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THREE FIXED POINT THEOREMS IN A G-METRIC SPACE

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ABSTRACT: Recently, Binayak Choudhary et al, see [1], introduced compatible and weakly compatible maps, and property (EA) in G-metric spaces. Using these ideas, in this paper, we prove some generalizations of results of Choudhary et al, see [1].

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Key Words: G-metric space, G-Cauchy sequence, compatible maps, weakly compatible maps, property (EA)

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1. INTRODUCTION

The study of contractive type conditions through metric spaces in fixed point theory plays vital role because it finds many applications in different areas like differential equations, integral equations, game theory, operational research and mathematical economics. The notion of G-metric space was introduced by Mustafa and Sims [2] as a generalization of a metric space in 2006. Let

(X, ρ) be a metric space and $a = \rho(x, y)$,$b = \rho(y, z)$ and $c = \rho(z, x)$ be the sides of a Δxyz with vertices x, y and z in the plane. The perimeter of Δxyz defines a G-metric on X. In fact, we have the following definition.

Definition 1.1. Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then the pair (X, G) is called a G-metric space with G-metric G on X. Axioms (G4) and (G5) are referred to as the symmetry and the rectangle inequality (of G) respectively. The topology induced by a G-metric consists of all the all G-balls of the form $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$ and is called a G-metric topology. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-convergent with limit $p \in X$ if it converges to p in the G-metric topology, $\tau(G)$. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy if for every $\epsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $l, m, n \geq N$.

Definition 1.2 (Choudhary et al [1])**.** Self-maps f and g on a G-metric space (X, G) are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 1.3 (Binayak Choudhary et al [1])**.** Self-maps f and g on a G-metric space (X, G) are said to be weakly compatible if they commute at coincidence points.

Definition 1.4 (Binayak Choudhary et al [1])**.** Self-maps f and g on a G-metric space (X, G) are said to satisfy property (EA) if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

From this definition of G-metric space, it immediately follows that

$$G(x,y,y) \leq 2G(x,x,y) \text{ for all } x,y \in X. \tag{1.1}$$

Lemma 1.1 (Mustafa and Sims, see [2]). *The following statements are equivalent in a G-metric space (X,G):*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$,

(b) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$,

(c) $\lim_{n\to\infty} G(x_n, p, p) = 0$.

In this paper, we prove generalizations of three results of Choudhary et al [1] using the notions of compatible maps, weakly compatible maps and property (EA) in a G-metric space.

2. MAIN RESULTS

Our first result is the following theorem.

Theorem 2.1. *Suppose (X,G) is a complete G-metric space, and f and g are two self maps on X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) f *or* g *is continuous;*

(c) For all $x,y,z \in X$, *we have*

$$\begin{aligned} G(fx,fy,fz) \leq & aG(fx,gy,gz) + bG(gx,fy,gz) + cG(gx,gy,fz) \\ & + dG(gx,gy,gz) \\ & + e\max\Big\{G(gx,fy,fy), G(gy,fx,fx), \\ & G(gy,fz,fz), G(gz,fy,fy), G(gz,fx,fx), \\ & G(gx,fz,fz)\Big\}, \end{aligned} \tag{2.1}$$

where $a,b,c,d,e \geq 0$ *such that* $a + 3b + 3c + d + 2e < 1$.

Then f and g will have a unique common fixed point in X, provided f and g are compatible.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, we can choose a $x_1 \in X$ such that $fx_0 = gx_1$. In general, by induction, we can choose a $x_{n+1} \in X$ such that $y_n = fx_n = gx_{n+1}$ where $n = 0, 1, 2, ...$

