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PERIODIC SOLUTIONS OF FRACTIONAL NABLA DIFFERENCE EQUATIONS

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ABSTRACT: In this article, we discuss periodic properties of nonlinear fractional nabla difference systems. First, we prove that any given system of fractional nabla difference equations doesn't possess a nonconstant periodic solution. Next, we establish sufficient conditions on existence and uniqueness of S-asymptotically periodic solutions for the same. Finally, we provide few examples to illustrate the applicability of established results.

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1. INTRODUCTION

Fractional nabla calculus is a new branch of mathematics that deals with the properties of nabla sums and differences of arbitrary order. Gray & Zhang [15] initiated the study of the theory of fractional nabla differences in 1988. Since then several authors have given valuable contributions to the development of the qualitative theory of fractional nabla difference equations [2, 10, 11, 12, 14, 18, 22, 27, 31, 33, 34, 35]. But a very little progress has been reported for fractional nabla difference systems [13, 25].

Atici and Eloe [13] studied linear systems of fractional nabla difference equations with constant coefficients and constructed the fundamental matrix for the homogeneous system and the causal Green's function for the nonhomogeneous system. Recently, the author established sufficient conditions on existence, uniqueness and

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stability of solutions of nonlinear fractional nabla difference systems [19, 20, 21].

Study of periodic solutions is one of the most important research directions in the theory of fractional differential equations, with applications ranging from celestial mechanics to biology and finance. Tavazoei et al. [29, 30] have shown analytically that a time invariant Caputo type fractional order system contrary to its integer order counterpart cannot generate exactly periodic signals. Belmekki et al. [28] introduced a new and proper concept of periodic boundary value conditions and studied the existence of periodic solutions for a class of fractional differential equations. Kaslik et al. [9] have also shown the nonexistence of exact periodic solutions in a wide class of fractional order dynamical systems using the Mellin transform approach. Wang et al. [23] obtained two basic existence and uniqueness results for asymptotically periodic solution of semi linear fractional Cauchy problem in an asymptotically periodic functions space. Cuevas et al. [5] studied the existence of pseudo S-asymptotically T-periodic mild solutions for a class of abstract fractional differential equation. Diblik et al. [6] have shown that a fractional delta difference equation with a periodic right hand side can not possess a periodic solution but it can have an S-asymptotically periodic solution. Recently, Area et al. [17] investigated quasi-periodic properties of fractional order sums and differences of periodic functions. But the study of periodic solutions for fractional nabla difference systems is not yet reported.

The present paper is organized as follows. Sections 2 and 3 contain preliminaries on fractional nabla calculus and N-transforms, respectively. We discuss periodic properties of integer order and fractional order nabla sums and differences in Section 4. In Section 5, we show that any given system of fractional nabla difference equations doesn't have a nonconstant periodic solution. We also establish sufficient conditions on existence and uniqueness of S-asymptotically periodic solutions of nonlinear fractional nabla difference systems.

2. PRELIMINARIES ON FRACTIONAL NABLA CALCULUS

Throughout, we shall use the following notations, definitions and known results of fractional nabla calculus. We denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For any $a \in \mathbb{R}$, define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$. Let $u = \{u(t)\}_{t \in \mathbb{N}_a}$ be a sequence of real numbers. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1. For any $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds, \quad t > 0,$$

$$\Gamma(t+1) = t\Gamma(t).$$

Definition 2.2. For any $\alpha \in \mathbb{R}$, $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of gamma and rising factorial functions.

Lemma 2.1. *[20] Assume the following factorial functions are well defined.*

1. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
2. *If* $t \leq r$ *then* $t^{\overline{\alpha}} \leq r^{\overline{\alpha}}$.
3. *If* $\alpha < t \leq r$ *then* $r^{\overline{-\alpha}} \leq t^{\overline{-\alpha}}$.

Lemma 2.2. *[36] For any* $\alpha \in [0,1]$, $t \in \mathbb{R} \setminus \{\cdots, -2, -1\}$ *such that* $(t+\alpha) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$,

$$(t+1)^{\alpha-1} \leq \frac{\Gamma(t+\alpha)}{\Gamma(t+1)} \leq t^{\alpha-1}.$$

Lemma 2.3. *[24] For any* a, b, $t \in \mathbb{R}$ *such that* $(t+a)$, $(t+b) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, *the quotient expansion of two gamma functions at infinityis is given by*

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = t^{a-b}\Big[1 + O\Big(\frac{1}{t}\Big)\Big], \quad |t| \to \infty.$$

Definition 2.3. Let $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

1. (Nabla Difference) [1] The first order backward difference or nabla difference of u is defined by
$$(\nabla_a u)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$
and the N^{th}-order nabla difference of u is defined recursively by
$$\big(\nabla_a^N u\big)(t) = \big(\nabla_a\big(\nabla_a^{N-1} u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$
In addition, we take ∇_a^0 as the identity operator.

2. (Fractional Nabla Sum) [11] The α^{th}-order nabla sum of u is given by
$$\big(\nabla_a^{-\alpha} u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a,$$
where $\rho(s) = s-1$. Also, we define the trivial sum by $\big(\nabla_a^{-0} u\big)(t) = u(t)$ for $t \in \mathbb{N}_a$.

3. (R-L Fractional Nabla Difference) [11] The α^{th}-order Riemann-Liouville type nabla difference of u is given by

$$\big(\nabla_a^{\alpha} u\big)(t) = \big(\nabla_a^{N}\big(\nabla_a^{-(N-\alpha)} u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$

For $\alpha = 0$, we set $\big(\nabla_a^0 u\big)(t) = u(t),\ t \in \mathbb{N}_a$.

4. (Caputo Fractional Nabla Difference) [33] The α^{th}-order Caputo type nabla difference of u is given by

$$\big(\nabla_{a*}^{\alpha} u\big)(t) = \big(\nabla_a^{-(N-\alpha)}\big(\nabla_a^{N} u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$

For $\alpha = 0$, we set $\big(\nabla_{a*}^0 u\big)(t) = u(t),\ t \in \mathbb{N}_a$.

Obviously,

1. $\big(\nabla_a^N u\big) = \Big\{\big(\nabla_a^N u\big)(t)\Big\}_{t\in\mathbb{N}_{a+N}}$.
2. $\big(\nabla_a^{-\alpha} u\big) = \Big\{\big(\nabla_a^{-\alpha} u\big)(t)\Big\}_{t\in\mathbb{N}_{a}}$.
3. $\big(\nabla_a^{\alpha} u\big) = \Big\{\big(\nabla_a^{\alpha} u\big)(t)\Big\}_{t\in\mathbb{N}_{a+N}}$.
4. $\big(\nabla_{a*}^{\alpha} u\big) = \Big\{\big(\nabla_{a*}^{\alpha} u\big)(t)\Big\}_{t\in\mathbb{N}_{a+N}}$

are also sequences of real numbers. The unified definition for fractional nabla sum and differences is as follows.

Definition 2.4. [11, 33] Let $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$.

