Abstract: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be (M,k) quasi class Q and (M,k) quasi * class Q operator and also the characterization of (M,k) quasi class Q and (M,k) quasi * class Q composition operators on weighted Hardy space.

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1. INTRODUCTION

Various properties of composition operators on weighted Hardy spaces have been studied by different authors, see [3], Cowen and Kriete obtained a nice correlation between hyponormality of composition operator on $H^2$. In [9], E.A. Nordgeen, studied some results on the hyponormality of composition operators and their adjoints. In [13], S. Panayappan D. Senthilkumar and Mohanraj have investigated on M-Quasihyponormality of composition operators.
and their adjoints. T. Veluchamy (see [15]) have investigated parahyponormal \* paranormal and posinormal operators.

In this paper, we are interested in k-quasi \* class Q Operators, we give a characterization of such operators and other known classes of operators.

2. PRELIMINARY NOTES

Let $f$ be an analytic map on the open disk $D$ given by the Taylor’s series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots.$$ 

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ as $n \to \infty$. The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\|f\|_{2, \beta}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ is a Hilbert space of functions analytic in the unit disk with the inner product $\langle f, g \rangle_{\beta} = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$ for $f$ as above and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let $D$ be the open unit disk in the complex plane and let $T : D \to D$ be an analytic self map of the unit disk and consider the corresponding composition operator $C$ acting on $H^2(\beta)$ i.e., $Cf = f \circ T$, $f \in H^2(\beta)$.

Let $(x, \sum, \lambda)$ be a sigma-finite measure space and let $T : X \to X$ be a non-singular measurable transformation. A bounded linear operator $Cf = f \circ T$ on $L^2(x, \sum, \lambda)$ is said to be a composition operator induced by $T$ when the measure $\lambda T^{-1}$ is absolutely continuous with respect to the measure $\lambda$ and the Random-Nikodym derivative $\frac{d\lambda T^{-1}}{d\lambda} = f_0$ is essentially bounded. The random-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to $\lambda$ is denoted by $f_0^{(k)}$ where $T^k$ is obtained by composing $T$-k times. Every essentially bounded complex valued measurable function $f_0$ induces the bounded operator $M_{f_0}$ on $L^2(\lambda)$ which is defined by $M_{f_0}f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_0}$ and $C^{*2}C^2 = M_{f_0^2}$.

Let $u$ be an essentially bounded function. Then the weighted composition operator $W(= W_{u,T})$ on the space $L^2(\mu)$ induced by $u$ and $T$ is given by $Wf = u \cdot f \circ T$ for each $f \in L^2(\mu)$.

A transformation $T$ is measurable if $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$. A measurable transformation $T$ is said to be non-singular if $\mu(T^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for $A \in \mathcal{A}$. If $T$ is a measurable transformation then $T^n$
is also a measurable transformation. If $T$ is non-singular, then we say that $\mu T^{-1}$ is absolutely continuous with respect to $\mu$ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to $\mu$. Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded measurable function $h_n$ such that

$$\mu(T^{-1})^n(A) = \int_A h_n \, d\mu$$

for every $A \in \mathcal{A}$, and $h_n$ is called the $n$th order radon-nikodym derivative and is denoted by $\frac{d\mu(T^{-1})^n}{d\mu}$. It can be seen that $h_n = h.hoT^{-1}.hoT^{-2}...hoT^{-(n-1)}$ and $h_n = h_{n-1}.hoT^{-(n-1)}$. Throughout this paper, we assume that $u$ is non-negative.

