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 = \mathcal{K}\boldsymbol{\xi}\left(A_{00} + \frac{\lambda}{\mu}A_2\right) = \mathcal{K}\boldsymbol{\xi}\tilde{A} = 0 \tag{3.10}$$

and from relation (3.7), we have

$$\begin{aligned} \mathbf{x}_{n-1}A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1}A_2 &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\left(\tfrac{\mu}{\lambda}A_0 + A_1 + \tfrac{\lambda}{\mu}A_2\right) \\ &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\left(\tfrac{\mu}{\lambda}A_0 + A_{00} - \tfrac{\mu}{\lambda}A_0 + \tfrac{\lambda}{\mu}A_2\right) \\ &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\tilde{A} = 0. \end{aligned} \tag{3.11}$$

Thus (3.9) satisfies (3.6) and (3.7). Now applying the normalizing condition $\mathbf{xe} = 1$, we get

$$\mathcal{K}\boldsymbol{\xi}\left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \ldots\right]\mathbf{e} = 1.$$

Hence under the condition that $\lambda < \mu$, we have

$$\mathcal{K} = 1 - \frac{\lambda}{\mu}. \tag{3.12}$$

Thus we arrive at our main result:

Theorem 3.2. *Under the necessary and sufficient condition $\lambda < \mu$ for stability, the components of the steady state probability vector of the CTMC Ω, with generator $\mathcal{Q}$, is given by (3.9) and (3.12). That is,*

$$\boldsymbol{x}_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \boldsymbol{\xi} \textit{ for } n \ge 0. \tag{3.13}$$

3.3. PROBABILITY THAT THE NEXT CYCLE STARTS WITH SERVICE COMPLETION/REALIZATION OF COMMON LIFE TIME

In this section we analyze the probability that next cycle starts with service completion / realization of common life time. First choose K such that

$$\sum_{n=0}^{K}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n > 1-\epsilon \text{ for any preassigned } \epsilon.$$

Consider the Markov chain $\{(I(t),N(t)),t\geq 0\}$ whose state space $\{(i,n),1\leq i\leq S,0\leq n\leq K\}\bigcup\{\Delta_\mu\}\bigcup\{\Delta_{CLT}\}$ where $\{\Delta_\mu\}$ is the absorbing state which means the replenishment order is placed on realization of event A and $\{\Delta_{CLT}\}$ represents the realization of common life time. Thus its infinitesimal generator is of the form

$$\mathcal{P}=\begin{pmatrix}\mathcal{T} & \mathcal{T}_\mu^0 & \mathcal{T}_{CLT}^0\\ \mathbf{0} & 0 & 0\end{pmatrix},$$

where

$$\mathcal{T}=\begin{array}{c}\\ S\\ S-1\\ S-2\\ \vdots\\ 3\\ 2\\ 1\end{array}\left(\begin{array}{ccccccc}S & S-1 & S-2 & \cdots & 3 & 2 & 1\\ B_1 & B_2 & & & & & \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2\\ & & & & & B_0^{(S-1)} & B_1^{(S-1)}\end{array}\right),$$

$$\mathcal{T}_\mu^0=\begin{pmatrix}0\\ \vdots\\ 0\\ B_2'\end{pmatrix},\quad \mathcal{T}_{CLT}^0=\begin{pmatrix}\alpha\mathbf{e}\\ \vdots\\ \alpha\mathbf{e}\\ \alpha\mathbf{e}\end{pmatrix},$$

and

$$B_2'=\begin{pmatrix}0\\ \mu\\ \vdots\\ \mu\end{pmatrix},\quad B_2=\begin{pmatrix}0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0\end{pmatrix},\quad B_0^{(i)}=\begin{pmatrix}i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta\end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, \quad B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix},$$

with $a_0 = -(\lambda+\alpha), a = -(\lambda+\mu+\alpha), a_K = -(\mu+\alpha), b_0 = -(\lambda+i\beta+\alpha), b = -(\lambda+\mu+i\beta+\alpha), b_K = -(\mu+i\beta+\alpha), 1 \leq i \leq S-1$.

Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_S, 0, ..., 0)$ be the initial probability vector of order $S(K+1)$ where $\boldsymbol{\gamma}_S = \frac{1}{(1-\rho^{K+1})}((1-\rho),(1-\rho)\rho,...,(1-\rho)\rho^K)$ with $\rho = \frac{\lambda}{\mu}$.

Thus we arrive at

Theorem 3.3. **(a)** *Probability that the inventory level becomes zero before realization of common life time,* $p_\mu = -\boldsymbol{\gamma}\mathcal{T}^{-1}\mathcal{T}_\mu^0$.

(b) *Probability that the common life time realizes before inventory level becomes zero,* $p_{CLT} = -\boldsymbol{\gamma}\mathcal{T}^{-1}\mathcal{T}_{CLT}^0 = -\boldsymbol{\gamma}\mathcal{T}^{-1}\alpha\boldsymbol{e}$.

(c) *Mean duration of the time until either the inventory level becomes zero or realization of common life time,* $\mu_T = -\boldsymbol{\gamma}\mathcal{T}^{-1}\boldsymbol{e}$.

3.4. SYSTEM PERFORMANCE MEASURES

In this section we consider system performance measures.

