GROWTH CONDITIONS FOR UNIQUENESS OF SMOOTH POSITIVE SOLUTIONS TO AN ELLIPTIC MODEL

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ABSTRACT: The uniqueness of positive solution to the elliptic model

$$\Delta u + u[a + g(u, v)] = 0 \ \text{ in } \ \Omega,$$
$$\Delta v + v[a + h(u, v)] = 0 \ \text{ in } \ \Omega,$$
$$u = v = 0 \ \text{ on } \ \partial\Omega,$$

were investigated.

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1. INTRODUCTION

A lot of research has been focused on reaction-diffusion equations modeling various systems in mathematical biology, especially the elliptic steady states of competitive and predator-prey interacting processes with various boundary conditions. In earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

Suppose two species of animals, rabbits and squirrels for instance, are competing in a bounded domain Ω. Let $u(x, t)$ and $v(x, t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the following biological interpretation of terms.

(A) The partial derivatives $u_t(x, t)$ and $v_t(x, t)$ mean the rate of change of densities with respect to time t.

(B) The laplacians $\Delta u(x,t)$ and $\Delta v(x,t)$ stand for the diffusion or migration rates.

(C) The rates of self-reproduction of each species of animals are expressed as multiples of some positive constants a, d and current densities $u(x,t), v(x,t)$, i.e. $au(x,t)$ and $dv(x,t)$ which will increase the rate of change of densities in (A), where $a > 0, d > 0$ are called the self-reproduction constants.

(D) The rates of self-limitation of each species of animals are multiples of some positive constants b, f and the frequency of encounters among themselves $u^2(x,t), v^2(x,t)$, i.e. $bu^2(x,t)$ and $fv^2(x,t)$ which will decrease the rate of change of densities in (A), where $b > 0, f > 0$ are called the self-limitation constants.

(E) The rates of competition of each species of animals are multiples of some positive constants c, e and the frequency of encounters of each species with the other $u(x,t)v(x,t)$, i.e. $cu(x,t)v(x,t)$ and $eu(x,t)v(x,t)$ which will decrease the rate of change of densities in (A), where $c > 0, e > 0$ are called the competition constants.

(F) We assume that none of the species of animals is staying on the boundary of Ω.

Combining all those together, we have the dynamic model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) - cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) - eu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{for} \ x \in \partial\Omega, \end{cases} \quad \text{in} \ \Omega \times [0,\infty),$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) - eu(x,t)) \\ u(x,t) = v(x,t) = 0 \ \text{for} \ x \in \partial\Omega, \end{cases} \quad \text{in} \ \Omega \times [0,\infty),$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in} \ \Omega, \tag{1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means the two species of animals can live peacefully and forever.

A lot of work about the existence and uniqueness of the coexistence state of the above steady state model has already been done during the last decade.(See [2], [3], [4], [6], [7], [14], [15].)

In [4], Cosner and Lazer established a sufficient and necessary conditions for the existence of positive solution to the competing system.

The following is their result:

Theorem 1.1. *In order that there exist positive smooth functions u and v in Ω satisfying (1) with $a = d$, it is necessary and sufficient that one of the following three sets of conditions hold, where λ_1 is as described in the Lemma 2.2.*

$$(1)a > \lambda_1, b > e, c < f$$
$$(2)a > \lambda_1, b = e, c = f$$
$$(3)a > \lambda_1, b < e, c > f$$

Furthermore, in case (1), there is a unique positive solution $u = \frac{f-c}{bf-ce}\theta_a, v = \frac{b-e}{bf-ce}\theta_a$.

Biologically, the Theorem 1.1 implies that they can coexist peacefully if their reproduction rates are large enough and their self-limitation and competition rates are balanced each other.

In this paper we study rather general types of the system. We are concerned with the existence and uniqueness of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of a positive steady state of

$$\begin{cases} \Delta u + u[a + g(u,v)] = 0 \\ \Delta v + v[a + h(u,v)] = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

where a is a positive constant, $g, h \in C^1$ are such that $g_u < 0, g_v < 0, h_u < 0, h_v < 0$, there exist constants $c_0 > 0, c_1 > 0$ such that $a + g(u,0) \leq 0$ for $u \geq c_0$ and $a + h(0,v) \leq 0$ for $v \geq c_1$, and $g(0,0) = h(0,0) = 0$, Ω is a bounded domain in R^n and u, v are densities of the two competitive species.