**Definition 2.1.** [10] An operator $T$ on a Hilbert space $H$ is said to be $k$-quasi class $Q$ if

$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^*kT^k \geq 0.$$  

**Definition 2.2.** An operator $T$ on a Hilbert space $H$ is said to be $(M,k)$-quasi class $Q$ if

$$M^2T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^*kT^k \geq 0.$$ 

**Definition 2.3.** An operator $T$ on a Hilbert space $H$ is said to be $M^*$-class $Q$ if

$$M^2T^{*2}T^2 - 2TT^* + I \geq 0.$$  

**Definition 2.4.** An operator $T$ on a Hilbert space $H$ is said to be $M$-quasi-$^*$-class $Q$ if

$$M^2T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq 0.$$ 

**Definition 2.5.** An operator $T$ on a Hilbert space $H$ is said to be $(M,k)$-quasi-$^*$-class $Q$ if

$$M^2T^{*(k+2)}T^{k+2} - 2T^{*k}TT^*T^k + T^{*k}T^k \geq 0.$$ 

**Proposition 1.** Change of variable: Let $X$ be a non-empty set and let $\mathcal{A}$ be a $\sigma$-algebra on $X$. Let $\mu$ and $\mu T^{-1}$ be measures on $\mathcal{A}$ let $h : X \to [0, \infty]$ be a measurable function. Then the following are equivalent:
(i) \( \mu T^{-1} \) is absolutely continuous with respect to \( \mu \) and \( h \) is Radon-nikodym derivative of \( \mu T^{-1} \) with respect to \( \mu \).

(ii) For every measurable function \( f : X \to [0, \infty] \), the equality

\[
\int_X f \, d\mu T^{-1} = \int_X f \, h \, d\mu
\]

holds.

The conditional expectation operator \( E(T^{-1}(A)) = E(f) \) is defined for each non-negative function \( f \) in \( L^p(1 \leq p < \infty) \) and is uniquely determined by the following set of conditions:

(i) \( E(f) \) is \( T^{-1}(A) \) measurable.

(ii) If \( B \) is any \( T^{-1}(A) \) measurable set for which \( \int_A f \, d\mu \) converges then we have

\[
\int_A f \, d\mu = \int_A E(f) \, d\mu.
\]

\( E \) is the projection operator onto the closure of the range of the composition operator \( C \) on \( L^2(\mu) \).

**Lemma 1.** [14] Let \( P \) be the projection of \( L^2(X, A, \mu) \) onto \( \overline{R(C)} \). Then:

(i) \( C^* Cf = hf \) and \( CC^* f = (hoT)Pf \) for all \( f \in L^2(\mu) \).

(ii) \( \overline{R(C)} = \{ f \in L^2(\mu) : f \text{ is } T^{-1}(A) \text{ measurable} \} \).

(iii) If \( f \) is \( T^{-1}(A) \) measurable and \( g \) and \( fg \) belong to \( L^2(\mu) \), then \( P(fg) = fP(g) \), \( f \) need not be in \( L^2(\mu) \).

In [12] Senthikumar has proved the conditions for composition and weighted composition operators to be k-quasi paranormal operator. A. Gupta and N. Bhatia [1] also proved the conditions for composition and weighted composition operators to be (n,k)-quasi paranormal and (n,k)-quasi -*-paranormal operators. In this paper we obtain the conditions for composition and weighted composition operators to be k quasi * class Q operator and quasi * class Q operators in terms of expectation operator and Radon-Nikodym derivative \( h \) (or \( h_n \)).
3. COMPOSITION OPERATOR

Let $C$ be the composition operator and $C^*$ be its adjoint which is given by

$$C^* f = h.E(f)oT^{-1}.$$ 

**Proposition 2.** [1] For every $n \in \mathbb{N}$:

(i) $(C^* C)^n f = h^n f$.

(ii) $(CC^*)^n f = (hoT)^n P(f)$.

(iii) $E$ is the identity operator on $L^2(\mu)$ iff $T^{-1}(A) = A$.

**Theorem 3.1.** Let $C$ be a composition operator on $L^2(\mu)$. Then $C$ is $(M,k)$-quasi class $Q$ if and only if

$$M^2 h^{k+2} - 2h^{k+1} + h^k \geq 0.$$ 

**Proof.** Suppose $C$ is $(M,k)$-quasi class $Q$ operator. Then for every $f \in L^2(\mu)$,

$$\langle (M^2 C^* (k+2) C^k + 2C^* (k+1) C^{k+1} + C^* C^{k}) f, f \rangle \geq 0.$$ 

Let $f = \chi_A$ with $\mu(A) < \infty$. Therefore

$$\langle (M^2 C^* (k+2) C^k + 2C^* (k+1) C^{k+1} + C^* C^{k}) \chi_A, \chi_A \rangle \geq 0 \iff \int ((M^2 h^{k+2} - 2h^{k+1} + h^k) \chi_A) d\mu \geq 0 \iff \int (M^2 h^{k+2} - 2h^{k+1} + h^k) d\mu \geq 0 \iff M^2 h^{k+2} - 2h^{k+1} + h^k \geq 0.$$ 

**Theorem 3.2.** Let $C$ be a composition operator on $L^2(\mu)$. Then $C$ is $(M,k)$-quasi * class $Q$ if and only if $M^2 h^{k+2} - 2h_k E(h) oT^{-1} + h^k \geq 0$.