Writing $x = x_n$ and $y = z = x_{n+1}$ in (2.1), we get

$$\begin{aligned}
G(fx_n, fx_{n+1}, fx_{n+1}) \leq & aG(fx_n, gx_{n+1}, gx_{n+1}) + bG(gx_n, fx_{n+1}, gx_{n+1}) \\
& + cG(gx_n, gx_{n+1}, fx_{n+1}) + dG(gx_n, gx_{n+1}, gx_{n+1}) \\
& + e \max\big\{G(gx_n, fx_{n+1}, fx_{n+1}), G(gx_{n+1}, fx_n, fx_n), \\
& G(gx_{n+1}, fx_{n+1}, fx_{n+1}), G(gx_{n+1}, fx_{n+1}, fx_{n+1}), \\
& G(gx_{n+1}, fx_n, fx_n), G(gx_n, fx_{n+1}, fx_{n+1})\big\} \\
= & aG(fx_n, fx_n, fx_n) + bG(fx_{n-1}, fx_{n+1}, fx_n) \\
& + cG(fx_{n-1}, fx_n, fx_{n+1}) + dG(fx_{n-1}, fx_n, fx_n) \\
& + e \max\big\{G(fx_{n-1}, fx_{n+1}, fx_{n+1}), G(fx_n, fx_n, fx_n), \\
& G(fx_n, fx_{n+1}, fx_{n+1}), G(fx_n, fx_{n+1}, fx_{n+1}), \\
& G(fx_n, fx_n, fx_n), G(fx_{n-1}, fx_{n+1}, fx_{n+1})\big\} \\
= & (b + c)G(fx_{n-1}, fx_{n+1}, fx_n) + dG(fx_{n-1}, fx_n, fx_n) \\
& + e \max\big\{G(fx_{n-1}, fx_{n+1}, fx_{n+1}), \\
& G(fx_n, fx_{n+1}, fx_{n+1})\big\}.
\end{aligned}$$

But, by (G5), we have

$$\begin{aligned}
G(fx_n, fx_{n+1}, fx_{n+1}) \leq & (b + c)[G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}] \\
& + dG(fx_{n-1}, fx_n, fx_n) \\
& + e \max\big\{[G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}], \\
& G(fx_n, fx_{n+1}, fx_{n+1})\big\}
\end{aligned}$$

or

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq q \cdot G(fx_{n-1}, fx_n, fx_n),$$

where

$$q = \frac{b + c + d + e}{1 - 2b - 2c - e} < 1.$$

By induction,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq q^n \cdot G(fx_0, fx_1, fx_1) \text{ for } n \geq 1.$$

Again by the rectangle inequality of G,

$$\begin{aligned}G(y_n, y_m, y_m) \leq & G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots \\ & + G(y_{m-1}, y_m, y_m) \\ \leq & (q^n + q^{n+1} + ... + q^{m-1})G(y_0, y_1, y_1) \text{ for all } m, n \text{ with } m > n.\end{aligned}$$

This implies that

$$G(y_n, y_m, y_m) \leq \tfrac{q^n}{1-q} \cdot G(y_0, y_1, y_1) \text{ for all } n \geq 1. \tag{2.2}$$

Employing the limit as $n \to \infty$ in this, we obtain that $\langle y_n \rangle_{n=1}^{\infty}$ is G-Cauchy in X. Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p.$$

Now, suppose that g is continuous. Then

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_{n+1} = gp. \tag{2.3}$$

Since f and g are compatible, it follows that $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gp. \tag{2.4}$$

Then from (2.1) with $x = gx_n$ and $y = z = x_n$, we have

$$\begin{aligned}G(fgx_n, fx_n, fx_n) \leq & aG(fgx_n, gx_n, gx_n) + bG(ggx_n, fx_n, gx_n) \\ & + cG(ggx_n, gx_n, fx_n) + dG(ggx_n, gx_n, gx_n) \\ & + e \max \big\{G(ggx_n, fx_n, fx_n), G(gx_n, fgx_n, fgx_n), \\ & G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n), \\ & G(gx_n, fgx_n, fgx_n), G(ggx_n, fx_n, fx_n)\big\}\end{aligned}$$

Applying the limit as $n \to \infty$ in this, and then using (2.3) and (2.4), we get

$$\begin{aligned}G(gp, p, p) \leq & aG(gp, p, p) + bG(gp, p, p) + cG(gp, p, p) + dG(gp, p, p) \\ & + e \max \big\{G(gp, p, p), G(p, gp, gp), G(p, p, p), \\ & G(gp, p, p), G(p, gp, gp), G(gp, p, p)\big\} \\ \leq & (a + b + c + d)G(gp, p, p) + e \max \big\{G(gp, p, p), 2G(gp, p, p)\big\}\end{aligned}$$

or $G(gp,p,p) \leq (a+b+c+d+2e)G(gp,p,p)$, which further implies that

$$(1-a-b-c-d-2e)G(gp,p,p) \leq 0 \text{ or } gp = p.$$

That is p is a fixed point of g.