In [8], Kang established the following sufficient conditions for the existence of positive solution to (2).

Theorem 1.2. *Suppose one of the following three sets of conditions holds.*

$$(1)a > \lambda_1, \inf(g_u) < \inf(h_u), \inf(g_v) > \inf(h_v)$$
$$(2)a > \lambda_1, \inf(g_u) = \inf(h_u), \inf(g_v) = \inf(h_v)$$
$$(3)a > \lambda_1, \inf(g_u) > \inf(h_u), \inf(g_v) < \inf(h_v)$$

Then (2) has a positive smooth solution.

In this paper, we focus on the uniqueness of positive solution to (2) in the case (1) of the Theorem 1.2 with some other additional conditions.

2. PRELIMINARIES

In this section, we state some preliminary results which will be useful for our later arguments.

Definition 2.1. (upper and lower solutions)

$$\begin{cases} \Delta u + f(x, u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where $f \in C^\alpha(\bar{\Omega} \times R)$ and Ω is a bounded domain in R^n.
(A) A function $\bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \bar{u} + f(x, \bar{u}) \le 0 & \text{in } \Omega, \\ \bar{u}|_{\partial\Omega} \ge 0, \end{cases}$$

is called an upper solution to (3).

(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x, \underline{u}) \ge 0 & \text{in } \Omega, \\ \underline{u}|_{\partial\Omega} \le 0 \end{cases}$$

is called a lower solution to (3).

Lemma 2.1. *Let $f(x, \xi) \in C^\alpha(\bar{\Omega} \times R)$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\bar{\Omega})$ be respectively, upper and lower solutions to (3) which satisfy $\underline{u}(x) \le \bar{u}(x), x \in \bar{\Omega}$. Then (3) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x) \le u(x) \le \bar{u}(x), x \in \bar{\Omega}$.*

We also need some information on the solutions of the following logistic equations.

Lemma 2.2. *(Established in [14])*
    Consider

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \le 0$ for $u \ge c_0$ and Ω is a bounded domain in R^n.
    If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary conditions whose corresponding eigenfunction is denoted by ϕ_1. We denote this unique positive solution as θ_f.

The most important property of this positive solution is that θ_f is increasing as f is increasing.

We specifically note that for $a > \lambda_1$, the unique positive solution of

$$\begin{cases} \Delta u + u(a - u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

is denoted by $\omega_a \equiv \theta_{a-x}$. Hence, θ_a is increasing as $a > 0$ is increasing.

3. MAIN UNIQUENESS RESULT

Theorem 3.1. *Suppose $a > \lambda_1, \inf(g_u) < \inf(h_u), \inf(g_v) > \inf(h_v)$ and $\sup(g_u) < \sup(h_u), \sup(g_v) > \sup(h_v)$.*

If

$$4\sup(g_u)\sup(h_v) > \frac{[\sup(g_v) - \sup(h_v)][\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)]}{[\inf(h_u) - \inf(g_u)][\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)]}[\inf(g_v)]^2$$
$$+ \frac{[\sup(h_u) - \sup(g_u)][\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)]}{[\inf(g_v) - \inf(h_v)][\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)]}[\inf(h_u)]^2 + 2\inf(g_v)\inf(h_u),$$

then (2) has a unique positive smooth solution.

Proof. By the Theorem 1.1, both of the following systems

$$\begin{aligned} \Delta u + u[a - (-\inf(g_u))u - (-\inf(g_v))v] &= 0 \\ \Delta v + v[a - (-\inf(h_u))u - (-\inf(h_v))v] &= 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} &= 0. \end{aligned} \quad \text{in } \Omega, \qquad (4)$$

and

$$\begin{aligned} \Delta u + u[a - (-\sup(g_u))u - (-\sup(g_v))v] &= 0 \\ \Delta v + v[a - (-\sup(h_u))u - (-\sup(h_v))v] &= 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} &= 0. \end{aligned} \quad \text{in } \Omega, \qquad (5)$$

have unique positive solutions.

