Communications in Applied Analysis, **21**, No. 1 (2017), 39-55 ISSN: 1083-2564

CONDITIONS FOR EXISTENCE OF POSITIVE SOLUTIONS OF FIRST ORDER BOUNDARY VALUE PROBLEMS WITH DELAY AND NONLINEAR NONLOCAL BOUNDARY CONDITIONS AND APPLICATION TO HEMATOPOIESIS

SMITA PATI[1], SESHADEV PADHI[2], AND P.D.N. SRINIVASU[3]

[1]Department of Mathematics
Amity University
Ranchi, 834001, INDIA

[2]Department of Mathematics
Birla Institute of Technology, Mesra
Ranchi, 835215, INDIA

[3]Department of Mathematics
Andhra University
Visakhapatnam, 530017, INDIA

ABSTRACT: In this paper, existence criteria for positive solutions of the following nonlinear first order boundary value problem with delay and nonlinear nonlocal boundary condition

$$\begin{aligned} x'(t) &= r(t)x(t) + p(t)\sum_{i=1}^{m} f_i(t, x(\alpha_i(t))), \quad t \in [0,1], \\ \lambda x(0) &= x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0,1], \end{aligned}$$

are established using Leray-Schauder theorem and Leggett-Williams fixed point theorem. These results are employed to provide a complete existence criteria for positive solutions of the following boundary value problem associated with the well known Hematopoiesis model

$$\begin{aligned} x'(t) &= r(t)x(t) + p(t)\frac{x^n(t-\alpha)}{1+x^m(t-\alpha)}, \quad t \in [0,1], \\ \lambda x(0) &= x(1) + \Lambda(\tau, x(\tau)), \quad \tau \in [0,1] \end{aligned}$$

where m and n are nonnegative parameters.

AMS Subject Classification: 34B08, 34B18, 34B15, 34B10
Key Words: delay and nonlinear nonlocal boundary conditions, first order boundary value problems, positive solutions, existence criteria, Leray-Schauder theorem, Leggett-Williams fixed point theorem

Received: October 11, 2016 ; **Accepted:** December 1, 2016 ;
Published: January 8, 2017. **doi:** 10.12732/caa.v21i1.4
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

Consider the first order boundary value problem (BVP in short) with nonlinear nonlocal boundary condition

$$x'(t) =r(t)x(t) + p(t)\sum_{i=1}^{m} f_i(t, x(\alpha_i(t))), \quad t \in [0,1], \tag{1}$$

$$\lambda x(0) =x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0,1], \tag{2}$$

where $r, p : [0,1] \to [0,\infty)$ are continuous; the nonlocal points satisfy $0 \leq \tau_1 < \tau_2 < ... < \tau_n \leq 1$, the nonlinear functions f_i are continuous mappings from $[0,1] \times [0,\infty) \to [0,\infty)$, α_i are continuous functions from $[0,1]$ to $[0,1]$ with $\alpha_i(t) \leq t$ on $[0,1]$ and $\alpha_i(t) \not\equiv t$ for $i = 1,2,...,m$, and Λ_j are continuous mappings from $[0,1] \times [0,\infty) \to [0,\infty)$ for $j = 1,2,...,n$ respectively, and satisfy

$$0 \leq \Lambda_j(t,x) \leq x\Psi_j(t,x), \;\; t \in [0,1],\, x \in [0,\infty) \tag{3}$$

for some positive continuous functions $\Psi_j : [0,1] \times [0,\infty) \to [0,\infty)$, and the scalar λ satisfies

$$\lambda > \big(1 + \sum_{j=1}^{n} \beta_j\big) e^{\int_0^1 r(\eta)\, d\eta}, \tag{4}$$

where

$$\beta_j = \max_{[0,1]\times[0,c]} \Psi_j(t,x)$$

for some positive real constant c .