- Expected number of customers in the system

$$E_N = \sum_{n=1}^{\infty}\sum_{i=1}^{S} n x_n(i) = \frac{\lambda}{\mu-\lambda}\sum_{i=1}^{S}\xi_i.$$

- Expected number of items in the inventory

$$E_I = \sum_{n=0}^{\infty}\sum_{i=1}^{S} i x_n(i) = \sum_{i=1}^{S} i\xi_i.$$

- Expected rate of purchase

$$E_{PR} = \mu\sum_{n=1}^{\infty}\sum_{i=1}^{S} x_n(i) = \lambda\sum_{i=1}^{S}\xi_i.$$

- Expected cancellation rate

$$E_{CR} = \sum_{n=0}^{\infty}\sum_{i=1}^{S}(S-i)\beta x_n(i) = \sum_{i=1}^{S}(S-i)\beta\xi_i.$$

- Expected number of purchase in a cycle

$$E_{PN} = \frac{E_{PR}}{\mu_T}.$$

- Expected number of cancellations in a cycle

$$E_{CN} = \frac{E_{CR}}{\mu_T}.$$

4. CASE OF POSITIVE LEAD TIME

In this section we consider the system with positive lead time. Thus on realization of CLT or when the inventory level reaches zero through a service completion, an order for replenishment is place. The lead time is exponentially distributed with parameter θ. Subsequently the inventory level reaches its maximum S (denoted by S^* for convenience in identification). When the inventory level is zero, new arrivals and cancellation of purchased items are not permitted. The above condition is imposed since inventory level can drop to zero through a demand. The significance of this assumption is that a passenger bus leaves the station with its maximum capacity and so cancellation thereafter has no meaning. Remaining assumptions are as in Section 2. We have the $CTMC$ $\{(N(t), I(t)), t \geq 0\}$ with state space

$$\{(n, i), n \geq 0, 0 \leq i \leq S\} \bigcup \{(n, S^*), n \geq 0\}.$$

Thus the infinitesimal generator is same as that given in (3.1). But with entries of A_0 as given in (i); that of A_2 as given in (ii) and that in $A_{0,0}$ and A_1 correspond to transition rates given by $(iii), (iv)$ and (v) below. In addition diagonal entries in A_{00} and A_1 are non-positive, having numerical value equal to the sum of other elements of the same row found in A_{00}, A_0, A_1 and A_2. All other transitions have rate zero.

(i) *Transitions due to arrival:*

$$\begin{array}{lll} (n,i) \to (n+1,i): & \text{rate } \lambda & \text{for } n \geq 0, \quad 1 \leq i \leq S \\ (n,S^*) \to (n+1,S^*): & \text{rate } \lambda & \text{for } n \geq 0. \end{array}$$

(ii) *Transitions due to service completions:*

$$\begin{array}{lll} (n,i) \to (n-1,i-1): & \text{rate } \mu & \text{for } n \geq 1, \quad 1 \leq i \leq S \\ (n,S^*) \to (n-1,S-1): & \text{rate } \mu & \text{for } n \geq 1. \end{array}$$

(iii) *Transitions due to common life time realization:*

$$(n,i) \rightarrow (n,0): \quad \text{rate } \alpha \quad \text{for } n \geq 0, 1 \leq i \leq S.$$

(iv) *Transition due to cancellation:*

$$(n,i) \rightarrow (n,i+1): \quad \text{rate } (S-i)\beta \quad \text{for } n \geq 0, 1 \leq i \leq S-1.$$

(v) *Transition due to lead time:*

$$(n,0) \rightarrow (n,S^*): \quad \text{rate } \theta \quad \text{for } n \geq 0.$$

Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $S+2$.

STABILITY CONDITION

Let $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_S, \phi_{S^*})$ be the steady state probability vector of $\mathcal{A} = A_0 + A_1 + A_2$. Then

$$\boldsymbol{\phi}\mathcal{A} = \mathbf{0}, \ \boldsymbol{\phi}\mathbf{e} = 1.$$

The Markov chain is stable if and only if (see Neuts [14]) the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\phi} A_0 \mathbf{e} < \boldsymbol{\phi} A_2 \mathbf{e}.$$

Using this relation we have the following

Theorem 4.1. *The system under study is stable if and only if* $\lambda < \mu$.

4.1. STOCHASTIC DECOMPOSITION OF SYSTEM STATES

Let $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ be the steady-state probability vector of $\mathcal{Q}$ where each component
$\mathbf{y}_n = (y_n(0), y_n(1), ..., y_n(S), y_n(S^*)), n \geq 0$. Then

$$\mathbf{y}\mathcal{Q} = \mathbf{0}, \ \mathbf{ye} = 1.$$

$$y_n(i) = \lim_{t\to\infty} Prob.(N(t) = n, I(t) = i), n \geq 0, 0 \leq i \leq S \text{ and } i = S^*.$$

Assume

$$\mathbf{y}_n = \mathcal{K}\rho^n \boldsymbol{\psi}, n \geq 0$$

where $\boldsymbol{\psi}$ is steady-state probability vector when the service time is negligible, $\mathcal{K}$ is a constant and $\rho = \dfrac{\lambda}{\mu}$.

Now we first consider the system with instantaneous service time. The infinitesimal generator is given by

$$\tilde{\mathcal{A}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \left(\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \\ -\theta & & & & & & & & \theta \\ \alpha+\lambda & f_{S-1} & h_{S-1} & & & & & & \\ \alpha & \lambda & f_{S-2} & h_{S-2} & & & & & \\ \alpha & & \lambda & f_{S-3} & h_{S-3} & & & & \\ \vdots & & & \ddots & \ddots & \ddots & & & \\ \alpha & & & & \lambda & f_2 & h_2 & & \\ \alpha & & & & & \lambda & f_1 & h_1 & \\ \alpha & & & & & & \lambda & f_0 & \\ & & & & & & \lambda & & -\lambda \end{array} \right)$$

with $f_i = -(\lambda + \alpha + i\beta), 0 \le i \le S-1$ and $h_j = j\beta, 1 \le j \le S-1$.