Again, from (2.1) with $x = x_n$ and $y = z = p$, we have

$$\begin{aligned} G(fx_n, fp, fp) \leq & aG(fx_n, gp, gp) + bG(gx_n, fp, gp) + cG(gx_n, gp, fp) \\ & + dG(gx_n, gp, gp) + e\max\big\{G(gx_n, fp, fp), G(gp, fx_n, fx_n), \\ & G(gp, fp, fp), G(gp, fp, fp), G(gp, fx_n, fx_n), G(gx_n, fp, fp)\big\}. \end{aligned}$$

In the limit as $n \to \infty$, this in view of (2.4) and $gp = p$, gives

$$\begin{aligned} G(p, fp, fp) \leq & aG(p,p,p) + bG(p, fp, p) + cG(p,p,fp) + dG(p,p,p) \\ & + e\max\big\{G(p, fp, fp), G(p,p,p), G(p, fp, fp),, \\ & G(p, fp, fp), G(p,p,p), G(p, fp, fp)\big\} \\ = & (b+c)G(p,p,fp) + eG(p, fp, fp) \\ \leq & 2(b+c)G(p, fp, fp) + eG(p, fp, fp) \end{aligned}$$

or $G(p, fp, fp) \leq (2b+2c+e)G(gp,p,p)$ so that $(1-2b-2c-e)G(p, fp, fp) \leq 0$ or $fp = p$. Hence p is a common fixed point of f and g.

Uniqueness: Suppose q is another common fixed of f and g. That is, $fq = q$ and $gq = q$. Then from (2.1), we have

$$\begin{aligned} G(fp, gq, gq) \leq & aG(fp, gq, gq) + bG(gp, fq, gq) + cG(gp, gq, fq) + dG(gp, gq, gq) \\ & + e\max\big\{G(gp, fq, fq), G(gq, fp, fp), G(gq, fq, fq), \\ & G(gq, fq, fq), G(gq, fp, fp), G(gp, fq, fq)\big\} \\ = & aG(p,q,q) + bG(p,q,q) + cG(p,q,q) + dG(p,q,q) \\ & + e\max\big\{G(p,q,q), G(q,p,p), G(q,q,q), \\ & G(q,q,q), G(q,p,p), G(p,q,q)\big\} \\ \leq & (a+b+c+d)G(p,q,q) + e\max\big\{G(p,q,q), 2G(p,q,q)\big\} \end{aligned}$$

or $G(p,q,q) \leq (a+b+c+d+2e)G(gp,q,q)$ so that $(1-a-b-c-d-2e)G(p,q,q) \leq 0$ or $p = q$. That is, p is an unique common fixed point of f and g. □

Taking $d = 0$ and $e = 0$ in Theorem 2.1, we receive the following result.

Corollary 2.1 (Theorem 2.1, see [1]). *Suppose (X, G) is a complete G-metric space and f and g are two self maps on X satisfying the conditions (a) and (b), and*

$$G(fx, fy, fz) \leq aG(fx, gy, gz) + bG(gx, fy, gz) + cG(gx, gy, fz), \quad \text{for all } x, y, z \in X, \tag{2.5}$$

where $a, b, c \geq 0$ such that $a + 3b + 3c < 1$.

If f and g are compatible, then f and g will have a unique common fixed point in X.

Our second result is the following theorem.