The motivation of this present study has come from the following problem

$$x'(t) = r(t)x(t) + p(t)\frac{x^n(t-\alpha)}{1+x^m(t-\alpha)}, \quad t \in [0,1], \tag{5}$$

$$\lambda x(0) = x(1) + \Lambda(\tau, x(\tau)), \quad \tau \in [0,1], \tag{6}$$

where $\lambda > \frac{3}{2}e^{\int_0^1 r(\eta)\,d\eta}$, $r, p : [0,1] \to [0,\infty)$ are continuous,

$$\Lambda(t,x) = \begin{cases} \frac{x}{4}\left(1 + e^{-\frac{1}{x(t)-1}}\right), & x > 1 \\ \frac{x}{2}, & x \le 1, \end{cases}$$

and m and n are nonnegative parameters satisfying the property

$$\int_0^1 p(s)e^{\int_s^1 r(\eta)\,d\eta}\,ds > (\lambda - e^{\int_0^1 r(\eta)\,d\eta})\lambda^{2n-1}\frac{m}{m-n+1}\left(\frac{m-n+1}{n-1}\right)^{\frac{n-1}{m}} \tag{7}$$

for $1 < n < m+1$. Here $f(t,x) = \frac{x^n}{1+x^m}$, $t \in [0,1]$. Clearly $\Lambda(t,x) < \frac{x}{2}$ with $\beta = \frac{1}{2}$. Consequently, $\lambda > (1+\beta)e^{\int_0^1 r(\eta)\,d\eta} = \frac{3}{2}e^{\int_0^1 r(\eta)\,d\eta}$.

The first equation (5) in the BVP (5)–(6) is a particular case of the Hematopoiesis model or model for blood cell production [8]:

$$x'(t) = a(t)x(t) + b(t)\frac{x^n(t-\tau)}{1+x^m(t-\tau)}. \tag{8}$$

In a remarkable work, Mackey and Glass [7], introduced the above equation (8) in order to model the process of Hematopoiesis. Here $x(t)$ is the density of the mature circulating cells, the delay $\tau > 0$ denotes the time of maturation which is necessary from the start of the production of immature cells in the bone marrow and their release into blood circulation. When $n = 0$, Eq.(8) describes the hematopoiesis with a monotone decreasing production rate and when $n = 1$, Eq.(8) describes the hematopoiesis with single-humped production rate. A clear description and various aspects on this model can be found in [5] and [8]. In the last decade, researchers have been focussed on the study of positive periodic solutions of the equation (8).

To study the qualitative properties of the periodic solutions of the equation (8), it has been assumed that the coefficients of (8) are periodic with a prescribed period T, $T \in [0,\infty)$. One may find a considerable amount of sufficient conditions on the existence and stability of positive periodic solutions of (8) in [5] and [8]. Many authors have used fixed point theorem of cone expansion and cone compression method, upper-lower solution method and iterative technique to find out at least one and at least two positive periodic solutions of (8), although it is very difficult to find upper and lower solutions for a general differential equation. It is easy to verify that a function $x(t)$ is a positive T-periodic solution of (8) if and only if it is a positive solution of the periodic boundary value problem

$$x'(t) = a(t)x(t) + b(t)\frac{x^n(t-\tau)}{1+x^m(t-\tau)},\ t \in [0,T], \tag{9}$$

$$x(0) = x(T). \tag{10}$$

The transformation $x(t) = y(\theta), \theta = t/T$ transforms the BVP (9) into the BVP

$$y'(\theta) = q(\theta)y(\theta) + g(\theta)\frac{y^n(\theta-\mu)}{1+y^m(\theta-\mu)},\ \theta \in [0,1],$$

$$y(0) = y(1).$$

In this work we study a nonlinear and nonlocal BVP associated with (8). In other words, we shall find the existence of number of positive solutions of the BVP (5)–(6) with various possibilities on the positive parameters m and n, as a consequence of studying the existence of positive solutions of the BVP (1)–(2). The method of study the existence of positive solutions of the BVP (5)–(6) has come from the works in [3] for continuous case, and from [9] in discrete case. We have obtained the following theorems on the existence of positive solutions for the BVP (1)–(2).

Theorem 1.1. *Let*

$$f_{i0} = \liminf_{x\to 0+,\, 0\le t\le 1} \frac{f_i(t,x)}{x} = 0,\ i = 1, 2, \cdots, m \tag{11}$$

holds. Then the BVP (1)–(2) has at least one positive solution.

Theorem 1.2. *Let*

$$f_{i\infty} = \liminf_{x\to\infty,\, 0\le t\le 1} \frac{f_i(t,x)}{x} = 0,\ i = 1, 2, \cdots, m \tag{12}$$

holds. Then the BVP (1)–(2) has at least one positive solution.