Let $\boldsymbol{\psi} = (\psi_0, \psi_1, ..., \psi_S, \psi_{S^*})$ be the steady state vector of $\tilde{\mathcal{A}}$. Then $\boldsymbol{\psi}$ satisfies the equations

$$\boldsymbol{\psi}\tilde{\mathcal{A}} = \mathbf{0}, \ \ \boldsymbol{\psi}\mathbf{e} = 1.$$

Each ψ_i can be obtained as

$$\psi_i = \begin{cases} \mathcal{U}_i \psi_1 & 0 \le i \le S, \\ \mathcal{U}_{S^*}\psi_1 & i = S^*, \\ \left[\displaystyle\sum_{i=0}^{S} \mathcal{U}_i + \mathcal{U}_{S^*} \right]^{-1} & i = 1, \end{cases} \tag{4.1}$$

where

$$\mathcal{U}_i = \begin{cases} \dfrac{\mu}{\theta}\mathcal{U}_{S^*} & i = 0, \\ 1 & i = 1, \\ \dfrac{\alpha + \mu + (S-1)\beta}{\mu}\mathcal{U}_1 & i = 2, \\ \dfrac{\alpha + \mu + (S-i+1)\beta}{\mu}\mathcal{U}_{i-1} - \dfrac{(S-i+2)\beta}{\mu}\mathcal{U}_{i-2} & 3 \le i \le S-1, \\ \dfrac{\beta}{\alpha+\mu}\mathcal{U}_{S-1} & i = S, \\ \dfrac{\alpha+\mu+\beta}{\mu}\mathcal{U}_{S-1} - \dfrac{2\beta}{\mu}\mathcal{U}_{S-2} - \mathcal{U}_S & i = S^*. \end{cases} \tag{4.2}$$

Now from $\mathbf{y}\mathcal{Q} = 0$ and $\mathbf{y}_n = \mathcal{K}\rho^n\boldsymbol{\psi}, n \ge 0$, we have

$$\mathbf{y}_0 A_{00} + \mathbf{y}_1 A_2 = \mathcal{K}\boldsymbol{\psi}\tilde{\mathcal{A}} = 0,$$

and

$$\mathbf{y}_{n-1}A_0 + \mathbf{y}_n A_1 + \mathbf{y}_{n+1}A_2 = \mathcal{K}\rho^n\boldsymbol{\psi}\tilde{\mathcal{A}} = 0.$$

Using $\mathbf{ye} = 1$ we get $\mathcal{K} = 1 - \rho$.

Theorem 4.2. *The steady-state probability vector* $\boldsymbol{y}$ *of* $\mathcal{Q}$ *is obtained as*

$$y_n(i) = (1-\rho)\rho^n\psi_i, n \geq 0, 0 \leq i \leq S \text{ and } i = S^* \text{ at the beginning of the new cycle} \tag{4.3}$$

where $\rho = \dfrac{\lambda}{\mu}$ *and* ψ_i *represents the inventory level probabilities when service time is negligible and are given in (4.1).*

4.2. PROBABILITY THAT THE NEXT CYCLE STARTS WITH SERVICE COMPLETION OR REALIZATION OF COMMON LIFE TIME

Unlike in Section 3.3, we cannot compute the probabilities of a new cycle starting with a service completion/ realization of CLT, with the help of the same infinitesimal generator since the lead time is positive. In this section we analyze the probabilities of the next cycle starting with service completion and realization of common life time. Choose K sufficiently large that

$$\sum_{n=0}^{K}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n > 1-\epsilon, \text{ for arbitrary small } \epsilon > 0.$$

Except for heavy traffic (that is, $\frac{\lambda}{\mu}$ close to 1) the above approximation is very much near to exact value.

First we compute the probability that the inventory level becomes zero before realization of CLT. Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space $\{(i,n), 0 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta_\mu\}$ where $\{\Delta_\mu\}$ is the absorbing state which means the replenishment order is placed after realization of event A. Thus its infinitesimal generator is of the form

$$\mathcal{P}_1 = \begin{pmatrix} \mathcal{T}_1 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$\mathcal{T}_1 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \left(\begin{array}{cccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \\ B_1^0 & B_2 & & & & & & \\ B_0^{(1)} & B_1^{0(1)} & B_2 & & & & & \\ & B_0^{(2)} & B_1^{0(2)} & B_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & B_0^{(S-3)} & B_1^{0(S-3)} & B_2 & & \\ & & & & B_0^{(S-2)} & B_1^{0(S-2)} & B_2 & \\ & & & & & B_0^{(S-1)} & B_1^{0(S-1)} & B_2 \\ & & & & & & & -\theta I \end{array} \right),$$

$$\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta\mathbf{e} \end{pmatrix} \text{ where } B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix}, B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1^0 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{0(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -\lambda, a = -(\lambda + \mu), a_K = -\mu, b_0 = -(\lambda + i\beta), b = -(\lambda + \mu + i\beta), b_K = -(\mu + i\beta), 1 \le i \le S - 1$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of order $(S+1)(K+1)$ where $\boldsymbol{\eta}_S = \dfrac{1}{(1-\rho^{K+1})}((1-\rho), (1-\rho)\rho, ..., (1-\rho)\rho^K)$.
Thus we have

Theorem 4.3. *Probability that the inventory level becomes zero before realization of common life time,* $p_\mu = -\boldsymbol{\eta}\,(\mathcal{T}_1)^{-1}\,\tilde{\mathcal{T}}^0_\mu$.

Next we compute the probability that CLT realized before inventory level becomes zero. Consider the Markov chain $\{(I(t), N(t)), t \ge 0\}$ whose state space $\{(i, n), 0 \le i \le S, 0 \le n \le K\} \bigcup \{\Delta_{CLT}\}$ where $\{\Delta_{CLT}\}$ is the absorbing state which means the replenishment order is placed after realization of event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_2 = \begin{pmatrix} \mathcal{T}_2 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where

$$\mathcal{T}_2 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \\ B_1 & B_2 & & & & & & \alpha I \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & & \alpha I \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & & \alpha I \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & & \alpha I \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 & \alpha I \\ & & & & & B_0^{(S-1)} & B_1^{'(S-1)} & \alpha I \\ & & & & & & & -\theta I \end{pmatrix},$$

$$\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta \mathbf{e} \end{pmatrix} \text{ where } B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1^{\prime(S-1)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b' & \lambda & & \\ & & \ddots & \ddots & \\ & & & b' & \lambda \\ & & & & b'_K \end{pmatrix}, B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -(\alpha + \lambda), a = -(\lambda + \alpha + \mu), a_K = -(\alpha + \mu), b_0 = -(\lambda + \alpha + i\beta), b = -(\lambda+\alpha+\mu+i\beta), b_K = -(\mu+\alpha+i\beta), 1 \leq i \leq S-2, b' = -(\lambda+\alpha+i\beta), b'_K = -(\alpha+i\beta)$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of order $(S+1)(K+1)$ where $\boldsymbol{\eta}_S = \dfrac{1}{(1-\rho^{K+1})}((1-\rho), (1-\rho)\rho, ..., (1-\rho)\rho^K)$.
The above discussion leads us to