Suppose (u, v) is a positive solution to (2). Then by the Mean Value Theorem,

$$\begin{aligned} & \Delta u + u[a - (-\inf(g_u))u - (-\inf(g_v))v] \\ = & \Delta u + u[a + \inf(g_u)u + \inf(g_v)v] \\ \leq & \Delta u + u[a + g(u, v) - g(0, v) + g(0, v) - g(0, 0)] \\ = & \Delta u + u[a + g(u, v)] \\ = & 0, \end{aligned}$$

and

$$\begin{aligned}
& \Delta v + v[a - (-\inf(h_u))u - (-\inf(h_v))v] \\
= \;& \Delta v + v[a + \inf(h_u)u + \inf(h_v)v] \\
\leq \;& \Delta v + v[a + h(u,v) - h(0,v) + h(0,v) - h(0,0)] \\
= \;& \Delta v + v[a + h(u,v)] \\
= \;& 0.
\end{aligned}$$

Hence, (u,v) is a supersolution to (4).

Since $a - \lambda_1 > 0$, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
& \Delta(\epsilon\phi_1) + \epsilon\phi_1[a - (-\inf(g_u))\epsilon\phi_1 - (-\inf(g_v))\epsilon\phi_1] \\
= \;& -\epsilon\lambda_1\phi_1 + \epsilon\phi_1[a - (-\inf(g_u))\epsilon\phi_1 - (-\inf(g_v))\epsilon\phi_1] \\
= \;& \epsilon\phi_1[-\lambda_1 + a - (-\inf(g_u))\epsilon\phi_1 - (-\inf(g_v))\epsilon\phi_1] \\
> \;& 0,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta(\epsilon\phi_1) + \epsilon\phi_1[a - (-\inf(h_u))\epsilon\phi_1 - (-\inf(h_v))\epsilon\phi_1] \\
= \;& -\epsilon\lambda_1\phi_1 + \epsilon\phi_1[a - (-\inf(h_u))\epsilon\phi_1 - (-\inf(h_v))\epsilon\phi_1] \\
= \;& \epsilon\phi_1[-\lambda_1 + a - (-\inf(h_u))\epsilon\phi_1 - (-\inf(h_v))\epsilon\phi_1] \\
> \;& 0,
\end{aligned}$$

so $(\epsilon\phi_1, \epsilon\phi_1)$ is a subsolution to (4).

But, by the uniqueness of positive solution to (4) and the Lemma 2.1, we have

$$\begin{aligned}
\frac{\inf(g_v) - \inf(h_v)}{\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)}\theta_a \leq u, \\
\frac{\inf(h_u) - \inf(g_u)}{\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)}\theta_a \leq v.
\end{aligned} \tag{6}$$

By the Mean Value Theorem again,

$$\begin{aligned}
& \Delta u + u[a - (-\sup(g_u))u - (-\sup(g_v))v] \\
= \;& \Delta u + u[a + \sup(g_u)u + \sup(g_v)v] \\
\geq \;& \Delta u + u[a + g(u,v) - g(0,v) + g(0,v) - g(0,0)] \\
= \;& \Delta u + u[a + g(u,v)] \\
= \;& 0,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta v + v[a - (-\sup(h_u))u - (-\sup(h_v))v] \\
= \;& \Delta v + v[a + \sup(h_u)u + \sup(h_v)v] \\
\geq \;& \Delta v + v[a + h(u,v) - h(0,v) + h(0,v) - h(0,0)] \\
= \;& \Delta v + v[a + h(u,v)] \\
= \;& 0.
\end{aligned}$$

Hence, (u,v) is a subsolution to (5).

Clearly, for sufficiently large constant $M > 0$, (M, M) is a supersolution to (5).