Now, we consider the BVP (5)–(6). First, suppose that $0 \le n \le 1$. Then $\lim_{x\to\infty} \frac{f(t,x)}{x} = 0$ implies that Theorem 1.2 can be applied to the BVP (5)–(6). On the other hand, $\liminf_{x\to 0+} \frac{f(t,x)}{x} \ne 0$ implies that Theorem 1.1 can not be applied to the BVP (5)–(6). Hence, by Theorem 1.2, the BVP (5)–(6) has a positive solution.

Next, we consider the case $n \ge m+1$. Then

$$\liminf_{x\to\infty, 0\le t\le 1} \frac{f(t,x)}{x} = \lim_{x\to\infty} \frac{x^{n-1}}{1+x^m} = \lim_{x\to\infty} \frac{x^{n-m-1}}{\frac{1}{x^m}+1} \ne 0$$

implies that Theorem 1.2 cannot be applied to this example. On the other hand,

$$\liminf_{x\to 0, 0\le t\le 1} \frac{f(t,x)}{x} = \lim_{x\to 0} \frac{x^{n-1}}{1+x^m} = 0 \tag{13}$$

implies, by Theorem 1.1, that the BVP (5)–(6) has a positive solution. Note that (13) holds for any $n > 1$. Thus, for any $n > 1$, the BVP (5)–(6) has a positive solution.

Finally, we consider the case $1 < n < m+1$. Since both the conditions (11) and (12) are satisfied, the above Theorems 1.1 and 1.2, although guarantee existence of a positive solution independently, they do not throw any light on the number of positive solutions admitted by the BVP (5)–(6). Therefore, we use the following theorem to find the number of positive solutions admitted by the BVP (5)–(6).

Theorem 1.3. *Suppose that the conditions (11) and (12) are satisfied, and there exists a positive constant $c_2 > 0$ such that*

$$\sum_{i=1}^{m} \int_0^1 p(s) e^{\int_s^1 r(\eta)\,d\eta} f_i(s, x(\alpha_i(s)))\,ds > c_2[\lambda - e^{\int_0^1 r(\eta)\,d\eta}] \tag{14}$$

holds for $c_2 \le x \le \lambda c_2$ and $0 \le t \le 1$. Then the BVP (1)–(2) has at least two positive solutions.

In order to apply Theorem 2.2 to the BVP (5)–(6) for the case $1 < n < m+1$, we need to find a constant $c_2 > 0$ such that

$$\int_0^1 p(s) e^{\int_s^1 r(\eta)\,d\eta} \frac{x^n(s-\alpha)}{1+x^m(s-\alpha)}\,ds > c_2[\lambda - e^{\int_0^1 r(\eta)\,d\eta}],$$

$$\text{for } c_2 \le \|x\| \le \lambda c_2 \quad (15)$$

holds. For $x \in K$ and $c_2 \le \|x\| \le \lambda c_2$, we have $\frac{c_2}{\lambda} \le x(t-\alpha) \le \|x\| \le \lambda c_2$ and hence

$$\int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta} \frac{x^n(s-\alpha)}{1+x^m(s-\alpha)}\, ds > \frac{c_2^n}{\lambda^n(1+\lambda^m c_2^m)} \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds.$$

This, in turn, implies that (15) holds if

$$\frac{\int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds}{\lambda - e^{\int_0^1 r(\eta)\, d\eta}} > \frac{\lambda^n(1+\lambda^m c_2^m)}{c_2^{n-1}} \tag{16}$$

holds. Set $c_2 = \frac{1}{\lambda}\left(\frac{n-1}{m-n+1}\right)^{1/m}$, which is the minimizer of $\frac{(1+\lambda^m c_2^m)}{c_2^{n-1}}$. Then the inequality (16) follows from (7). Thus for the case $1 < n < m+1$, by Theorem 2.2, the BVP (5)–(6) admits at least two positive solutions.

Anderson in [1] and Padhi et. al in [10] applied Leggett-Williams multiple fixed point theorem [6] to obtain sufficient conditions for the existence of three positive solutions of the BVP (1)–(2). Padhi et al. [10] improved the results obtained in [1]. Although, the conditions of Theorem 2.2 imply the conditions obtained in [1] and [10], and the conditions of Theorem 2.2 are easy to use. An extensive use the Leggett-Williams fixed point theorem can be found in [8] on the existence of positive periodic solutions of first order functional differential equations with applications in population dynamics. Motivated by the work of Anderson [1], Cetin and Topal in [2] used monotone iteration method and established an iterative scheme for existence and approximation of two positive solutions for the following nonlinear nonlocal first-order multipoint BVP with sign changing nonlinearities