1. The α^{th}-order nabla sum of u is given by

$$\big(\nabla_a^{-\alpha} u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t - \rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

2. The α^{th}-order Riemann-Liouville type nabla difference of u is given by

$$\big(\nabla_a^{\alpha} u\big)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t} (t - \rho(s))^{\overline{-\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+N}.$$

3. The α^{th}-order Caputo type nabla difference of u is given by

$$\big(\nabla_{a*}^{\alpha} u\big)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{-\alpha-1}} u(s) - \sum_{k=0}^{N-1} \frac{(t-a+1)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \big(\nabla_a^k u\big)(a),\ t \in \mathbb{N}_{a+N}.$$

Theorem 2.4. *(Power Rule) [35] Let α, $\mu \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$. Assume the following factorial functions are well defined.*

1. $\nabla_a^N (t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-N+1)}(t-a+1)^{\overline{\mu-N}},\ t \in \mathbb{N}_{a+N}$.
2. $\nabla_a^{-\alpha} (t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+1)^{\overline{\mu+\alpha}},\ t \in \mathbb{N}_{a}$.
3. $\nabla_a^{\alpha} (t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a+1)^{\overline{\mu-\alpha}},\ t \in \mathbb{N}_{a+N}$.
4. $\nabla_{a*}^{\alpha} (t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a+1)^{\overline{\mu-\alpha}},\ t \in \mathbb{N}_{a+N}$.

Definition 2.5. u is said to be T-periodic if there exists a least positive integer T such that

$$u(t+T) = u(t) \text{ for all } t \in \mathbb{N}_a.$$

Definition 2.6. u is said to be S-asymptotically T-periodic if it is bounded and there exists a least positive integer T such that

$$\lim_{t\to\infty}[u(t+T) - u(t)] = 0.$$

3. N-TRANSFORMS

Now we recall the definition of nabla Laplace transform, also known as N-transform.

Definition 3.1. [13] The N-transform of u is defined by

$$N_a\big[u(t)\big] = \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} = U(z)$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 3.2. u is said to be exponentially bounded if there exist an $M > 0$ and a $r \in (0,1)$ such that

$$|u(t)| \leq M(1-r)^{-t}, \quad t \in \mathbb{N}_a.$$

Theorem 3.1. *If u is exponentially bounded, then the N-transform of u exists.*

Proof. We have

$$\begin{aligned}\Big|N_a\big[u(t)\big]\Big| \leq \sum_{j=a}^{\infty}\Big|u(j)(1-z)^{j-1}\Big| &\leq \sum_{j=a}^{\infty}|u(j)||1-z|^{j-1} \\ &\leq M\sum_{j=a}^{\infty}(1-r)^{-j}|1-z|^{j-1}\end{aligned}$$

$$\leq \frac{M}{1-r}\sum_{j=a}^{\infty}\left|\frac{1-z}{1-r}\right|^{j-1}. \quad (3.1)$$

If $a = 0$, the sum in RHS of (3.1) converges for $0 < \left|\frac{1-z}{1-r}\right| < 1$. If $a \geq 1$, the sum converges for $\left|\frac{1-z}{1-r}\right| < 1$. Thus, the N-transform of u exists. □

Remark 1. Note that every periodic sequence of real numbers is bounded and every bounded sequence of real numbers is exponentially bounded. Consequently, if u is periodic then, $U(z)$ exists.

We observe the following properties of N-transforms.

Theorem 3.2. *[13] Let u be exponentially bounded. For any $\alpha \in \mathbb{R}\backslash\{\cdots,-3,-2,-1\}$,*

1. $N_a\big[(t-a+1)^{\overline{\alpha}}\big] = (1-z)^{a-1}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}}$.
2. $N_a\Big[\big(\nabla_a^{-\alpha}u\big)(t)\Big] = z^{-\alpha}N_a\big[u(t)\big]$.

Theorem 3.3. *Let u be exponentially bounded. Then,*

1. $N_a\big[u\big(\rho(t)\big)\big] = (1-z)N_{a-1}\big[u(t)\big]$.
2. $N_a\big[u\big(\rho(t)\big)\big] = u(a-1)(1-z)^{a-1} + (1-z)N_a\big[u(t)\big]$.

Proof. (1) Consider

$$\begin{aligned}
N_a\big[u\big(\rho(t)\big)\big] = \sum_{j=a}^{\infty}u\big(\rho(j)\big)(1-z)^{j-1} &= \sum_{j=a-1}^{\infty}u\big(\rho(j+1)\big)(1-z)^{j} \\
&= (1-z)\sum_{j=a-1}^{\infty}u(j)(1-z)^{j-1} \\
&= (1-z)N_{a-1}\big[u(t)\big].
\end{aligned}$$

(2) Consider

$$\begin{aligned}
N_a\big[u\big(\rho(t)\big)\big] &= \sum_{j=a}^{\infty}u\big(\rho(j)\big)(1-z)^{j-1} \\
&= u\big(\rho(a)\big)(1-z)^{a-1} + \sum_{j=a}^{\infty}u\big(\rho(j+1)\big)(1-z)^{j} \\
&= u(a-1)(1-z)^{a-1} + (1-z)\sum_{j=a}^{\infty}u(j)(1-z)^{j-1} \\
&= u(a-1)(1-z)^{a-1} + (1-z)N_a\big[u(t)\big].
\end{aligned}$$

□

Theorem 3.4. *Let u be exponentially bounded and T-periodic. Then,*

$$N_a\big[u(t)\big] = \frac{1}{1-(1-z)^T}\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}.$$

Proof. Consider

$$\begin{aligned}N_a\big[u(t)\big] &= \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} + \sum_{j=a+T}^{a+2T-1} u(j)(1-z)^{j-1} + \sum_{j=a+2T}^{a+3T-1} u(j)(1-z)^{j-1} + \cdots \\ &= \Big[1+(1-z)^T+(1-z)^{2T}+\cdots\cdots\Big]\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} \\ &= \frac{1}{1-(1-z)^T}\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}.\end{aligned}$$ □

Theorem 3.5. *(Final Value Theorem) Let u be exponentially bounded and $u(\infty)$ exists. Then*

$$\lim_{t\to\infty} u(t) = \lim_{z\to 0^+} zU(z).$$

Proof. Consider

$$\begin{aligned}N_a\big[\big(\nabla_a u\big)(t)\big] &= \sum_{j=a}^{\infty}\big[u(j)-u(j-1)\big](1-z)^{j-1} \\ &= \lim_{t\to\infty}\sum_{j=a}^{t}\big[u(j)-u(j-1)\big](1-z)^{j-1} \\ &= \lim_{t\to\infty}\big[u(t)-u(a-1)\big] \\ &= \lim_{t\to\infty} u(t) - u(a-1).\end{aligned} \tag{3.2}$$

On the other hand, using Theorem 3.3 (2), we have

$$\begin{aligned}N_a\big[\big(\nabla_a u\big)(t)\big] &= N_a\big[u(t)\big] - N_a\big[u(t-1)\big] \\ &= N_a\big[u(t)\big] - N_a\big[u(\rho(t))\big] \\ &= N_a\big[u(t)\big] - u(a-1)(1-z)^{a-1} - (1-z)N_a\big[u(t)\big] \\ &= zN_a\big[u(t)\big] - u(a-1)(1-z)^{a-1}.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we get

$$zN_a\big[u(t)\big] - u(a-1)(1-z)^{a-1} = \lim_{t\to\infty} u(t) - u(a-1).$$

Letting $z \to 0^+$, the proof is complete. □

We now discuss the boundedness of $\big(\nabla_a^{-\alpha} u\big)$.