Theorem 4.4. *Probability that common life time realized before the inventory level becomes zero,* $p_{CLT} = -\boldsymbol{\eta}\,(\mathcal{T}_2)^{-1}\,\tilde{\mathcal{T}}^0$.

4.3. MEAN DURATION OF THE TIME BETWEEN TWO SUCCESSIVE REPLENISHMENT

Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space $\{(i, n), 0 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state which means the replenishment order is placed after realization of event A or event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_3 = \begin{pmatrix} \mathcal{T}_3 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where

$$\mathcal{T}_3 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \\ B_1 & B_2 & & & & & & \alpha I \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & & \alpha I \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & & \alpha I \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & & \alpha I \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 & \alpha I \\ & & & & & B_0^{(S-1)} & B_1^{(S-1)} & \alpha I + B_2 \\ & & & & & & & -\theta I \end{pmatrix},$$

$$\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta \mathbf{e} \end{pmatrix} \text{ where } B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix}, B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -(\alpha+\lambda), a = -(\lambda+\alpha+\mu), a_K = -(\alpha+\mu), b_0 = -(\lambda+\alpha+i\beta), b = -(\lambda+\alpha+\mu+i\beta), b_K = -(\mu+\alpha+i\beta), 1 \le i \le S-1$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of order $(S+1)(K+1)$ where $\boldsymbol{\eta}_S = \frac{1}{(1-\rho^{K+1})}((1-\rho), (1-\rho)\rho, ..., (1-\rho)\rho^K)$.
The above considerations lead us to

Lemma 4.5. *Mean duration of the time between two successive replenishment,* $\mu_T = -\boldsymbol{\eta}\,(\mathcal{T}_3)^{-1}\,\boldsymbol{e}$.

4.3.1. MEAN DURATION OF THE TIME UNTIL THE INVENTORY LEVEL REACHES ZERO THROUGH REALIZATION OF EVENT A OR EVENT B

Consider the Markov chain $\{(I(t), N(t)), t \ge 0\}$ whose state space $\{(i,n), 1 \le i \le S, 0 \le n \le K\} \bigcup \{\Delta'\}$ where $\{\Delta'\}$ is the absorbing state which means the inventory level becomes zero after realization of event A or event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_4 = \begin{pmatrix} \mathcal{T}_4 & \tilde{\mathcal{T}}'^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$\mathcal{T}_4 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \end{array} \left(\begin{array}{ccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 \\ B_1 & B_2 & & & & & \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 \\ & & & & & B_0^{(S-1)} & B_1^{(S-1)} \end{array} \right),$$

$$\tilde{\mathcal{T}}'^0 = \begin{pmatrix} \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ \vdots \\ \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ B_2' \end{pmatrix} \text{ with } B_2' = \begin{pmatrix} \alpha \\ \alpha+\mu \\ \vdots \\ \alpha+\mu \end{pmatrix}.$$

Let $\boldsymbol{\eta}' = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of $\mathcal{T}_4$ (see Section 4.3) is of order $S(K+1)$.

The above discussion lead us to

Lemma 4.6. *The mean time until the inventory level becomes zero,* $\mu_T' = -\boldsymbol{\eta}' (\mathcal{T}_4)^{-1} \boldsymbol{e}$.

4.4. WAITING TIME DISTRIBUTION OF A TAGGED CUSTOMER

For deriving the waiting time distribution of a tagged customer who joins the queue as the r^{th} customer, $r > 0$, we consider the Markov process $W(t) = \{(N'(t), I(t)), t \geq 0\}$ where $N'(t)$ is the rank of the customer and $I(t)$ is the size of the inventory at time t. The rank $N'(t)$ of the customer is assumed to be i if he is the i^{th} customer in the queue at time t. His rank decreases to 1 as the customers ahead of him leave the system after completing their service. Since the customers who arrive after the tagged customer can not change the rank, level changing transitions in $W(t)$ can only take place to one side of the diagonal. We arrange the state space of $W(t)$ as $\{r, r-1, \ldots, 2, 1\} \times \{0, 1, 2, \ldots, S-1, S, S^*\} \cup \{\Delta\}$, where $\{\Delta\}$ is the absorbing state denoting that the tagged customer is selected for service. Thus the infinitesimal generator $\mathbf{W}$ of the process $W(t)$ assumes the form

$$\mathbf{W} = \begin{pmatrix} \tilde{\mathbf{T}} & \tilde{\mathbf{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$\tilde{\mathbf{T}} = \begin{pmatrix} A_1 & A_2 & & & \\ & A_1 & A_2 & & \\ & & \ddots & \ddots & \\ & & & A_1 & A_2 \\ & & & & A_1 \end{pmatrix} \text{ and } \tilde{\mathbf{T}}^0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ A_2' \end{pmatrix} \text{ with } A_2' = \begin{pmatrix} 0 \\ \mu \\ \vdots \\ \mu \end{pmatrix}.$$

Note that A_1 and A_2 are the same matrices defined in the beginning of Section 4.