**Proof.** Consider

$$C^{*k} C C^* C^k f = C^{*k} C C^* (f oT^k) = C^{*k} C (h.E(f oT^k) oT^{-1}) = C^{*k} (hoT^{-1}. f oT^k) oT = h_k E(h) oT^{-1}. f.$$ 

$C$ is $(M,k)$-quasi * class $Q$ if and only if for every $f \in L^2(\mu)$ and $\lambda > 0$, 

$$\langle (M^2 C^* (k+2) C^{k+2} - 2C^{*k} C C^* C^{k} + C^{*k} C^{k}) f, f \rangle \geq 0.$$ 

\[ \Leftrightarrow (M^2h^{k+2} - 2h_kE(h)oT^{-1} + h^k)f \geq 0 \]
\[ \Leftrightarrow M^2h^{k+2} - 2h_kE(h)oT^{-1} + h^k \geq 0. \]

**Example 3.1.** Consider the space \( l^2(\omega) = L^2(N, 2^N, \mu)(\omega) \) where \( \omega = \langle m_n \rangle_{n=1}^{\infty} \) is a sequence of positive real numbers. \( \mu \) is a measure given by \( \mu(n) = m_n \). Let \( T : N \to N \) be a non-singular measurable transformation. Then \( T^n \) is also a non-singular measurable transformation for \( n \in N \). Now,

\[ h_k(s) = \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j, \]

\[ h^k(s) = \frac{1}{m_s^k} \left( \sum_{j \in T^{-1}(s)} m_j \right)^k, \]

\[ E(f)(k) = \frac{\sum_{j \in T^{-1}T(k)} f_j m_j}{\sum_{j \in T^{-1}T(k)} m_j}, \]

for all non-negative sequence \( f = \langle f_n \rangle_{n=1}^{\infty} \) and \( s, k \in N \). by theorem (3.1), C is (M,k)-quasi class Q if and only if

\[ M^2 \frac{1}{m_s^{k+2}} \left( \sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2 \frac{1}{m_s^{k+1}} \left( \sum_{j \in T^{-1}(s)} m_j \right)^{k+1} + \frac{1}{m_s^k} \left( \sum_{j \in T^{-1}(s)} m_j \right)^k \geq 0. \]

for all non-negative sequence \( f = \langle f_n \rangle_{n=1}^{\infty} \) and \( s, k \in N \). by theorem (3.2), C is (M,k)-quasi * class Q if and only if

\[ M^2 \frac{1}{m_s^{k+2}} \left( \sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2 \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j \frac{1}{m_{T^{-k}(s)}} \sum_{j \in T^{-k+1}(s)} m_j + \frac{1}{m_s^k} \left( \sum_{j \in T^{-1}(s)} m_j \right)^k \geq 0. \]

**Proposition 3.** If \( P \) denote the projection of \( L^2(\mu) \) on \( \overline{R(C)} \), then \( C^*Cf = f_0f \) and \( CC^*f = (f_0T)Pf \). For all \( f \in L^2(\mu) \), where \( P \) denote the projection of \( L^2 \) on \( \overline{R(C)} \) and \( \overline{R(C)} = \{ f \in L^2 : f \text{ is } T^{-1} \sum \text{ measurable} \} \).
Theorem 3.3. Let $C \in B(L^2(\mu))$. Then $C$ is of $M$ quasi * class $Q$ if and only if $M^2 f_0^{(3)} - 2(f_0)^2 + f_0 \geq 0$.

Proof. Let $C \in B(L^2(\mu))$. Then $C$ is of $M$ quasi * class $Q$ operator

$$
\Leftrightarrow M^2 C^{*3}C^3 - 2(C^*C)^2 + C^*C \geq 0
\Leftrightarrow \left\langle (M^2 C^{*3}C^3 - 2(C^*C)^2 + C^*C) \chi_E, \chi_E \right\rangle \geq 0,
$$

for every characteristic function $\chi_E$ of $E$ in $\sum$ such that $\lambda(E) < \infty$.

Since, $C^{*2}C^2 = Mf_0^{(2)}$, $C^*C = Mf_0$,

$$
\Leftrightarrow \int_E (M^2 f_0^{(3)} - 2(f_0)^2 + f_0) d\mu \geq 0
$$

for every $E$ in $\sum$.