Theorem 3.6. *Let $0 < \alpha < 1$ and u be exponentially bounded such that*

$$\sum_{j=a}^{\infty} u(j) < \infty. \tag{3.4}$$

Then $\left(\nabla_a^{-\alpha} u\right)$ is bounded.

Proof. We have

$$\lim_{z\to 0^+} U(z) = \sum_{j=a}^{\infty} u(j) < \infty.$$

Consider

$$\begin{aligned}\lim_{t\to\infty}\Big[\big(\nabla_a^{-\alpha}u\big)(t)\Big] = \lim_{z\to 0^+}\Big[z\ N_a\big[\big(\nabla_a^{-\alpha}u\big)(t)\big]\Big] &= \lim_{z\to 0^+}\Big[\frac{1}{z^{\alpha}}\big(zU(z)\big)\Big]\\ &= \lim_{z\to 0^+}\Big[z^{1-\alpha}U(z)\Big] = 0,\end{aligned}$$

implies $\left(\nabla_a^{-\alpha} u\right)$ is bounded. □

Theorem 3.7. *Let u be exponentially bounded and T-periodic such that*

$$\sum_{j=a}^{a+T-1} u(j) = 0. \tag{3.5}$$

Then $\left(\nabla_a^{-\alpha} u\right)$ is bounded for $0 < \alpha < 1$.

Proof. Using Theorem 3.4, (3.5) and L'Hôspital's rule, we have

$$\lim_{z\to 0^+} U(z) = \lim_{z\to 0^+}\Big[\frac{1}{1-(1-z)^T}\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}\Big] = -\frac{1}{T}\sum_{j=a}^{a+T-1} ju(j) < \infty.$$

Consider

$$\lim_{t\to\infty}\Big[\big(\nabla_a^{-\alpha}u\big)(t)\Big] = \lim_{z\to 0^+}\Big[\frac{1}{z^{\alpha}}\big(zU(z)\big)\Big] = \lim_{z\to 0^+}\Big[z^{1-\alpha}U(z)\Big] = 0,$$

implies $\left(\nabla_a^{-\alpha} u\right)$ is bounded. □

4. PERIODIC PROPERTIES OF FRACTIONAL NABLA DIFFERENCES

In this section we discuss periodic properties of integer order and fractional order nabla differences.

Theorem 4.1. *Let u be a nonconstant exponentially bounded T-periodic sequence of real numbers, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$. Then,* *1.* $\left(\nabla_a^N u\right)$ *is also nonconstant* T*-periodic.*

2. $\left(\nabla_a^{-\alpha} u\right)$ *cannot be* T*-periodic.*

3. $\left(\nabla_a^{\alpha} u\right)$ *cannot be* T*-periodic.*

4. $\left(\nabla_{a*}^{\alpha} u\right)$ *cannot be* T*-periodic.*

Proof. The proof of (1) is trivial. Now, we prove (2). Suppose there exists a nonconstant T-periodic sequence u such that $\left(\nabla_a^{-\alpha} u\right)$ is also T-periodic. Let

$$\left(\nabla_a^{-\alpha} u\right)(t) = v(t), \quad t \in \mathbb{N}_a.$$

Applying N-transform on both sides, we get

$$z^{-\alpha} N_a[u(t)] = N_a[v(t)].$$

Since u and v are T-periodic, using Theorem 3.4, we have

$$\frac{1}{1-(1-z)^T} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} = \frac{z^\alpha}{1-(1-z)^T} \sum_{j=a}^{a+T-1} v(j)(1-z)^{j-1},$$

or

$$\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} = z^\alpha \sum_{j=a}^{a+T-1} v(j)(1-z)^{j-1}.$$

Comparing the coefficients of $(1-z)^{j-1}$ $(j = a, a+1, \cdots, a+T-1)$ on both sides, we have

$$u(j) = z^\alpha v(j), \quad j = a, a+1, \cdots, a+T-1.$$

Letting $z \to 0^+$, we get

$$u(j) = 0, \quad j = a, a+1, \cdots, a+T-1.$$

This is a contradiction. Hence, our assumption is false, and therefore, $\left(\nabla_a^{-\alpha} u\right)$ cannot be a T-periodic sequence. Replacing α by $-\alpha$ in (2), we get (3). From definition 2.3, the Caputo type fractional nabla difference of u is given by

$$\left(\nabla_{a*}^{\alpha} u\right)(t) = \left(\nabla_a^{-(N-\alpha)}\left(\nabla_a^N u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Given that u is a nonconstant T-periodic sequence. Then, from (1), $\left(\nabla_a^N u\right)$ is also a nonconstant T-periodic sequence and hence from (2), $\left(\nabla_{a*}^{\alpha} u\right)$ cannot be a T-periodic sequence. □

Theorem 4.2. *Let* $0 < \alpha < 1$ *and* $u = \{u(t)\}_{t \in \mathbb{N}_a}$ *be a nonconstant exponentially bounded* T*-periodic sequence of real numbers such that (3.5) holds. Then,*

1. $\big(\nabla_a u\big)$ *is S-asymptotically T-periodic.*

2. $\big(\nabla_a^{-\alpha} u\big)$ *is S-asymptotically T-periodic.*

3. $\big(\nabla_a^{\alpha} u\big)$ *is S-asymptotically T-periodic.*

4. $\big(\nabla_{a*}^{\alpha} u\big)$ *is S-asymptotically T-periodic.*

Proof. Clearly $\big(\nabla_a u\big)$ is T-periodic. Since every T-periodic sequence is S-asymptotically T-periodic, we have (1). To prove (2), we consider

$$\begin{aligned}
\lim_{t\to\infty}\Big[\big(\nabla_a^{-\alpha}u\big)(t+T)-\big(\nabla_a^{-\alpha}u\big)(t)\Big] &= \lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}(t+T-\rho(s))^{\overline{\alpha-1}}u(s)\Big]\\
&= \lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}\Big(\frac{\Gamma(t+T-s+\alpha)}{\Gamma(t+T-s+1)}\Big)u(s)\Big]\\
&= \Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}u(s)\Big]\lim_{t\to\infty}\Big[t^{\alpha-1}+O\Big(\frac{1}{t}\Big)\Big]\\
&= 0.
\end{aligned}$$

We have already shown that $\big(\nabla_a^{-\alpha} u\big)$ is bounded (Theorem 3.7). Thus, $\big(\nabla_a^{-\alpha} u\big)$ is S-asymptotically T-periodic. Proofs of (3) and (4) are similar to the proofs of (3) and (4) of Theorem 4.1. So, we omit it. □