Now, the waiting time W of a customer, who joins the queue as the r^{th} customer is the time until absorption of the Markov chain $W(t)$. Thus the waiting time of this particular customer is a PH-variate with representation $PH(\boldsymbol{\phi}, \tilde{\mathbf{T}})$, where $\boldsymbol{\phi} = (\boldsymbol{\psi}, \mathbf{0}, ..., \mathbf{0})$ with $\boldsymbol{\psi} = (0, \psi_1', \psi_2', \ldots, \psi_S', \psi_{S^*}')$ and $\psi_i' = \frac{\psi_i}{1-\psi_0}$ for $i \in \{1, 2, ..., S, S^*\}$ (see Section 4.1). Thus we have arrived at

Theorem 4.7. *The waiting time distribution function and the expected waiting time of a tagged customer are given by*

$$F(t) = 1 - \boldsymbol{\phi} \exp\{\tilde{\boldsymbol{T}} t\} \boldsymbol{e}$$

and

$$E_W^T = -\boldsymbol{\phi}(\tilde{\boldsymbol{T}})^{-1} \boldsymbol{e}$$

respectively.

For the computation $F(t)$ in the above theorem we employ the uniformization procedure (see Latouche and Ramaswami [12]).

Essentially, the uniformization approach associates the infinitesimal generator $\mathbf{W}$ of the Markov chain with another matrix $\mathbf{K}$ which can be viewed as the transition matrix for a discrete time Markov chain. The two matrices are related via $\mathbf{K} = (1/c)\mathbf{W} + I = \begin{pmatrix} \tilde{P} & \tilde{\mathbf{p}} \\ \mathbf{0} & 0 \end{pmatrix}$ where c is at least as big as the maximum of the absolute values of the diagonal elements of $\mathbf{W}$; ordinarily it equals this maximum. Now we have

$$F(t) = 1 - \sum_{k=0}^{\infty} e^{-ct} \frac{(ct)^k}{k!} \boldsymbol{\phi} \tilde{P} \mathbf{e}.$$

Algorithm to compute the distribution function of a continuous $PH(\boldsymbol{\phi}, \tilde{\mathbf{T}})$ random variable

$$M := \boldsymbol{\phi}(I - \tilde{P})^{-1}\mathbf{e};$$

$$a_0 := \boldsymbol{\phi}\mathbf{e};$$

$$k := 0;$$

$\nu := \tilde{P}\mathbf{e};$

repeat

$k := k + 1;$

$a_k := \boldsymbol{\phi}\nu;$

$\nu := \tilde{P}\nu;$

until $\left|\sum_{i=0}^{k} a_i - M\right| < \epsilon;$

$K_1 := k;$

for any t of interest **do**

$p := \exp(-ct);$

$F_1 := pa_0;$

for $k := 1$ **to** K_1 **do**

$p := ctp/k;$

$F_1 := F_1 + pa_k$

end

end

$F := 1 - F_1.$

p	t	$F(t)$
0.1	0.1460	0.7555
0.2	0.1020	0.6268
0.3	0.0763	0.5219
0.4	0.0581	0.4299
0.5	0.0439	0.3465
0.6	0.0324	0.2692
0.7	0.0226	0.1967
0.8	0.0141	0.1281
0.9	0.0067	0.0627
1	0	0

Table 1: Values of $F(t)$: Fix $(S, \lambda, \mu, \beta, \alpha, \theta) = (8, 2, 3, 0.25, 0.1, 0.2)$

4.5. SYSTEM PERFORMANCE MEASURES

In this section we obtain system performance measures as under.

- Expected number of customers in the system

$$E'_N = \sum_{n=1}^{\infty}\sum_{i=0}^{S} n y_n(i) = \frac{\lambda}{\mu-\lambda}\sum_{i=0}^{S}\psi_i$$

- Expected number of items in the inventory

$$E'_I = \sum_{n=0}^{\infty}\sum_{i=1}^{S} i y_n(i) = \sum_{i=1}^{S} i\psi_i$$

- Expected rate of purchase

$$E'_{PR} = \mu\sum_{n=1}^{\infty}\sum_{i=1}^{S} y_n(i) = \lambda\sum_{i=1}^{S}\psi_i$$

- Expected cancellation rate

$$E'_{CR} = \sum_{n=0}^{\infty}\sum_{i=1}^{S}(S-i)\beta y_n(i) = \sum_{i=1}^{S}(S-i)\beta\psi_i$$

- Expected loss rate of customers during the lead time

$$E'_{LR} = \lambda\sum_{n=0}^{\infty} y_n(0)$$

- Expected number of purchase up to (excluding) order placement in a cycle

$$E'_{PN} = \frac{E'_{PR}}{\mu'_T}$$

- Expected number of cancellations up to (excluding) order placement in a cycle

$$E'_{CN} = \frac{E'_{CR}}{\mu'_T}$$

- Expected number of customers lost during the positive lead time

$$E'_{LN} = \frac{E'_{LR}}{\theta}$$

5. NUMERICAL ILLUSTRATION

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

λ	E_N	E_I	E_{PR}	E_{CR}
4	0.5714	17.6074	3.9024	3.8095
5	0.8333	17.2267	4.9018	4.7623
6	1.1993	16.8134	5.8999	5.7170
7	1.7420	16.3790	6.8895	6.6762
8	2.6027	15.9334	7.8473	7.6310
9	4.0594	15.5035	8.7253	8.5332

(a) Zero lead time

λ	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
4	0.5714	17.1882	3.8095	3.7188	0.0238
5	0.8333	16.8146	4.7845	4.6483	0.0239
6	1.1993	16.4099	5.7583	5.5798	0.0240
7	1.7420	15.9850	6.7238	6.5157	0.0241
8	2.6027	15.5495	7.6582	7.4472	0.0241
9	4.0594	15.1292	8.5146	8.3273	0.0241

(b) Positive lead time for $\theta = 4$

Table 2: Effect of λ: Fix $(S, \mu, \beta, \alpha) = (20, 11, 2, 0.1)$

EFFECT OF ARRIVAL RATE λ

Changes in arrival rate has no significant impact on the measures irrespective of the lead time (see Table 2 (a) and (b)). This is so since no customer joins when inventory is zero.