Hence $C$ is $M$-quasi * class $Q$ operator if and only if $M^2 f_0^{(3)} - 2(f_0)^2 + f_0 \geq 0$. \qed

Corollary 1. Let $C \in B(L^2(\mu))$ with dense range. Then $C \in M$-quasi * class $Q$ if and only if $M^2 f_0^{(3)} - 2(f_0)^2 + f_0 \geq 0$.

Theorem 3.4. Let $C \in B(L^2(\mu))$. Then $C^* \in M$-quasi * class $Q$ if and only if

$$
M^2 (f_0oT)^3 P_1 - 2((f_0oT)P_1)^2 + (f_0oT)P_1 \geq 0,
$$

where $P_1$ is the projection of $L^2$ on to $\overline{R(C)}$.

Proof. Let $C^*$ is of $M$-quasi * class $Q$ operator if and only if

$$
M^2 C^{*3}C^3 - 2(CC^*)^2 + CC^* \geq 0,
$$

$$
\left\langle (C^{*3}C^3 - 2(CC^*)^2 + CC^*) f, f \right\rangle \geq 0,
$$

for every $f \in L^2$. $< CC^*f, f > = < (f_0oT)P_1 f, f >$ where $P_1$ is the projections of $L^2$ on to $\overline{R(C)}$. Thus $C^*$ is of $M$-quasi * class $Q$ Operator if and only if

$$
\left\langle M^2 (f_0oT)^3 P_1 f, f \right\rangle - 2 \left\langle ((f_0oT)P_1)^2 f, f \right\rangle + \left\langle (f_0oT)P_1 f, f \right\rangle \geq 0
$$

for every $f \in L^2$,

$$
M^2 (f_0oT)^3 P_1 - 2((f_0oT)P_1)^2 + (f_0oT)P_1 \geq 0. \qed$$
Corollary 2. Let $C \in B(L^2(\mu))$ with dense range. Then $C \in M$-quasi * class $Q$ operator if and only if $M^2(f_0T)^3P_1 - 2((f_0T)P_1)^2 + (f_0T)P_1 \geq 0$

4. WEIGHTED COMPOSITION OPERATORS

Let $W$ be the weighted composition operator on $L^2(\mu)$. Let $W^*$ be its adjoint which is given by $W^* f = h.E(u.f)oT^{-1}$ for $f \in L^2(\mu)$. For a positive integer $n$, $u_n = u.(uoT)^2...(uoT)^{(n-1)}$. For $f \in L^2(\mu)$, $W^n f = u_n.f oT^{-n}$ and $W^*n f = h_n.E(u_n.f)oT^{-n}$.

Proposition 4. [2] For $u \geq 0$:

(i) $W^*W f = h.E[(u^2)]oT^{-1}f$.

(ii) $WW^* f = u(hoT)E(uf)$.

Theorem 4.1. Let $W$ be a weighted composition operator on $L^2(\mu)$. Then $W$ is $(M,k)$-quasi class $Q$ operator if and only if

$$M^2h_{k+2}.E(u_{k+2}^2)oT^{-(2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-1} + h_k.E(u_k^2) \geq 0.$$ 

Proof. Suppose $W$ is $(M,k)$-quasi class $Q$ operator. Then for $f \in L^2(\mu)$.

$$\left\langle (M^2W^kW^2W^k - 2W^kW^*W^k + W^kW^k)f, f \right\rangle \geq 0.$$ 

Let $f = \chi_A$ with $\mu(A) < \infty$. Then:

$$\left\langle (M^2W^kW^2W^k - 2W^kW^*W^k + W^kW^k)\chi_A, \chi_A \right\rangle \geq 0$$

$$\Leftrightarrow \left\langle (M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k})\chi_A, \chi_A \right\rangle \geq 0$$

$$\Leftrightarrow \int (M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k})\chi_A d\mu \geq 0$$

$$\Leftrightarrow \int M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k}d\mu \geq 0$$

$$\Leftrightarrow M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k} \geq 0.$$
Corollary 3. If $W$ is a weighted composition operator on $L^2(\mu)$ and $T^{-1}(A) = A$. Then $W$ is $(M,k)$-quasi class $Q$ operator if and only if
\[ M^2 h_{k+2} u_{k+2}^2 oT^{-2} - 2 h_{k+1} u_{k+1}^2 oT^{-1} + h_k u_k^2 \geq 0 \]