Theorem 4.3. *Let $0<\alpha<1$ and $u=\{u(t)\}_{t\in\mathbb{N}_a}$ be a nonconstant exponentially bounded S-asymptotically T-periodic sequence of real numbers such that (3.4) and (3.5) hold. Then,*

1. $\big(\nabla_a u\big)$ *is S-asymptotically T-periodic.*

2. $\big(\nabla_a^{-\alpha} u\big)$ *is S-asymptotically T-periodic.*

3. $\big(\nabla_a^{\alpha} u\big)$ *is S-asymptotically T-periodic.*

4. $\big(\nabla_{a*}^{\alpha} u\big)$ *is S-asymptotically T-periodic.*

Proof. Let $v(t)=u(t+T)-u(t)$. Since u is S-asymptotically T-periodic,

$$\lim_{t\to\infty}\big[u(t+T)-u(t)\big]=0\Rightarrow\lim_{t\to\infty}\big[v(t)\big]=0.$$

Consequently, from Theorem 3.5, we have

$$\lim_{z\to 0^+} zV(z)=0.$$

Consider

$$\lim_{t\to\infty}\Big[\big(\nabla_a u\big)(t+T)-\big(\nabla_a u\big)(t)\Big] = \lim_{t\to\infty}\Big[\big(\nabla_a v\big)(t)\Big]$$

$$\begin{aligned}
&= \lim_{t\to\infty}\Big[v(t)-v(t-1)\Big] \\
&= \lim_{z\to 0^+}\Big[z^2V(z)-v(a-1)z(1-z)^{a-1}\Big] \\
&= 0.
\end{aligned}$$

Also, $\big(\nabla_a u\big)$ is bounded. Thus, $\big(\nabla_a u\big)$ is an S-asymptotically T-periodic sequence. Next, we prove (2). We have

$$\begin{aligned}
V(z)=N_a\big[v(t)\big] &= N_a\big[u(t+T)-u(t)\big] \\
&= N_a\big[u(t+T)\big]-N_a\big[u(t)\big] \\
&= \sum_{j=a}^{\infty}u(j+T)(1-z)^{j-1}-U(z) \\
&= (1-z)^{-T}\sum_{j=a+T}^{\infty}u(j)(1-z)^{j-1}-U(z) \\
&= (1-z)^{-T}\Big[\sum_{j=a}^{\infty}u(j)(1-z)^{j-1}-\sum_{j=a}^{a+T-1}u(j)(1-z)^{j-1}\Big]-U(z) \\
&= (1-z)^{-T}\Big[U(z)-\sum_{j=a}^{a+T-1}u(j)(1-z)^{j-1}\Big]-U(z) \\
&= \Big[\frac{1}{(1-z)^T}-1\Big]U(z)-(1-z)^{-T}\sum_{j=a}^{a+T-1}u(j)(1-z)^{j-1}.
\end{aligned}$$

From (3.4), we have

$$\lim_{t\to\infty}\big[u(t)\big]=0\Rightarrow\lim_{z\to 0^+}zU(z)=0.$$

Consequently,

$$\lim_{z\to 0^+}V(z)=\lim_{z\to 0^+}\Big[\frac{1}{(1-z)^T}\Big(\frac{1-(1-z)^T}{z}\Big)\big(zU(z)\big)\Big]-\sum_{j=a}^{a+T-1}u(j)=0.$$

Now, consider

$$\begin{aligned}
&\lim_{t\to\infty}\Big[\big(\nabla_a^{-\alpha}u\big)(t+T)-\big(\nabla_a^{-\alpha}u\big)(t)\Big] \\
&= \lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t+T}(t+T-\rho(s))^{\overline{\alpha-1}}u(s)-\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\alpha-1}}u(s)\Big] \\
&= \lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\alpha-1}}v(s)\Big]+\lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}(t+T-\rho(s))^{\overline{\alpha-1}}u(s)\Big] \\
&= \lim_{z\to 0^+}\Big[z\Big(\frac{1}{z^\alpha}V(z)\Big)\Big]+\lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}\frac{\Gamma(t+T-s+\alpha)}{\Gamma(t+T-s+1)}u(s)\Big] \\
&= \lim_{z\to 0^+}\Big[z^{1-\alpha}V(z)\Big]+\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}u(s)\Big]\lim_{t\to\infty}\Big[t^{\alpha-1}+O\Big(\frac{1}{t}\Big)\Big]
\end{aligned}$$

$$= \quad 0.$$

Also, from Theorem 3.6, $\left(\nabla_a^{-\alpha}u\right)$ is bounded. Thus, $\left(\nabla_a^{-\alpha}u\right)$ is S-asymptotically T-periodic. Proofs of (3) and (4) are similar to the proofs of (3) and (4) of Theorem 4.1. So, we omit it. □

5. PERIODIC SOLUTIONS OF FRACTIONAL NABLA DIFFERENCE SYSTEMS

Let $0 < \alpha < 1$. Consider the following initial value problems

$$\begin{aligned} \left(\nabla_0^{\alpha}\mathbf{u}\right) &= \mathbf{f}(t,\mathbf{u}), \quad t \in \mathbb{N}_1, \\ \left(\nabla_0^{-(1-\alpha)}\mathbf{u}\right)(t)\Big|_{t=0} &= \mathbf{u}(0) = \mathbf{c}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \left(\nabla_{0*}^{\alpha}\mathbf{u}\right) &= \mathbf{f}(t,\mathbf{u}), \quad t \in \mathbb{N}_1, \\ \mathbf{u}(0) &= \mathbf{c}, \end{aligned} \tag{5.2}$$

where ∇_0^{α} and ∇_{0*}^{α} are the Riemann-Liouville and Caputo type fractional nabla difference operators, respectively; $\mathbf{u} = \{\mathbf{u}(t)\}_{t\in\mathbb{N}_0}$ is a sequence of n-vectors and $\mathbf{f} : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$.

The main results presented in the previous section lead us to the following consequence.

Theorem 5.1. *There are no non-constant T-periodic solutions of (5.1) and (5.2).*

Proof. Assume that there exists a non-constant T-periodic solution $\mathbf{u}$ of (5.1) (or (5.2)). From the T-periodicity of the function $\mathbf{f}$ with respect to its first argument, it follows that $\left(\nabla_{0*}^{\alpha}\mathbf{u}\right)$ is also T-periodic, which is contradiction to Theorem 4.1. □

Now, we establish sufficient conditions on existence and uniqueness of S-asymptotically T-periodic solutions of (5.1) and (5.2) using Banach's fixed point theorem.

Definition 5.1. (Contraction Mapping) Let (X,ρ) be a complete metric space and $P : X \to X$. The map P is said to be contraction if there exists a positive constant $a < 1$ such that for each pair $x, y \in X$ we have

$$\rho(Px, Py) \leq a\rho(x,y). \tag{5.3}$$

The constant a is called the contraction constant of P.