EFFECT OF SERVICE RATE μ

Table 3 (a) and (b) tell us that there is significant impact of lead time on expected inventory held and moderate impact on expected purchase and cancellation rates with respect to service time parameter.

μ	E_N	E_I	E_{PR}	E_{CR}
9	3.3118	20.2354	6.8135	6.6075
10	2.2982	20.2516	6.8692	6.5899
11	1.7420	20.2631	6.8895	6.5754
12	1.3979	20.2687	6.8969	6.5680
13	1.1661	20.2712	6.8997	6.5648
14	0.9998	20.2722	6.9007	6.5634

(a) Zero lead time

μ	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
9	3.3125	16.9061	5.6925	5.5204	0.1645
10	2.2984	16.9174	5.7382	5.5050	0.1646
11	1.7420	16.9262	5.7550	5.4925	0.1647
12	1.3979	16.9305	5.7610	5.4863	0.1647
13	1.1661	16.9325	5.7633	5.4835	0.1647
14	0.9998	16.9333	5.7641	5.4824	0.1647

(b) Positive lead time for $\theta = 0.5$

Table 3: Effect of μ: Fix $(S, \lambda, \beta, \alpha) = (25, 7, 1.5, 0.1)$

EFFECT OF CANCELLATION RATE β

Impact of lead time with respect to cancellation rate β, is significant on measures such as expected inventory, expected purchase, loss and cancellation rates (see Table 4 (a) and (b)).

β	E_N	E_I	E_{PR}	E_{CR}
1	4.0594	21.5214	8.7258	8.1372
1.5	4.0594	23.9795	8.7260	8.5197
2	4.0594	25.2908	8.7260	8.7378
2.5	4.0594	26.1062	8.7261	8.8842
3	4.0594	26.6623	8.7261	8.9930
3.5	4.0594	27.0659	8.7261	9.0795

(a) Zero lead time

β	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
1	4.0592	14.3995	5.8383	5.4448	0.3309
1.5	4.0611	16.0525	5.8414	5.7036	0.3306
2	4.0619	16.9342	5.8428	5.8509	0.3304
2.5	4.0624	17.4825	5.8436	5.9497	0.3303
3	4.0628	17.8564	5.8441	6.0230	0.3303
3.5	4.0630	18.1279	5.8445	6.0813	0.3302

(b) Positive lead time for $\theta = 0.2$

Table 4: Effect of β for $(S, \lambda, \mu, \alpha) = (30, 9, 11, 0.1)$

EFFECT OF COMMON LIFE TIME PARAMETER $\boldsymbol{\alpha}$

A look at Tables 4(a) and (b) tell the sharp difference between zero lead time and positive lead time. Since during the lead time inventory level stays at zero, the sharp decrease seen in Table 4(b) is justified in contrast to quite moderate decrease rate indicated in Table 4(a). The common life time parameter α plays a significant role

α	E_N	E_I	E_{PR}	E_{CR}
0.1	3.3118	15.3454	6.8132	6.5482
0.2	3.3118	15.3216	6.7177	6.1714
0.3	3.3118	15.2779	6.6248	5.8359
0.4	3.3118	15.2182	6.5344	5.5352
0.5	3.3118	15.1457	6.4465	5.2642
0.6	3.3118	15.0629	6.3609	5.0187

(a) Zero lead time

α	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
0.1	3.3120	10.2792	4.5639	4.3865	0.3301
0.2	3.3142	7.7755	3.4091	3.1320	0.4925
0.3	3.3154	6.2744	2.7207	2.3968	0.5893
0.4	3.3161	5.2717	2.2636	1.9175	0.6536
0.5	3.3165	4.5531	1.9380	1.5826	0.6994
0.6	3.3168	4.0120	1.6942	1.3367	0.7336

(b) Positive lead time for $\theta = 0.2$

Table 5: Effect of α for $(S, \lambda, \mu, \beta) = (20, 7, 9, 1.5)$

on measures such as p_μ, p_{CLT}, μ_T. We see from Table 6 that for the zero lead time case the measures p_μ and μ_T decrease sharply with respect to increasing α, whereas p_{CLT} shows a fast increasing trend with increasing value of α. The latter tendency is an account of faster CLT realization.

5.1. COST ANALYSIS

Based on the above performance measures we define the following two revenue (profit) functions as:

$$F(\alpha, \beta, S) = C_1 E_{PR} + C_2 E_{CR} - C_3 E_I - C_4 E_N \text{ for zero lead time}$$

$$F_{PL}(\alpha, \beta, S) = C_1 E'_{PR} + C_2 E'_{CR} - C_3 E'_I - C_4 E'_N - C_5 E'_{LR} \text{ for positive lead time}$$

where

α	p_μ	p_{CLT}	μ_T
0.1	0.8212	0.1788	1.7878
0.2	0.6768	0.3232	1.6158
0.3	0.5597	0.4403	1.4675
0.4	0.4644	0.5356	1.3389
0.5	0.3866	0.6134	1.2269
0.6	0.3227	0.6773	1.1288
0.7	0.2702	0.7298	1.0426
0.8	0.2268	0.7732	0.9665
0.9	0.1909	0.8091	0.8989
1	0.1611	0.8388	0.8388

Table 6: Effect of α on p_μ, p_{CLT}, μ_T

- C_1 = revenue to the system due to per unit purchase (by a customer at the end of his service)
- C_2 = revenue to the system due to per unit cancellation
- C_3 = holding cost per inventoried item per unit time
- C_4 = holding cost per customer per unit time
- C_5 = cost due to customer lost per unit time (applicably only to positive lead time case)

In order to study the variation in different parameters on profit function we first take the values $(C_1, C_2, C_3, C_4, C_5) = (\$100, \$30, \$10, \$2, \$10)$.