Theorem 2.2. *Suppose that (X, G) is a G-metric space and f and g are self-maps on X satisfying (a) and (2.1). If $f(X)$ or $g(X)$ is G-complete and (f, g) is a weakly compatible pair, then f and g will have a unique common fixed point.*

Proof. From Theorem 2.1 $\langle y_n \rangle_{n=1}^{\infty}$ is a G-Cauchy sequence in X. Suppose that $g(X)$ is a complete subspace of X. Then the subsequence of $\langle y_n \rangle_{n=1}^{\infty}$ has a limit in $g(X)$, say p so that $gu = p$ for some $u \in X$.

Since $\langle y_n \rangle_{n=1}^{\infty}$ is a G-Cauchy in X containing the convergent subsequence, $\langle y_n \rangle_{n=1}^{\infty}$ will also be convergent. Now, from (2.1) with $x = u$ and $y = z = x_n$, we have

$$\begin{aligned} G(fu, fx_n, fx_n) \leq & aG(fu, gx_n, gx_n) + bG(gu, fx_n, gx_n) \\ & + cG(gu, gx_n, fx_n) + dG(gu, gx_n, gx_n) \\ & + e \max \big\{ G(gu, fx_n, fx_n), G(gx_n, fu, fu), \\ & G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n), \\ & G(gx_n, fu, fu), G(gu, fx_n, fx_n) \big\} \end{aligned}$$

Now, let $n \to \infty$ in the above, and then using $gu = p$,

$$\begin{aligned} G(fu, p, p) \leq & aG(fu, p, p) + bG(p, p, p) + cG(p, p, p) + dG(p, p, p) \\ & + e \max \big\{ G(p, p, p), G(p, fu, fu), G(p, p, p), \end{aligned}$$

$$G(p,p,p), G(p,fu,fu), G(p,p,p)\big\}$$
$$\le aG(fu,p,p) + e2G(fu,p,p)$$

or $G(fu,p,p) \le (a+2e)G(fu,p,p)$ so that $(1-a-2e)G(fu,p,p) \le 0$ or $fu = p$. Hence $fu = gu = p$. That is, u is coincidence point of f and g. Since f and g are weakly compatible, $fgu = gfu$ or $fp = gp$.

Now, (2.1) with $x = p$, $y = z = u$, in view of $fu = gu = p$ and $fp = gp$, gives

$$\begin{aligned} G(fp,p,p) =& G(fp,fu,fu) \\ \le& aG(fp,gu,gu) + bG(gp,fu,gu) + cG(gp,gu,fu) + dG(gp,gu,gu) \\ &+ e\max\big\{G(gp,fu,fu), G(gu,fp,fp), G(gu,fu,fu), \\ &G(gu,fu,fu), G(gu,fp,fp), G(gp,fu,fu)\big\} \\ =& aG(fp,p,p) + bG(fp,p,p) + cG(fp,p,p) + dG(fp,p,p) \\ &+ e\max\big\{G(fp,p,p), G(p,fp,fp), G(p,p,p), \\ &G(p,p,p), G(p,fp,fp), G(fp,p,p)\big\} \\ =& (a+b+c+d)G(fp,p,p) + e\max\big\{G(fp,p,p), 2G(fp,p,p)\big\}, \end{aligned}$$

or $G(fp,p,p) \le (a+b+c+d+2e)G(fp,p,p)$ so that

$$(1-a-b-c-d-2e)G(fp,p,p) \le 0,$$

or $fp = p$. Thus p is a fixed point of f, and hence is a common fixed point of f and g. Uniqueness of the common fixed point follows easily from Theorem 2.1. □

Taking $d = 0$ and $e = 0$ in Theorem 2.2, we get

Corollary 2.2 (Theorem 2.2, see [1]). *Let f and g be self maps on a G-metric space (X,G), satisfying the conditions* (a) *and* (2.5). *If $f(X)$ or $g(X)$ is G-complete and (f,g) is a weakly compatible pair, then f and g will have a unique common fixed point.*

Our final result is the next theorem.