Theorem 5.2. *(Banach Fixed Point Theorem) Let (X,ρ) be a complete metric space and let $P : X \to X$ be contraction. Then P has a unique fixed point $u \in X$, that is,*

$$Pu = u. \tag{5.4}$$

Theorem 5.3. *Let (X,ρ) be a complete metric space containing an open ball having center x_0 and radius r. Let $P : B_r(x_0) \to X$ be a contraction map with a positive number $a < 1$ as the contraction constant. If*

$$\rho(Px_0, x_0) < (1-a)r, \tag{5.5}$$

then P has a unique fixed point in $B_r(x_0)$.

Definition 5.2. $\mathbb{R}^n$ is the space of all n-vectors. Clearly, $\mathbb{R}^n$ is a Banach space with respect to the Euclidean norm $\| \cdot \|$.

Definition 5.3.

$$\mathbf{l}^\infty = \mathbf{l}^\infty(\mathbb{R}^n) = \{\mathbf{u} : \mathbf{u} = \{\mathbf{u}(t)\}_{t\in\mathbb{N}_0},\ \mathbf{u}(t) \in \mathbb{R}^n \text{ with } \|\mathbf{u}\|_\infty < \infty\}$$

denotes the Banach space comprising bounded sequences of n-vectors with respect to the supremum norm $\| \cdot \|_\infty$ defined by

$$\|\mathbf{u}\|_\infty = \sup_{t\in\mathbb{N}_0} \|\mathbf{u}(t)\|.$$

Definition 5.4. Define

$$\mathbf{l}^\infty_{ST} = \mathbf{l}^\infty_{ST}(\mathbb{R}^n) = \{\mathbf{u} \in \mathbf{l}^\infty : \mathbf{u} \text{ is } S\text{-asymptotically } T\text{-periodic with } \|\mathbf{u}\|_\infty < \infty\}.$$

Clearly, $\mathbf{l}^\infty_{ST}$ is a subspace of $\mathbf{l}^\infty$. We observe that $\mathbf{l}^\infty_{ST}$ is a Banach space with respect to the supremum norm $\| \cdot \|_\infty$ defined by

$$\|\mathbf{u}\|_\infty = \sup_{t\in\mathbb{N}_0} \|\mathbf{u}(t)\|.$$

Using the supremum norm $\| \cdot \|_\infty$, we define a contraction map on $\mathbf{l}^\infty_{ST}$ that yield fixed points as nonconstant S-asymptotically T-periodic solutions to the initial value problems (5.1) and (5.2).

We know that, $\mathbf{u}$ is a solution of (5.1) if and only if

$$\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.6}$$

Similarly, $\mathbf{u}$ is a solution of (5.2) if and only if

$$\mathbf{u}(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.7}$$

Define the operators

$$(F\mathbf{u})(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0, \tag{5.8}$$

$$(F'\mathbf{u})(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.9}$$

It is evident from (5.5) - (5.8) that $\mathbf{u}$ is a fixed point of F if and only if $\mathbf{u}$ is a solution of (5.1) and $\mathbf{u}$ is a fixed point of F' if and only if $\mathbf{u}$ is a solution of (5.2).

Now we make the following assumptions to establish the main result of this section.

Define

$$D = \{(t,\mathbf{u}) : \|\mathbf{u}(t)\| \le M\} \subseteq \mathbb{N}_0 \times \mathbb{R}^n$$

and

$$B_p^{\infty}(\mathbf{0}) = \Big\{\mathbf{u} \in \mathbf{l}_{ST}^{\infty} : \|\mathbf{u}\|_{\infty} < p\Big\},\ p = \frac{M}{(1-L)^2},$$

is an open ball with radius r centered on the null vector sequence in $\mathbf{l}_{ST}^{\infty}$. Now we make the following assumptions to establish main results of this section.

(C) $\mathbf{f}$ is continuous with respect to the second argument.

(L) There exists a nonnegative sequence of real numbers $\{a(t)\}_{t\in\mathbb{N}_0}$ such that, for all $(t,\mathbf{u})$, $(t,\mathbf{v}) \in \mathbb{N}_0 \times \mathbb{R}^n$,

$$\|\mathbf{f}(t,\mathbf{u}) - \mathbf{f}(t,\mathbf{v})\| \le a(t)\|\mathbf{u}-\mathbf{v}\|, \quad t \in \mathbb{N}_0,$$

and

$$L = \sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\Big] < 1.$$

(L') There exists a nonnegative sequence of real numbers $\{a(t)\}_{t\in\mathbb{N}_0}$ such that, for all $(t,\mathbf{u})$, $(t,\mathbf{v}) \in D$,

$$\|\mathbf{f}(t,\mathbf{u}) - \mathbf{f}(t,\mathbf{v})\| \le a(t)\|\mathbf{u}-\mathbf{v}\|, \quad t \in \mathbb{N}_0,$$

and

$$L = \sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\Big] < 1.$$

(A) Assume that

$$\sup_{t\in\mathbb{N}_0}\Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{0})\Big\| = M < \infty.$$

(A') Assume that

$$\sup_{t\in\mathbb{N}_0}\left\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{0})\right\| < \frac{M}{1-L}.$$

(B) There exists a nonnegative sequence of real numbers $\{b(t)\}_{t\in\mathbb{N}_0}$ such that, for all $(t,\mathbf{u})$, $(t,\mathbf{v})\in\mathbb{N}_0\times\mathbb{R}^n$,

$$\|\mathbf{f}(t+T,\mathbf{u})-\mathbf{f}(t,\mathbf{u})\| \le b(t)\big[\|\mathbf{u}\|_\infty+1\big],\quad t\in\mathbb{N}_0,$$

and

$$\lim_{t\to\infty}\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}b(s)\right]=0.$$

(B') There exists a nonnegative sequence of real numbers $\{b(t)\}_{t\in\mathbb{N}_0}$ such that, for all $(t,\mathbf{u})$, $(t,\mathbf{v})\in D$,

$$\|\mathbf{f}(t+T,\mathbf{u})-\mathbf{f}(t,\mathbf{u})\| \le b(t)\big[\|\mathbf{u}\|_\infty+1\big],\quad t\in\mathbb{N}_0,$$

and

$$\lim_{t\to\infty}\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}b(s)\right]=0.$$

Theorem 5.4. *(Global Existence & Uniqueness) Let (C), (L), (A) and (B) hold. Then there exists a unique S-asymptotically T-periodic solution of (5.1) and (5.2) in $\mathbf{l}^\infty_{ST}$.*