So, by the uniqueness of positive solution to (5) and the Lemma 2.1 again, we have

$$u \le \frac{\sup(g_v) - \sup(h_v)}{\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)}\theta_a,$$

$$v \le \frac{\sup(h_u) - \sup(g_u)}{\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)}\theta_a. \tag{7}$$

By (6) and (7), we conclude that for any positive solution (u, v) to (2),

$$\frac{\inf(g_v) - \inf(h_v)}{\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)}\theta_a \le u \le \frac{\sup(g_v) - \sup(h_v)}{\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)}\theta_a$$

$$\frac{\inf(h_u) - \inf(g_u)}{\inf(g_u)\inf(h_v) - \inf(g_v)\inf(h_u)}\theta_a \le v \le \frac{\sup(h_u) - \sup(g_u)}{\sup(g_u)\sup(h_v) - \sup(g_v)\sup(h_u)}\theta_a. \tag{8}$$

Suppose (u_1, v_1) and (u_2, v_2) are positive solutions to (2).

Let $p = u_1 - u_2$ and $q = v_1 - v_2$. Then

$$
\begin{aligned}
\Delta p + (a + g(u_1, v_1))p &= \Delta u_1 - \Delta u_2 + (a + g(u_1, v_1))(u_1 - u_2) \\
&= -\Delta u_2 - (a + g(u_1, v_1))u_2 \\
&= -\Delta u_2 - u_2(a + g(u_2, v_2) - g(u_2, v_2) + g(u_1, v_1)) \\
&= -u_2(-g(u_2, v_2) + g(u_1, v_1)) \\
&= -u_2(-g(u_2, v_2) + g(u_1, v_2) - g(u_1, v_2) + g(u_1, v_1)) \\
&= -u_2(\frac{\partial g(\tilde{x}, v_2)}{\partial u}p + \frac{\partial g(u_1, \bar{x})}{\partial v}q) \quad \text{in } \Omega,
\end{aligned}
$$

where $\tilde{x}, \bar{x}$ are from Mean Value Theorem depending on u_1, u_2, v_1, v_2. Hence,

$$\Delta p + (a + g(u_1, v_1))p + u_2(p\frac{\partial g(\tilde{x}, v_2)}{\partial u} + q\frac{\partial g(u_1, \bar{x})}{\partial v}) = 0 \quad \text{in } \Omega. \tag{9}$$

Similarly, we can get

$$\Delta q + (a + h(u_2, v_2))q + v_1(p\frac{\partial h(\tilde{y}, v_1)}{\partial u} + q\frac{\partial h(u_2, \bar{y})}{\partial v}) = 0 \quad \text{in } \Omega, \tag{10}$$

where $\tilde{y}, \bar{y}$ are from Mean Value Theorem depending on u_1, u_2, v_1, v_2. Since $\lambda_1(-a - g(u_1, v_1)) = 0$, by the Variational Characterization of the first eigenvalue we obtain

$$\int_\Omega z(-\Delta z - (a + g(u_1, v_1))z)dx \ge 0 \tag{11}$$

for any $z \in C^2(\bar{\Omega})$ and $z|_{\partial\Omega} = 0$. The same argument shows that

$$\int_\Omega w(-\Delta w - (a + h(u_2, v_2))w)dx \ge 0 \tag{12}$$

for any $w \in C^2(\bar{\Omega})$ and $w|_{\partial\Omega} = 0$. From (9) and (10) we have

$$
\begin{cases}
-p\Delta p - (a + g(u_1, v_1))p^2 - u_2p(p\dfrac{\partial g(\tilde{x}, v_2)}{\partial u} + q\dfrac{\partial g(u_1, \bar{x})}{\partial v}) = 0 \\
-q\Delta q - (a + h(u_2, v_2))q^2 - v_1q(p\dfrac{\partial h(\tilde{y}, v_1)}{\partial u} + q\dfrac{\partial h(u_2, \bar{y})}{\partial v}) = 0
\end{cases}
\quad \text{in } \Omega.
$$

Using (11) and (12), we conclude

$$\int_\Omega [-u_2 p(p\frac{\partial g(\tilde{x}, v_2)}{\partial u} + q\frac{\partial g(u_1, \bar{x})}{\partial v}) - v_1 q(p\frac{\partial h(\tilde{y}, v_1)}{\partial u} + q\frac{\partial h(u_2, \bar{y})}{\partial v})] \le 0.$$

Hence,

$$\int_\Omega [-u_2\frac{\partial g(\tilde{x}, v_2)}{\partial u}p^2 + (-u_2\frac{\partial g(u_1, \bar{x})}{\partial v} - v_1\frac{\partial h(\tilde{y}, v_1)}{\partial u})pq - v_1\frac{\partial h(u_2, \bar{y})}{\partial v}q^2] \le 0.$$