Theorem 2.3. *Let f and g be weakly compatible self-maps on a G-metric space (X,G), satisfying* (2.1). *If* *(d) f and g satisfy the property (EA);*

(e) $g(X)$ is a closed subspace of X.

Then f and g will have a unique common fixed point.

Proof. Since f and g satisfy the property (EA), there is a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$. Since $g(X)$ is a closed, we find that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p = gu$ for some $u \in X$ so that $p = gu \in g(X)$.

Now writing $x = u$ and $y = z = x_n$ in (2.1), we have

$$\begin{aligned} G(fu, fx_n, fx_n) \leq & aG(fu, gx_n, gx_n) + bG(gu, fx_n, gx_n) + cG(gu, gx_n, fx_n) \\ & + dG(gu, gx_n, gx_n) \\ & + e\max\big\{G(gu, fx_n, fx_n), G(gx_n, fu, fu), G(gx_n, fx_n, fx_n), \\ & G(gx_n, fx_n, fx_n), G(gx_n, fu, fu), G(gu, fx_n, fx_n)\big\}. \end{aligned}$$

Let $n \to \infty$, the above inequality and $gu = p$ imply that

$$\begin{aligned} G(fu, p, p) \leq & aG(fu, p, p) + bG(p, p, p) + cG(p, p, p) + dG(p, p, p) \\ & + e\max\big\{G(p, p, p), G(p, fu, fu), G(p, p, p), \\ & G(p, p, p), G(p, fu, fu), G(p, p, p)\big\} \\ = & aG(fu, p, p) + eG(p, fu, fu) \leq aG(fu, p, p) + e2G(fu, p, p), \end{aligned}$$

or $G(fu, p, p) \leq (a+2e)G(fu, p, p)$, which implies that $(1-a-2e)G(fu, p, p) \leq 0$ or $fu = p$. Thus $fu = gu = p$. That is, u is coincidence point of f and g. From the weak compatibility of (f, g), it immediately follows that $fp = gp$.

Again, (2.1) with $x = p$ and $y = z = u$, and a view of $fu = gu = p$ and $fp = gp$, give

$$\begin{aligned} G(fp, p, p) = & G(fp, fu, fu) \\ \leq & aG(fp, gu, gu) + bG(gp, fu, gu) + cG(gp, gu, fu) + dG(gp, gu, gu) \\ & + e\max\big\{G(gp, fu, fu), G(gu, fp, fp), G(gu, fu, fu), \\ & G(gu, fu, fu), G(gu, fp, fp), G(gp, fu, fu)\big\} \\ \leq & aG(fp, gu, gu) + bG(gp, fu, gu) + cG(gp, gu, fu) + dG(gp, gu, gu) \\ & + e\max\big\{G(gp, fu, fu), G(gu, fp, fp), G(gu, fu, fu), \end{aligned}$$

$$
\begin{aligned}
&G(gu,fu,fu),G(gu,fp,fp),G(gp,fu,fu)\big\}\\
=&aG(fp,p,p)+bG(fp,p,p)+cG(fp,p,p)+dG(fp,p,p)\\
&+e\max\big\{G(fp,p,p),G(p,fp,fp),G(p,p,p),\\
&G(p,p,p),G(p,fp,fp),G(fp,p,p)\big\}\\
=&(a+b+c+d)G(fp,p,p)+e\max\big\{G(fp,p,p),2G(fp,p,p)\big\},
\end{aligned}
$$

or $G(fp,p,p) \leq (a+b+c+d+2e)G(fp,p,p)$ so that

$$(1-a-b-c-d-2e)G(fp,p,p) \leq 0,$$

or $fp = p$.

Thus p is a fixed point of f, and hence a common fixed point of f and g. The uniqueness of the common fixed point follows from (2.1). □

If, we set $d = 0$ and $e = 0$, then the following result follows from Theorem 2.3.

Corollary 2.3 (Theorem 3.1, see [1]). *Let f and g be self maps on a G-metric space (X, G), satisfying the conditions* (d), (e) *and* (2.5). *If (f, g) is a weakly compatible pair, then f and g will have a unique common fixed point.*

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