Proof. We use Banach's fixed point theorem (Theorem 5.2) to establish global existence and uniqueness of S-asymptotically T-periodic solutions of (5.1) and (5.2) in $\mathbf{l}^\infty_{ST}$. We know that $\mathbf{l}^\infty_{ST}$ is a complete metric space with respect to the sup-metric defined by

$$\rho(\mathbf{u},\mathbf{v})=\sup_{t\in\mathbb{N}_0}\|\mathbf{u}(t)-\mathbf{v}(t)\|,$$

for each pair $\mathbf{u}$, $\mathbf{v}\in\mathbf{l}^\infty_{ST}$. First, we show that F' maps $\mathbf{l}^\infty_{ST}$ into $\mathbf{l}^\infty_{ST}$. Let $\mathbf{u}\in\mathbf{l}^\infty_{ST}$. Then, $\mathbf{u}$ is S-asymptotically T-periodic. By definition 2.6, $\mathbf{u}$ is bounded, continuous and for any $\epsilon>0$, there exists $N=N(\epsilon)$ such that

$$\sup_{t\in\mathbb{N}_N}\|\mathbf{u}(t+T)-\mathbf{u}(t)\|<\epsilon.$$

Using Lemma 2.1, Theorem 2.4, (L) and (A), we have

$$\big\|\big(F'\mathbf{u}\big)(t)\big\| \;\le\; \|\mathbf{c}\|+\left\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\right\|$$

$$
\begin{aligned}
&= \|\mathbf{c}\| + \Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}[\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0}) + \mathbf{f}(s,\mathbf{0})]\Big\| \\
&\leq \|\mathbf{c}\| + \Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}[\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0})]\Big\| \\
&\quad + \Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{0})\Big\| \\
&\leq \|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0})\| \\
&\quad + \Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{0})\Big\| \\
&\leq \|\mathbf{c}\|_\infty + L\|\mathbf{u}\|_\infty + M, \quad t \in \mathbb{N}_0,
\end{aligned}
$$

implies $F'\mathbf{u}$ is bounded. Since $\mathbf{f}$ is continuous, $F'\mathbf{u}$ is also continuous. Consider

$$
\begin{aligned}
&\big\|\big(F'\mathbf{u}\big)(t+T) - \big(F'\mathbf{u}\big)(t)\big\| \\
&= \frac{1}{\Gamma(\alpha)}\Big\|\sum_{s=1}^{t+T}(t+T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)) - \sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\Big\| \\
&= \frac{1}{\Gamma(\alpha)}\Big\|\sum_{s=1-T}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s+T,\mathbf{u}(s+T)) - \sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\Big\| \\
&\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big\|\mathbf{f}(s+T,\mathbf{u}(s+T)) - \mathbf{f}(s,\mathbf{u}(s+T))\big\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big\|\mathbf{f}(s,\mathbf{u}(s+T)) - \mathbf{f}(s,\mathbf{u}(s))\big\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{T}(t+T-\rho(s))^{\overline{\alpha-1}}\big\|\mathbf{f}(s,\mathbf{u}(s))\big\| \\
&= S_1 + S_2 + S_3. \qquad (5.10)
\end{aligned}
$$

Using (B), we have

$$
S_1 \leq \big(\|\mathbf{u}\|_\infty + 1\big)\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}b(s)\Big]. \qquad (5.11)
$$

Using (L), we have

$$
\begin{aligned}
S_2 &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s+T) - \mathbf{u}(s)\| \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{N-1}(t-\rho(s))^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s+T) - \mathbf{u}(s)\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\sum_{s=N}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s+T)-\mathbf{u}(s)\| \\
\leq\ & \frac{2\|\mathbf{u}\|_\infty}{\Gamma(\alpha)}\sum_{s=1}^{N-1}(t-\rho(s))^{\overline{\alpha-1}}a(s)+\frac{\epsilon}{\Gamma(\alpha)}\sum_{s=N}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s). \qquad (5.12)
\end{aligned}
$$

We know that $(t-s)\geq\frac{t}{N}(N-s)$ for $1\leq s\leq N$. Then, for each $1\leq s\leq N$,

$$
\begin{aligned}
(t-\rho(s))^{\overline{\alpha-1}} &\leq (t-s)^{\alpha-1} \\
&\leq \left(\frac{t}{N}\right)^{\alpha-1}(N-s)^{\alpha-1} \\
&= \left(\frac{t}{N}\right)^{\alpha-1}\frac{\Gamma(N-1-s+\alpha)}{\Gamma(N-s)} \\
&= \left(\frac{t}{N}\right)^{\alpha-1}(N-1-\rho(s))^{\overline{\alpha-1}}. \qquad (5.13)
\end{aligned}
$$

Using (5.13) in (5.12), we get

$$
\begin{aligned}
S_2 &\leq \frac{2\|\mathbf{u}\|_\infty}{\Gamma(\alpha)}\left(\frac{t}{N}\right)^{\alpha-1}\sum_{s=1}^{N-1}(N-1-\rho(s))^{\overline{\alpha-1}}(N-1)^{\overline{-\beta_1}}a(s) \\
&\quad+\frac{\epsilon}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s) \\
&\leq L\Big[2\|\mathbf{u}\|_\infty\left(\frac{t}{N}\right)^{\alpha-1}+\epsilon\Big]. \qquad (5.14)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
S_3 &\leq \frac{\|\mathbf{f}\|_\infty}{\Gamma(\alpha)}\sum_{s=1}^{T}(t+T-\rho(s))^{\overline{\alpha-1}} \\
&\leq (t+1)^{\overline{\alpha-1}}\frac{\|\mathbf{f}\|_\infty}{\Gamma(\alpha)}\sum_{s=1}^{T}1 \\
&= \frac{T\|\mathbf{f}\|_\infty}{\Gamma(\alpha)}\frac{\Gamma(t+\alpha)}{\Gamma(t+1)} \\
&\leq \frac{T\|\mathbf{f}\|_\infty}{\Gamma(\alpha)}t^{\alpha-1}. \qquad (5.15)
\end{aligned}
$$

Using (5.11), (5.14) and (5.15) in (5.10), for $t\in\mathbb{N}_N$, (i.e. as $t\to\infty$), we get

$$
\begin{aligned}
\big\|\big(F'\mathbf{u}\big)(t+T)-\big(F'\mathbf{u}\big)(t)\big\| &\leq \big(\|\mathbf{u}\|_\infty+1\big)\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}b(s)\Big] \\
&\quad+L\Big[2\|\mathbf{u}\|_\infty\left(\frac{t}{N}\right)^{\alpha-1}+\epsilon\Big]+\frac{T\|\mathbf{f}\|_\infty}{\Gamma(\alpha)}t^{\alpha-1},
\end{aligned}
$$

implies

$$\lim_{t\to\infty}\big\|\big(F'\mathbf{u}\big)(t+T)-\big(F'\mathbf{u}\big)(t)\big\|=\mathbf{0}.$$

Thus, $F'\mathbf{u}$ is S-asymptotically T-periodic and hence $F' : \mathbf{l}^\infty_{ST} \to \mathbf{l}^\infty_{ST}$. Similarly, we prove that $F : \mathbf{l}^\infty_{ST} \to \mathbf{l}^\infty_{ST}$. Now we show that F' is a contraction map. For all $\mathbf{u}$, $\mathbf{v} \in \mathbf{l}^\infty_{ST}$, using Lemma 2.1, Theorem 2.4, (L) and (A), we have

$$\begin{aligned}\big\|\big(F'\mathbf{u}\big)(t) - \big(F'\mathbf{v}\big)(t)\big\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{v}(s))\big\| \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\big\|\mathbf{u}(s) - \mathbf{v}(s)\big\| \\ &\leq L\|\mathbf{u}-\mathbf{v}\|_\infty, \quad t \in \mathbb{N}_0,\end{aligned}$$

implies $\big\|F'\mathbf{u} - F'\mathbf{v}\big\|_\infty \leq L\|\mathbf{u}-\mathbf{v}\|_\infty$. Since $L < 1$, F' is contraction. Hence by Theorem 5.2, F' has a unique fixed point $\mathbf{u} \in \mathbf{l}^\infty_{ST}$. Similarly we can prove that F has a unique fixed point $\mathbf{u} \in \mathbf{l}^\infty_{ST}$. Hence the proof. □