ZERO LEAD TIME

Table 7 is indicative of the fact that as cancellation rate increases optimal S value decreases. This could be explained as follows: with cancellation rate increasing, the trend for accumulation of lesser quantity of inventory increases at the time of realization of CLT, the items left in the inventory also tend to be longer which brings down the profit (see figure 1(a)).

In Table 8 (see figure 2(a)) we notice that optimal S value stays at 11 as rate of realization of CLT moves from 0.1 to 0.35. This could be attributed to the fact that the expected number of cancellations is brought down thereby the left over items at CLT realization becomes smaller and smaller.

$S \downarrow \beta \rightarrow$	0.6	0.7	0.8	0.9	1	1.1
12	695.7005	715.6908	733.6266	748.914	761.226	770.5668
13	705.4946	726.1356	743.6778	757.4794	767.4486	**774.0164**
14	714.3682	734.999	751.1854	762.5007	**769.4356**	773.0453
15	722.2044	741.9401	755.7128	**763.7802**	767.5236	768.5599
16	728.7723	746.5474	**756.9911**	761.5639	762.5276	761.7047
17	733.7456	**748.473**	755.1026	756.5191	755.4446	753.4817
18	736.7633	747.5921	750.5101	749.4988	747.1415	744.5712
19	**737.5331**	744.0955	743.905	741.2767	738.2117	735.3537
20	735.9427	738.447	735.9863	732.4085	728.994	726.0119

Table 7: Effect of S and β on $F(\alpha, \beta, S)$ for $(\lambda, \mu, \alpha) = (7, 9, 0.1)$

$S \downarrow \alpha \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35
10	783.0723	774.2038	765.587	757.2099	749.0611	741.1301
11	**787.4177**	**777.6512**	**768.2145**	**759.088**	**750.2537**	**741.6951**
12	786.0369	775.6812	765.7166	756.1166	746.8573	737.917
13	780.696	770.0332	759.7997	749.9644	740.4992	731.379
14	773.0865	762.3077	751.9776	742.0626	732.5323	723.3596
15	764.3822	753.5897	743.2543	733.3410	723.8184	714.6585

Table 8: Effect of S and α on revenue $F(\alpha, \beta, S)$ for $(\lambda, \mu, \beta) = (7, 9, 1.5)$

Table 9 (figure 3(a)) shows a decreasing trend for profit for fixed cancellation rate(s) as CLT is varied from 0.1 to 0.35. This is so since the number of cancellations decrease thereby decreasing the revenue from canceled items.

$\alpha \downarrow \beta \rightarrow$	1	1.5	2	2.5	3	3.5
0.1	767.5236	764.3822	758.8378	755.3327	753.0308	751.4426
0.15	754.3746	753.5897	749.7732	747.3268	745.732	744.6488
0.2	741.9535	743.2543	740.999	739.529	738.5952	737.9888
0.25	730.191	733.341	732.4984	731.9296	731.6143	731.4584
0.3	719.0269	723.8184	724.2561	724.52	724.7837	725.0534
0.35	708.408	714.6585	716.2579	717.2917	718.0978	718.7699

Table 9: Effect of α and β on profit $F(\alpha, \beta, S)$ for $(\lambda, \mu, S) = (7, 9, 15)$

POSITIVE LEAD TIME

Tabulations in Tables 10 to 12 (see figure 1(b) to 3(b))pertain to positive lead time. A comparison between Tables 7 and 10 reveal that revenue is less for positive lead time case. This could be attributed to loss of customers during time in the latter. Within Table 10 we notice that there is a decreasing trend in the optimal value of S with increase in cancellation rate for which the same explanation as given for Table is valid.

$S \downarrow \beta \rightarrow$	0.6	0.7	0.8	0.9	1	1.1
14	308.8123	352.6232	395.7066	433.407	462.5295	482.5812
15	339.5486	387.1692	429.6957	462.0075	483.024	495.012
16	369.6888	418.3271	456.1647	480.1111	492.7432	**498.406**
17	398.0574	443.9862	473.7589	488.8044	**494.8443**	496.5527
18	423.1877	462.6233	482.9697	**490.6662**	492.4892	492.1152
19	443.6235	473.9276	**485.6684**	488.3703	487.8934	486.5574
20	458.3777	478.8463	484.0916	483.8823	482.2969	480.558
21	467.2944	**479.0377**	480.0756	478.3637	476.2921	474.3943
22	**471.0636**	476.1888	474.8192	472.4018	470.1294	468.169
23	470.8868	471.6039	468.9873	466.2601	463.9054	461.9177

Table 10: Effect of S and β on profit $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, \alpha, \theta) = (7, 9, 0.1, 0.2)$

The results in Table 11 could be explained on the same lines as that for Table 8.

$S \downarrow \alpha \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35
10	418.2949	367.6324	327.1204	294.0112	266.465	243.2032
11	466.8609	403.2336	353.9116	314.5891	282.5283	255.9052
12	495.1468	422.7614	367.7864	324.6489	289.9221	261.3835
13	506.7439	429.9542	**372.2739**	**327.3945**	**291.5059**	**262.1714**
14	**508.2789**	**429.9933**	371.484	326.1333	289.9766	260.4944
15	505.0019	426.6814	368.2638	323.0535	287.0522	257.7253