$$\begin{aligned} x'(t) + r(t)x(t) &= \sum_{i=1}^{m} f_i(t, x(t)), \;\; t \in [0,1], \\ x(0) = x(1) &+ \sum_{j=1}^{n} g_j(t_j, x(t_j)), \;\; t_j \in [0,1], \end{aligned}$$

where $r : [0,1] \to [0,\infty)$ is continuous, $f_i : [0,1] \times [0,\infty) \to (-\infty,\infty)$, $i = 1, 2, \cdots, m$, $g_j : [0,1] \times [0,\infty) \to [0,\infty)$, $j = 1, 2, ..., n$ are continuous and the nonlocal points t_j satisfy $0 \le t_1 < t_2 < ... < t_n \le 1$.

This work has been divided into three sections. Section 1 contains the motivation, statements of the main theorems, namely Theorems 1.1–2.2, and the illustrative example considered in this paper. Leray-Schauder theorem has been used to prove Theorems 1.1 and 1.2, whereas Leggett-Williams multiple fixed point theorem is used to prove Theorem 2.2. Some required basic notations and the statements for this fixed point theorems are given in Section 2. The proof of the main Theorems 1.1–2.2 are given in Section 3.

2. PRELIMINARIES

We shall use the following notations for our use in the sequel. Let X be a Banach space. For any cone K on X, we denote $K_a = \{x \in K : ||x|| < a\}$, $\overline{K}_a = \{x \in K : ||x|| \leq a\}$ and

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b \,\text{and}\, ||x|| \leq c\}$$

for any constants $a > 0, b > 0$ and $c > 0$. With the above notations, we now state the following fixed point theorems.

Theorem 2.1. *([4], Leray-Schauder) Let K be a convex subset of a Banach space X, $0 \in K$, $A : K \to K$ be a completely continuous operator. Then, either (i) A has at least one fixed point in K; or (ii) the set $\{x \in K : x = \mu Ax, 0 < \mu < 1\}$ is unbounded.*

Theorem 2.2. ([6, Theorem 3.3]) *(Leggett-Williams Fixed Point Theorem) Let $X = (X, \|.\|)$ be a Banach space and $K \subset X$ be a cone, and $c_4 > 0$ be a constant. Suppose there exists a concave nonnegative continuous function ψ on K with $\psi(x) \leq \|x\|$ for $x \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that:*

(i) $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_3)$;

(ii) $\|Ax\| < c_1$ for all $x \in \overline{K}_{c_1}$;

(iii) $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$.
Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{K}_{c_4}$. Furthermore, we have $x_1 \in \overline{K}_{c_1}$, $x_2 \in \{x \in K(\psi, c_2, c_4) : \psi(x) > c_2\}$, and $x_3 \in \overline{K}_{c_4} \backslash \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

3. PROOF OF MAIN RESULTS

Clearly, the BVP (1)–(2) is equivalent to the integral equation

$$x(t) = \sum_{i=1}^{m} \int_0^1 G(t,s)p(s)f_i(s, x(\alpha_i(s)))\, ds + \frac{e^{\int_0^t r(\eta)\, d\eta} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{\lambda - e^{\int_0^1 r(\eta)\, d\eta}}, \quad (17)$$

where $G(t,s)$ is the Green's Kernel, given by

$$G(t,s) = \frac{e^{\int_s^t r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \times \begin{cases} \lambda & ; \quad if\, 0 \le s \le t \le 1 \\ e^{\int_0^1 r(\eta)\, d\eta}; & \quad if\, 0 \le t \le s \le 1. \end{cases} \quad (18)$$

Proof of Theorem 1.1: Let $X = C[0,1]$; then X is a Banach space endowed with the sup. norm. From (11), there exist constants ϵ and $B > 0$ such that

$$f_i(t, x(t)) < \epsilon x(t) \;\; for \;\; 0 < x(t) \le B, \;\; 0 \le t \le 1, \;\; 1 \le i \le m \text{ and } x \in X \quad (19)$$

holds, where $\epsilon > 0$ is chosen such that it satisfies

$$0 < \epsilon \le \frac{\lambda - \left(1 + \sum_{j=1}^{n} \beta_j\right) e^{\int_0^1 r(\eta)\, d\eta}}{\lambda m \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds)}. \quad (20)$$