Theorem 5.5. *(Local Existence & Uniqueness) Let (C), (L'), (A') and (B') hold. Then there exists a unique S-asymptotically T-periodic solution of (5.1) and (5.2) in $B^\infty_p(\mathbf{0})$.*

Proof. We use Theorem 5.3 to establish local existence and uniqueness of solutions of (5.1) and (5.2). Clearly F and F' map $B^\infty_p(\mathbf{0})$ into $\mathbf{l}^\infty$. We have already proved that F and F' are contractions with contraction constant $L < 1$. Now consider

$$\begin{aligned}\big\|F'\mathbf{0} - \mathbf{0}\big\| &\leq \Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{0})\Big\| \\ &< \frac{M}{1-L} \\ &= (1-L)p,\end{aligned}$$

implies $\big\|F'\mathbf{0} - \mathbf{0}\big\|_\infty < (1-L)p$. Hence by Theorem 5.3, F' has a unique fixed point $\mathbf{u} \in B^\infty_p(\mathbf{0})$. Similarly we can prove that F has a unique fixed point $\mathbf{u} \in B^\infty_p(\mathbf{0})$. □

The following example demonstrates the applicability of Theorem 5.4.

Example 1. Let d, b, α, $\beta \in \mathbb{R}^+$ such that $0 < \alpha < \beta < 1$ and $d\Gamma(1-\beta) < 1$. Consider the scalar initial value problem

$$\big(\nabla^\alpha_0 u\big) = \frac{d\sin u}{(t+1)^\beta} + b\sin\Big(\frac{\pi}{2}\Big)t, \quad t \in \mathbb{N}_1, \tag{5.16}$$

$$\big(\nabla^{-\alpha}_0 u\big)(t)\Big|_{t=0} = u(0) = c. \tag{5.17}$$

(C) $f(t,u) = d(t+1)^{-\beta}\sin u + b\sin\big(\frac{\pi}{2}\big)t$ is continuous with respect to the second argument.

(L) For any (t,u), $(t,v)\in\mathbb{N}_0\times\mathbb{R}^n$,

$$|f(t,u)-f(t,v)|\le d(t+1)^{-\beta}|u-v|\le d(t+1)^{\overline{-\beta}}|u-v|.$$

Here $a(t)=d(t+1)^{\overline{-\beta}}$ is a nonnegative function defined on $\mathbb{N}_0$ and

$$\begin{aligned}
L &= \sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\Big]\\
&= \sup_{t\in\mathbb{N}_0}\Big[\frac{d}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s+1)^{\overline{-\beta}}\Big]\\
&= \sup_{t\in\mathbb{N}_0}\Big[\frac{d}{\Gamma(\alpha)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s+1)^{\overline{-\beta}}-\frac{d}{\Gamma(\alpha)}(t+1)^{\overline{\alpha-1}}(1)^{\overline{-\beta}}\Big]\\
&= \sup_{t\in\mathbb{N}_0}\Big[d\nabla_0^{-\alpha}(t+1)^{\overline{-\beta}}-d\frac{\Gamma(1-\beta)}{\Gamma(\alpha)}(t+1)^{\overline{\alpha-1}}\Big]\\
&= d\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\sup_{t\in\mathbb{N}_0}\Big[(t+1)^{\overline{\alpha-\beta}}\Big]-d\frac{\Gamma(1-\beta)}{\Gamma(\alpha)}\inf_{t\in\mathbb{N}_0}\Big[(t+1)^{\overline{\alpha-1}}\Big]\\
&= d\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}(1)^{\overline{\alpha-\beta}}-0\\
&= d\Gamma(1-\beta)\\
&< 1.
\end{aligned}$$

(A) Consider

$$\sup_{t\in\mathbb{N}_0}\Big|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,0)\Big|=b\sup_{t\in\mathbb{N}_0}\Big|\nabla_1^{-\alpha}\sin\Big(\frac{\pi}{2}\Big)t\Big|.$$

Clearly, $\{\sin\left(\frac{\pi}{2}\right)t\}_{t\in\mathbb{N}_1}$ is an exponentially bounded 4-periodic sequence of real numbers such that

$$\sum_{s=1}^{4}\sin\Big(\frac{\pi}{2}\Big)s=0.$$

Then, from Theorem 3.7, $\big\{\nabla_1^{-\alpha}\sin\left(\frac{\pi}{2}\right)t\big\}_{t\in\mathbb{N}_0}$ is a bounded sequence. Consequently,

$$\sup_{t\in\mathbb{N}_0}\Big|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,0)\Big|<\infty.$$

(B) Consider

$$\begin{aligned}
|f(t+4,u)-f(t,u)| &\le d\big|(t+5)^{-\beta}-(t+1)^{-\beta}\big|\big|\sin u\big|+b\Big|\sin\Big(\frac{\pi}{2}\Big)(t+4)-\sin\Big(\frac{\pi}{2}\Big)t\Big|\\
&\le d(t+1)^{-\beta}\Big[\Big(\frac{t+1}{t+5}\Big)^{\beta}+1\Big]\big[|u|+1\big]\\
&\le 2d(t+1)^{\overline{-\beta}}\big[|u|+1\big].
\end{aligned}$$

Here $b(t) = 2d(t+1)^{\overline{-\beta}}$ is a nonnegative function defined on $\mathbb{N}_0$ and

$$\begin{aligned}
&\lim_{t\to\infty}\Big[\frac{2d}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s+1)^{\overline{-\beta}}\Big]\\
=\;&\lim_{t\to\infty}\Big[\frac{2d}{\Gamma(\alpha)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s+1)^{\overline{-\beta}}-\frac{2d}{\Gamma(\alpha)}(t+1)^{\overline{\alpha-1}}(1)^{\overline{-\beta}}\Big]\\
=\;&\lim_{t\to\infty}\Big[2d\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\frac{\Gamma(t+1+\alpha-\beta)}{\Gamma(t+1)}-2d\frac{\Gamma(1-\beta)}{\Gamma(\alpha)}\frac{\Gamma(t+\alpha)}{\Gamma(t+1)}\Big]\\
=\;&\lim_{t\to\infty}\Big[2d\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}t^{\alpha-\beta}\Big(1+O\Big(\frac{1}{t}\Big)\Big)-2d\frac{\Gamma(1-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\Big(1+O\Big(\frac{1}{t}\Big)\Big)\Big]\\
=\;&0.
\end{aligned}$$

Thus, all the assumptions of Theorem 5.4 hold and hence the initial value problem (5.16) - (5.17) has a unique S-asymptotically 4-periodic solution in l^∞.