Therefore, $p \equiv q \equiv 0$ if we can show that

$$(u_2\frac{\partial g(u_1, \bar{x})}{\partial v} + v_1\frac{\partial h(\tilde{y}, v_1)}{\partial u})^2 - 4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} < 0 \text{ in } \Omega,$$

which is true if

$$u_2^2(\frac{\partial g(u_1, \bar{x})}{\partial v})^2 + v_1^2(\frac{\partial h(\tilde{y}, v_1)}{\partial u})^2 + 2u_2 v_1\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u}$$
$$- 4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} < 0 \text{ in } \Omega,$$

i.e.,

$$4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} > u_2^2(\frac{\partial g(u_1, \bar{x})}{\partial v})^2 + v_1^2(\frac{\partial h(\tilde{y}, v_1)}{\partial u})^2$$
$$+ 2u_2 v_1\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u} \text{ in } \Omega,$$

or

$$4\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} > \frac{u_2}{v_1}(\frac{\partial g(u_1, \bar{x})}{\partial v})^2 + \frac{v_1}{u_2}(\frac{\partial h(\tilde{y}, v_1)}{\partial u})^2$$
$$+ 2\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u} \text{ in } \Omega.$$

This is the case from the hypothesis of the theorem and (8), and so the uniqueness is proved.

REFERENCES

[1] S.W. Ali and C. Cosner, On the uniqueness of the positive steady state for Lotka-Volterra Models with diffusion, *Journal of Mathematical Analysis and Application*, **168** (1992), 329-341.

[2] R.S. Cantrell and C. Cosner, On the steady-state problem for the Volterra-Lotka competition model with diffusion, *Houston Journal of Mathematics*, **13** (1987), 337-352.

[3] R.S. Cantrell and C. Cosner, On the uniqueness and stability of positive solutions in the Volterra-Lotka competition model with diffusion, *Houston J. Math.*, **15** (1989) 341-361.

[4] C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *Siam J. Appl. Math.*, **44** (1984), 1112-1132.

[5] D. Dunninger, *Lecture Note for Applied Analysis*, Michigan State University.

[6] C. Gui and Y. Lou, Uniqueness and nonuniqueness of coexistence states in the Lotka-Volterra competition model, *Comm Pure and Appl. Math.*, **XVL2**, No. 12 (1994), 1571-1594.

[7] J. L.-Gomez and R. Pardo, Existence and uniqueness for some competition models with diffusion, *C.R. Acad. Sci. Paris, Ser. I Math.*, **313** (1991), 933-938.

[8] J. Kang, Positive equibibrium solutions to general population model, *International Journal of Pure and Applied Mathematics*, **85**, No. 6 (2013), 1009-1019.

[9] J. Kang and Y. Oh, A sufficient condition for the uniqueness of positive steady state to a reaction diffusion system, *Journal of Korean Mathematical Society*, **39**, No. 39 (2002), 377-385.

[10] J. Kang and Y. Oh, Uniqueness of coexistence state of general competition model for several competing species, *Kyungpook Mathematical Journal*, **42**, No. 2 (2002), 391-398.

[11] J. Kang, Y. Oh, and J. Lee, The existence, nonexistence and uniqueness of global positive coexistence of a nonlinear elliptic biological interacting model, *Kangweon-Kyungki Math. Jour.*, **12**, No. 1 (2004), 77-90.

[12] P. Korman and A. Leung, A general monotone scheme for elliptic systems with applications to ecological models, *Proceedings of the Royal Society of Edinburgh*, **102A** (1986), 315-325

[13] P. Korman and A. Leung, On the existence and uniqueness of positive steady states in the Volterra-Lotka ecological models with diffusion, *Applicable Analysis*, **26**, 145-160.

[14] L. Li and R. Logan, Positive solutions to general elliptic competition models, *Differential and Integral Equations*, **4** (1991), 817-834.

[15] A. Leung, Equilibria and stabilities for competing-species, reaction-diffusion equations with Dirichlet boundary data, *J. Math. Anal. Appl.*, **73** (1980), 204-218.