Theorem 4.2. Let $W$ be a weighted composition operator on $L^2(\mu)$. Then $W$ is $(M,k)$-quasi * class $Q$ operator if and only if
\[ M^2 h_{k+2} E(u_{k+2}^2) oT^{-2} - 2 h_k h oT^{-(k-1)}.E(u_{k+1}^2) + h_k E(u_k^2) \geq 0. \]

Proof. Suppose $W$ is $(M,k)$-quasi * class $Q$ operator. Then for $f \in L^2(\mu)$.
\[ \left\langle (M^2 W^{*k}W^2W^k - 2W^{*k}WW^{*}W^k + W^{*k}W^k) f, f \right\rangle \geq 0. \]

Let $f = \chi_A$ with $\mu(A) < \infty$. Then
\[ \left\langle (M^2 W^{*k}W^2W^k - 2W^{*k}WW^{*}W^k + W^{*k}W^k) \chi_A, \chi_A \right\rangle \geq 0 \]
\[ \iff (M^2 h_{k+2} E(u_{k+2}^2) oT^{-(k+2)} - 2 h_k h E(u_{k+1} h oT)E(u_{k+1})) oT^{-k} + h_k E(u_k^2) oT^{-k} \chi_A, \chi_A \geq 0 \]
\[ \iff \int_A (M^2 h_{k+2} E(u_{k+2}^2) oT^{-(k+2)} - 2 h_k h E(u_{k+1} h oT)E(u_{k+1})) oT^{-k} + h_k E(u_k^2) oT^{-k} d\mu \geq 0 \]
\[ \iff M^2 h_{k+2} E(u_{k+2}^2) oT^{-(k-2)} - 2 h_k h oT^{-(k-1)} E(u_{k+1}^2) + h_k E(u_k^2) \geq 0. \]

Corollary 4. If $W$ is a weighted composition operator on $L^2(\mu)$ and $T^{-1}(A) = A$. Then $W$ is $(M,k)$-quasi * class $Q$ operator if and only if
\[ M^2 h_{k+2} u_{k+2}^2 oT^{-2} - 2 h_k h oT^{-(k-1)} u_{k+1}^2 + h_k u_k^2 \geq 0 \]

Theorem 4.3. Let $T^{-1} \sum = \sum$, $W \in B(L^2(\mu))$. Then $W$ is of $M$-quasi * class $Q$ if and only if
\[ M^2 f_0^{(3)} E(\pi_3^2) oT^{-3} - 2(f_0^{(1)} E(\pi_1^2) oT^{-1})^2 + f_0^{(1)} E(\pi_1^2) oT^{-1} \geq 0 \]

Proof. Since $W$ is weighted composition operator with weight $\pi = (\frac{f_0}{f_0^2})^{\frac{2}{\pi}}$ it follows that $W$ is of $M$-quasi * class $Q$ operator if and only if $M^2 f_0^{(3)} E(\pi_3^2) oT^{-3} - 2(f_0^{(1)} E(\pi_1^2) oT^{-1})^2 + f_0^{(1)} E(\pi_1^2) oT^{-1} \geq 0$. \qed
The second Aluthge transformation of $T$ described by B.P. Duggal is given by

$$\tilde{T} = \left| \tilde{T} \right|^\frac{1}{2} V \left| \tilde{T} \right|^\frac{1}{2},$$

where $\tilde{T} = V \left| \tilde{T} \right|$ is the polar decomposition of $\tilde{T}$, $\tilde{C} = \left| W \right|^\frac{1}{2}$, where $W = V \left| W \right|$ is the polar decomposition of the generalized Aluthge transformation $W : 0 < s < 1$ is a weighted composition operator with weight

$$\omega' = J^\frac{1}{4} \pi \left( \chi_{\sup} J o T \right),$$

where $J = f_0 E(\pi^2) o T^{-1}$.

**Corollary 5.** Let $T^{-1} \sum = \sum, W \in B(L^2(\mu))$. Then $W$ is of quasi * class $Q$ if and only if $f_0(3) E(\omega_3^2) o T^{-3} - 2(f_0(1) E(\omega_1^2) o T^{-1})^2 + f_0(1) E(\omega_1^2) o T^{-1} \geq 0$.