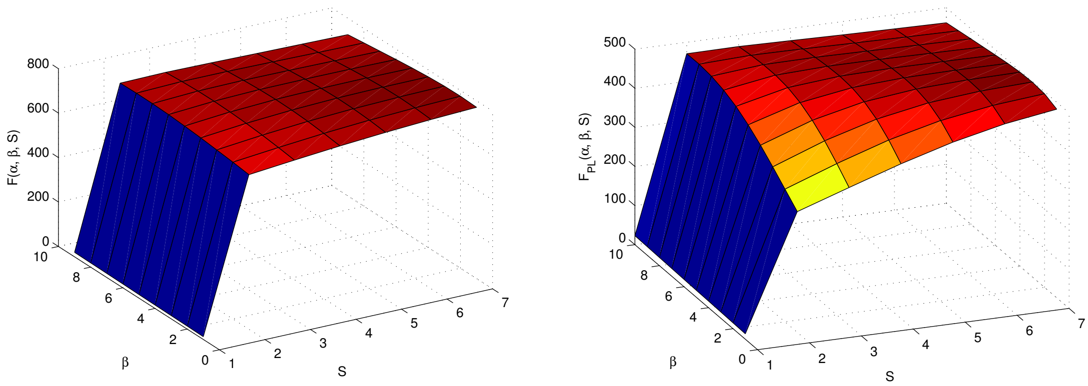
Table 11: Effect of S and α on revenue $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, \beta, \theta) = (7, 9, 1.5, 0.2)$

Finally, coming to Table 12, we notice that for fixed cancellation rates, the revenue decreases with increase in the rate of realization of CLT, which is on expected lines. On the other hand for fixed rates of realization of CLT, profit is seen to reaches a maximum and then starts decreasing, with increasing cancellation rates, until α

grows up to 0.25. This should be due to a trend of holding cost and revenue from cancellations. However, for higher rates of CLT realization, the profit due to increase in cancellation dominate the loss due to higher rate of realization of CLT.

$\alpha \downarrow \beta \rightarrow$	1	1.5	2	2.5	3	3.5
0.1	483.024	505.0019	502.6008	500.4557	499.0076	498.0052
0.15	411.3161	426.6814	425.3642	424.096	423.2434	422.6629
0.2	356.7699	368.2638	367.7428	367.1023	366.6791	366.4049
0.25	313.9579	323.0535	323.1262	322.9488	322.845	322.8004
0.3	279.5138	287.0522	287.5732	287.7448	287.8844	288.016
0.35	251.2406	257.7253	258.588	259.0269	259.3551	259.6245

Table 12: Effect of α and β on revenue $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, S, \theta) = (7, 9, 15, 0.2)$



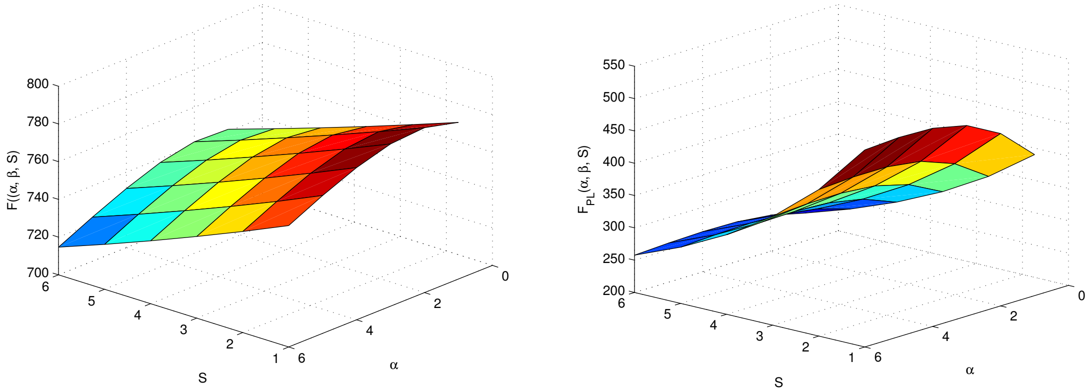
(a) zero lead time (b) positive lead time

Figure 1: Effect of S and β on revenue

A comparison between values in Tables (for example 7 and 10) indicate that lead time plays a crucial role in the revenue generation of the system. For zero lead time revenue is much larger than that corresponding to positive lead time. This is due to customer loss during lead time. Thus if we additionally introduce a cost for reduction in lead time, we will be able to have a trade off between duration of lead time and customer loss.

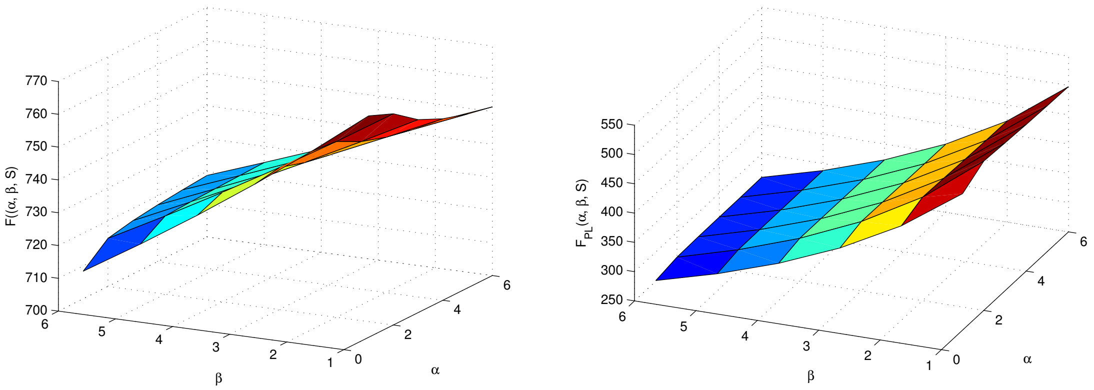
CONCLUSION

In this paper we considered a queueing-inventory model with reservation (purchase), cancellation (return of purchased items) when the items in a batch have common life



(a) zero lead time (b) positive lead time

Figure 2: Effect of S and α on profit



(a) zero lead time (b) positive lead time

Figure 3: Effect of α and β on profit function

time. The cases of both zero lead time as well as positive lead time were examined. In these two cases we arrived at the stochastic decomposition of the system state and further product form solution in the long run - that asymptotic independence of number of items in the inventory and number of customers in the system. Several performance characteristics of the system were studied. A significant application of the model is indicated in the transport system.

In a follow up paper we propose to analyze the effect of lead time when it is arbitrarily distributed.

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