Since $0 \le \alpha_i(t) \le t \le 1$ on $0 \le t \le 1$, then from $0 < x(t) \le B$, it follows that $0 \le x(\alpha_i(t)) \le B$ on $0 \le t \le 1$. Hence from (19) we have

$$f_i(t, x(\alpha_i(t))) < \epsilon B \quad \text{for } 0 < x(t) \le B, \;\; 0 \le t \le 1, \;\; 1 \le i \le m \text{ and } x \in X. \quad (21)$$

On X, we define a convex set K by

$$K = \{x(t) : x(t) \in X, x(t) \ge 0, x(t) \text{ is nondecreasing}, x(t) \le B\}, \quad (22)$$

and an operator $A : K \to X$ by

$$(Ax)(t) = \sum_{i=1}^{m} \int_0^1 G(t,s)p(s)f_i(s, x(\alpha_i(s)))\, ds$$

$$+\frac{e^{\int_0^t r(\eta)\,d\eta}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{\lambda-e^{\int_0^1 r(\eta)\,d\eta}}, \quad (23)$$

where $G(t,s)$ is the Green's Kernel given in (18). The operator A in (23) can be rewritten as

$$\begin{aligned} Ax(t) =& \sum_{i=1}^{m}\int_0^t \frac{\lambda e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} p(s) f_i(s,x(\alpha_i(s)))\,ds \\ &+ \sum_{i=1}^{m}\int_t^1 \frac{e^{\int_0^1 r(\eta)\,d\eta} e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} p(s) f_i(s,x(\alpha_i(s)))\,ds \\ &+ \frac{e^{\int_0^t r(\eta)\,d\eta}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{\lambda-e^{\int_0^1 r(\eta)\,d\eta}}. \end{aligned} \quad (24)$$

Let $x \in K$. Then the positivity of f_i, $i = 1,2,...,m$ and Λ_j, $j = 1,2,...,n$ shows that $(Ax)(t) \geq 0$ for all $t \in [0,1]$. Now, we show that the fixed points of the operator A are the solutions of the BVP (1)–(2). In fact, if $x = Ax$, then from (24), we have

$$\begin{aligned} \lambda x(0) - x(1) =& \sum_{i=1}^{m}\int_0^1 \frac{\lambda e^{\int_s^1 r(s)\,ds}}{(\lambda-e^{\int_0^1 r(s)\,ds})} p(s) f_i(s,x(\alpha_i(s)))\,ds \\ &+ \lambda\frac{\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} \\ &- \sum_{i=1}^{m}\int_0^1 \frac{\lambda e^{\int_s^1 r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} p(s) f_i(s,x(\alpha_i(s)))\,ds \\ &- \frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j)), \\ =& \sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j)), \quad \tau_j \in [0,1]. \end{aligned}$$

Hence the boundary condition (2) is satisfied. Again differentiating (24) with respect to t, with $Ax = x$, we obtain

$$\begin{aligned} x'(t) =& \sum_{i=1}^{m}\int_0^t \frac{\lambda r(t) e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} p(s) f_i(s,x(\alpha_i(s)))\,ds \\ &+ \sum_{i=1}^{m}\frac{\lambda}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})} p(t) f_i(t,x(\alpha_i(t))) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\int_t^1 \frac{r(t)e^{\int_0^1 r(\eta)\,d\eta}e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}p(s)f_i(s,x(\alpha_i(s)))\,ds\\
&-\sum_{i=1}^{m}\frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}p(t)f_i(t,x(\alpha_i(t)))\\
&+\frac{r(t)e^{\int_0^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))\\
=&r(t)\left[\sum_{i=1}^{m}\int_0^t \lambda\frac{e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}p(s)f_i(s,x(\alpha_i(s)))\,ds\right]\\
&+r(t)\left[\sum_{i=1}^{m}\int_t^1 \frac{e^{\int_0^1 r(\eta)\,d\eta}e^{\int_s^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}p(s)f_i(s,x(\alpha_i(s)))\,ds\right]\\
&+r(t)\left[\frac{e^{\int_0^t r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))\right]+p(t)\sum_{i=1}^{m}f_i(t,x(\alpha_i(t)))\\
=&r(t)x(t)+p(t)\sum_{i=1}^{m}f_i(t,x(\alpha_i(t))),\ \text{(using (17))}
\end{aligned}
$$

which shows that $x(t)$ satisfies (1). Moreover,

$$
(Ax)'(t)=r(t)x(t)+p(t)\sum_{i=1}^{m}f_i(t,x(\alpha_i(t))),
$$

$t\in[0,1]$ shows that Ax is nondecreasing, $t\in[0,1]$.