5. **(M,K)-QUASI -*-CLASS Q COMPOSITION OPERATOR ON WEIGHTED HARDY SPACE**

The operator $C_T$ are not necessarily defined on all of $H^2(\beta)$. They are ever where defined in some special cases in the classical Hardy spaces $H^2$ (the case when $\beta_n = 1$ for all $n$).

Let $\omega$ be a point on the open disk. Define $k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{z^n \omega^n}{\beta_n}$. Then the function $k_\omega^\beta$ is a point evaluation for $H^2(\beta)$. Then $k_\omega^\beta$ is in $H^2(\beta)$ and $\left\| k_\omega^\beta \right\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^n}{\beta_n}$. Thus, $\left\| k_\omega \right\|$ is an increasing function of $|\omega|$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $\left< f, k_\omega^\beta \right> = f(\omega)$ for all $f$ and $k_\omega^\beta$ is known as the point evaluation kernel at $\omega$. It can be easily shown that $C_T k_\omega^\beta = k_\omega^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

Now we introduce the class of (M,k) -quasi -*- class Q operators, which is a common generalization of M-* class Q and M-quasi *-class Q operators, defined as follows: An operator $T \in B(H)$ is said to be (M,k) -quasi-* class Q operator if $M^2 T^{*k} (T^{*2} T^2 - 2 T T^{*} + I) T^k \geq 0$, where $k$ is a natural number.

**Theorem 5.1.** If $C_T$ is a (M,k)-quasi * class Q operator in $H^2(\beta)$ then

$$M^2 C_T^{*k+2} C_T^{k+2} - 2 C_T^{*k} C_T C_T^{*k} C_T^k + C_T^{*k} C_T^k \geq 0.$$
Proof. An operator $C_T$ is $(M,k)$-quasi $*$ class $Q$, then
\[
M^2 C_T^{* (k+2)} C_T^{k+2} - 2C_T^{* k} C_T C_T^{* k} C_T + C_T^{* k} C_T^2 \geq 0,
\]
\[
\left\langle (M^2 C_T^{* (k+2)} C_T^{k+2} - 2C_T^{* k} C_T C_T^{* k} C_T + C_T^{* k} C_T^2) f, f \right\rangle \geq 0,
\]
\[
\left\langle (M^2 C_T^{* (k+2)} C_T^{k+2}) f, f \right\rangle - 2 \left\langle (C_T^{* k} C_T C_T^{* k}) f, f \right\rangle + \left\langle (C_T^{* k} C_T^2) f, f \right\rangle \geq 0,
\]
\[
M^2 \left\| C_T^{k+2} f \right\|^2 - 2 \left\| C_T^{* k} C_T f \right\|^2 + \left\| C_T^{k} f \right\|^2 \geq 0,
\]
\[
M^2 \left\| C_T^{k+1} (C_T f) \right\|^2 - 2 \left\| C_T^{* k} (C_T f) \right\|^2 + \left\| C_T^{k-1} (C_T f) \right\|^2 \geq 0.
\]
Let $f = k_0^\beta$, then
\[
M^2 \left\| C_T^{k+1} (C_T k_0^\beta) \right\|^2 - 2 \left\| C_T^{* (k+2)} (C_T k_0^\beta) \right\|^2 + \left\| C_T^{k-1} (C_T k_0^\beta) \right\|^2 \geq 0,
\]
\[
M^2 \left\| C_T^{k+1} (k_0^\beta) \right\|^2 - 2 \left\| C_T^{* k} (k_0^\beta) \right\|^2 + \left\| C_T^{k-1} (k_0^\beta) \right\|^2 \geq 0.
\]
Repeating the steps for $k$ times and $T(0) = 0$ we get
\[
M^2 \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \geq 0.
\]

Theorem 5.2. If $C_T^*$ is a $(M,k)$-quasi $*$ class $Q$ operator in $H^2(\beta)$ then
\[
M^2 C_T^{k+2} C_T^{* (k+2)} - 2C_T^{* k} C_T C_T^{* k} C_T + C_T^{* k} C_T^2 \geq 0.
\]