The following example demonstrates the applicability of Theorem 5.5.

Example 2. Consider the scalar initial value problem

$$\big(\nabla_{0*}^{(0.3)}u\big) = (0.6)e^{-(0.9)t}u^2 + (0.25)\cos\pi t, \quad t\in\mathbb{N}_1, \tag{5.18}$$

$$u(0) = c. \tag{5.19}$$

(C) $f(t,u) = (0.6)e^{-(0.9)t}u^2 + (0.25)\cos\pi t$ is continuous with respect to its second argument.

(L') Let $M > 0$ be an arbitrary constant. For any pair of elements (t,u), $(t,v)\in D$,

$$\begin{aligned}
|f(t,u)-f(t,v)| &\le (0.6)e^{-(0.9)t}\big|u^2-v^2\big|\\
&= (0.6)e^{-(0.9)t}|u-v||u+v|\\
&\le (0.6)(t+1)^{-(0.9)}|u-v|[|u|+|v|]\\
&\le (1.2)M(t+1)^{\overline{-(0.9)}}|u-v|.
\end{aligned}$$

Here $a(t) = (1.2)M(t+1)^{\overline{-0.9}}$ is a nonnegative function defined on $\mathbb{N}_0$ and

$$\begin{aligned}
L &= (1.2)M\sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(0.3)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{0.3-1}}(s+1)^{\overline{-0.9}}\Big]\\
&= (1.2)M\sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(0.3)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{0.3-1}}(s+1)^{\overline{-0.9}}-\frac{1}{\Gamma(0.3)}(t+1)^{\overline{0.3-1}}(1)^{\overline{-0.9}}\Big]\\
&= (1.2)M\sup_{t\in\mathbb{N}_0}\Big[\nabla_0^{-0.3}(t+1)^{\overline{-0.9}}-\frac{\Gamma(0.1)}{\Gamma(0.3)}(t+1)^{\overline{-0.7}}\Big]\\
&= (1.2)M\frac{\Gamma(0.1)}{\Gamma(0.4)}\sup_{t\in\mathbb{N}_0}\Big[(t+1)^{\overline{-0.6}}\Big]-(1.2)M\frac{\Gamma(0.1)}{\Gamma(0.3)}\inf_{t\in\mathbb{N}_0}\Big[(t+1)^{\overline{-0.7}}\Big]
\end{aligned}$$

$$= (1.2)M\frac{\Gamma(0.1)}{\Gamma(0.4)}(1)^{\overline{-0.6}} - 0$$
$$= (1.2)M\Gamma(0.1).$$

We choose $M = 0.08 < \frac{1}{(1.2)\Gamma(0.1)}$ so that $L < 1$.

(A') Clearly, $\{\cos\pi t\}_{t\in\mathbb{N}_1}$ is an exponentially bounded 2-periodic sequence of real numbers such that

$$\sum_{s=1}^{2}\cos\pi s = 0.$$

Then, from Theorem 3.7, $\left\{\nabla_1^{-\alpha}\cos\pi t\right\}_{t\in\mathbb{N}_0}$ is a bounded sequence. Consider

$$\begin{aligned}
\nabla_1^{-\alpha}(\cos\pi t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\cos\pi s \\
&= \sum_{s=1}^{t}\binom{t-s+\alpha-1}{t-s}(-1)^{s} \\
&= \sum_{s=1}^{t}\binom{s+\alpha-1}{s}(-1)^{t-s} \\
&= (-1)^{t}\sum_{s=1}^{t}\binom{-\alpha}{s} \\
&= (-1)^{t}\sum_{s=1}^{t}\frac{\Gamma(1-\alpha)}{\Gamma(s+1)\Gamma(1-s-\alpha)}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{t}\Gamma(1-\alpha)\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{1}{\Gamma(s+\alpha)\Gamma(1-(s+\alpha))} \\
&= \frac{(-1)^{t}\Gamma(1-\alpha)}{\pi}\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\sin(\pi(s+\alpha)) \\
&= \frac{(-1)^{t}\Gamma(1-\alpha)\sin\pi\alpha}{\pi}\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\cos\pi s.
\end{aligned}$$

Let $A(s) = \cos\pi s$ and $B(s) = \frac{\Gamma(s+\alpha)}{\Gamma(s+1)}$. Clearly,

$$B(1) \geq B(2) \geq B(3)\cdots \geq B(t) > 0$$

and

$$-1 \leq \sum_{s=1}^{t}\cos\pi s \leq 1.$$

Then, by Abel's Lemma, we have

$$-\Gamma(\alpha+1) \leq \sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\cos\pi s \leq \Gamma(\alpha+1).$$

or

$$\Big|\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\cos\pi s\Big| \le \Gamma(\alpha+1).$$

Now, consider

$$\begin{aligned}
&\sup_{t\in\mathbb{N}_0}\Big|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,0)\Big| \\
=\;& (0.25)\sup_{t\in\mathbb{N}_0}\Big|\nabla_1^{-\alpha}(\cos\pi t)\Big| \\
=\;& (0.25)\sup_{t\in\mathbb{N}_0}\Big|\frac{(-1)^t\Gamma(1-\alpha)\sin\pi\alpha}{\pi}\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\cos s\pi\Big| \\
=\;& (0.25)\sup_{t\in\mathbb{N}_0}\Big[\frac{\Gamma(1-\alpha)}{\pi}\big|(-1)^t\sin\pi\alpha\big|\Big|\sum_{s=1}^{t}\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\cos\pi s\Big|\Big] \\
=\;& (0.25)\Gamma(\alpha+1) \\
=\;& (0.25)\Gamma(1.3) \\
=\;& 0.2242 \\
<\;& \frac{M}{1-L}.
\end{aligned}$$

(B') Consider

$$\begin{aligned}
|f(t+2,u)-f(t,u)| &\le (0.6)\big|e^{-(0.9)(t+2)}-e^{-(0.9)t}\big|\big|u^2\big| + (0.25)\big|\cos\pi(t+2)-\cos\pi t\big| \\
&\le (0.6)e^{-(0.9)t}\big[e^{-(1.8)}+1\big]|u|^2 \\
&\le (1.2)M^2(t+1)^{-(0.9)} \\
&\le (1.2)M^2(t+1)^{\overline{-(0.9)}}.
\end{aligned}$$

Here $b(t)=(1.2)M^2(t+1)^{\overline{-(0.9)}}$ is a nonnegative function defined on $\mathbb{N}_0$ and

$$\lim_{t\to\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}b(s)\Big]=0.$$

Thus, all the assumptions of Theorem 5.5 are satisfied and hence the initial value problem (5.18) - (5.19) has unique S-asymptotically 2-periodic solution in an open ball having center 0 and radius $p=11$.

Remark 2. The problem in the previous example does not satisfy (L) and thus Theorem 5.4 does not apply.

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