Proof. An operator $C_T^*$ is $(M,k)$-quasi $*$ class $Q$,
\[
M^2 C_T^{k+2} C_T^{* (k+2)} - 2C_T^{* k} C_T C_T^{* k} C_T + C_T^{* k} C_T^2 \geq 0,
\]
\[
\left\langle (M^2 C_T^{k+2} C_T^{* (k+2)} - 2C_T^{* k} C_T C_T^{* k} C_T + C_T^{* k} C_T^2) f, f \right\rangle \geq 0,
\]
\[
\left\langle (M^2 C_T^{k+2} C_T^{* (k+2)}) f, f \right\rangle - 2 \left\langle (C_T^{* k} C_T C_T^{* k}) f, f \right\rangle + \left\langle (C_T^{* k} C_T^2) f, f \right\rangle \geq 0,
\]
\[
M^2 \left\| C_T^{* (k+2)} f \right\|^2 - 2 \left\| C_T^{* k} C_T f \right\|^2 + \left\| C_T^{k} f \right\|^2 \geq 0,
\]
\[
M^2 \left\| C_T^{* (k+1)} (C_T f) \right\|^2 - 2 \left\| C_T^{* k} (C_T f) \right\|^2 + \left\| C_T^{k-1} (C_T f) \right\|^2 \geq 0.
\]
Let $f = k_0^\beta$, then
\[
M^2 \left\| C_T^{* (k+1)} (C_T k_0^\beta) \right\|^2 - 2 \left\| C_T^{* k} (C_T k_0^\beta) \right\|^2 + \left\| C_T^{* (k-1)} (C_T k_0^\beta) \right\|^2 \geq 0,
\]
\[
M^2 \left\| C_T^{* (k+1)} (k_0^\beta) \right\|^2 - 2 \left\| C_T^{* k} (k_0^\beta) \right\|^2 + \left\| C_T^{* (k-1)} (k_0^\beta) \right\|^2 \geq 0.
\]
Repeating the steps for $k$ times and $T(0) = 0$ we get
\[
M^2 \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \geq 0.
\]
Example 5.1. Let $f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$ and $\phi : D \to D$ be defined by $\phi(z) = \frac{e^{i\theta}}{2} z$, where $0 \leq \theta \leq 2\pi$ is fixed. Then

$$(C^*_\phi f)(\omega) = \langle C^*_\phi f, k_\omega \rangle$$

$$= \langle f, k_{\omega o\phi} \rangle$$

$$= \langle f, k_{\varphi(\omega)} \rangle \quad \text{where} \quad \varphi(\omega) = \frac{e^{-i\theta}}{2} \omega$$

$$= f\left(\frac{e^{-i\theta}}{2} \omega\right).$$

Now

$$(C^*_\phi C_\phi f)(z) = (C_\phi f)\left(\frac{e^{-i\theta}}{2} z\right)$$

$$= f\left(\frac{e^{i\theta}}{2}, -\frac{e^{-i\theta}}{2} z\right)$$

$$= f\left(\frac{1}{4} z\right),$$

and

$$(C_\phi C^*_\phi f)(z) = (C^*_\phi f)\left(\frac{e^{i\theta}}{2} z\right)$$

$$= f\left(-\frac{e^{-i\theta}}{2}, \frac{e^{i\theta}}{2} z\right)$$

$$= f\left(\frac{1}{4} z\right),$$

$$(M^2 C^*_{\phi(k+2)} C_{\phi k+2} - 2 C^*_{\phi k} C_{\phi} C^*_{\phi k} + C^*_{\phi k} C^*_{\phi} f)(z) \geq 0,$$

$$(M^2 C^*_{\phi(k+2)} C_{\phi k+2} f)(z) - 2(C^*_{\phi k} C_{\phi} C^*_{\phi k} f)(z) + (C^*_{\phi k} C^*_{\phi} f)(z) \geq 0,$$

$$(M^2 f)\left(\frac{e^{-i(k+2)\theta}}{2k+2} z\right) - 2(C^*_{\phi k} C_{\phi} C^*_{\phi k} f)\left(\frac{e^{-i(k+2)\theta}}{2k+2} z\right) + (C^*_{\phi k} C^*_{\phi} f)\left(\frac{e^{-i(k+2)\theta}}{2k+2} z\right) \geq 0,$$

$$(M^2 f)\left(\frac{e^{i(k+2)\theta}}{2k+2} z\right) - 2(f)\left(\frac{e^{i(k+2)\theta}}{2k+2} z\right) + f\left(\frac{e^{i(k+2)\theta}}{2k+2} z\right) \geq 0,$$

$$(M^2 f)\left(\frac{1}{2k+4} z\right) - 2(f)\left(\frac{1}{2k+4} z\right) + f\left(\frac{1}{2k+4} z\right) \geq 0.$$
REFERENCES