Next, for $0<x\le B$, we have

$$
\begin{aligned}
&(Ax)(t)\le ||Ax||=Ax(1)=\sum_{i=1}^{m}\int_0^1 G(1,s)p(s)f_i(s,x(\alpha_i(s)))\,ds\\
&+\frac{e^{\int_0^1 r(\eta)\,d\eta}\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\\
\le&\frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\left[\lambda\sum_{i=1}^{m}\int_0^1 p(s)e^{-\int_0^s r(\eta)\,d\eta}f_i(s,x(\alpha_i(s)))\,ds+x(t)\sum_{j=1}^{n}\beta_j\right]
\end{aligned}
\tag{25}
$$

$$
\le\frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda-e^{\int_0^1 r(\eta)\,d\eta})}\left[B\lambda\epsilon\sum_{i=1}^{m}\int_0^1 p(s)e^{-\int_0^s r(\eta)\,d\eta}\,ds+\|x\|\sum_{j=1}^{n}\beta_j\right]\ \text{(using (21))}
$$

$$\leq \frac{Be^{\int_0^1 r(\eta)\,d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\,d\eta})}\left[\lambda\epsilon m\int_0^1 p(s)e^{-\int_0^s r(\eta)\,d\eta}\,ds + \sum_{j=1}^{n}\beta_j\right]$$
$$\leq B. \tag{26}$$

This proves that $A(K) \subset K$.

Now, we shall show that the operator $A : K \to K$ is completely continuous. Let $x_n, x_0 \in K$ with $\|x_n - x_0\| \to 0$ as $n \to \infty$. Then

$$\|Ax_n - Ax_0\| \leq \max_{t\in[0,1]}\int_0^1 G(t,s)\sum_{i=1}^{m}|f_i(x, x_n(\alpha_i(s))) - f_i(x, x_0(\alpha_i(s)))|\,ds$$
$$+ \max_{t\in[0,1]}\frac{e^{\int_0^t r(\eta)\,d\eta}}{\lambda - e^{\int_0^1 r(\eta)\,d\eta}}\sum_{j=1}^{n}|\Lambda_j(\tau_j, x_n(\tau_j)) - \Lambda_j(\tau_j, x_0(\tau_j))|$$
$$\leq \frac{\lambda e^{\int_0^1 r(\eta)\,d\eta}}{\lambda - e^{\int_0^1 r(\eta)\,d\eta}} m\int_0^1 \max_{s\in[0,1],1\leq i\leq m}|f_i(x, x_n(\alpha_i(s))) - f_i(x, x_0(\alpha_i(s)))|\,ds$$
$$+ \frac{e^{\int_0^1 r(\eta)\,d\eta}}{\lambda - e^{\int_0^1 r(\eta)\,d\eta}} n\max_{\tau_j\in[0,1],1\leq j\leq n}|\Lambda_j(\tau_j, x_n(\tau_j)) - \Lambda_j(\tau_j, x_0(\tau_j))|.$$

Since $f_i, i = 1, 2, \cdots, m$ and $\Lambda_j, j = 1, 2, \cdots, n$ are continuous, then we have $\|Ax_n - Ax_0\| \to 0$ as $n \to \infty$. Hence A is continuous. Let $\Omega \in K$ be a bounded set and let $\|x\| \leq M$ for all $x \in \Omega$. Set

$$N = \max_{1\leq i\leq m}\max_{t\in[0,1],0\leq x\leq M}|f_i(t,x)|, \quad Q = \max_{1\leq j\leq n}\max_{t\in[0,1],0\leq x\leq M}|\Lambda_j(\tau_j, x(\tau_j))|,$$

and

$$\mu = \frac{\lambda e^{\int_0^1 r(\eta)\,d\eta}}{\lambda - e^{\int_0^1 r(\eta)\,d\eta}}.$$

Then

$$\|Ax\| \leq \mu(mN\int_0^1 p(s)\,ds + nQ)$$

implies that $A(\Omega)$ is bounded and hence A is uniformly bounded on the bounded subset of Ω in K. Now, we shall show that A is equi-continuous. Let $x \in \Omega$ with $t_1, t_2 \in [0,1]$. Then

$$|(Ax)(t_2) - (Ax)(t_1)| \leq mN\int_0^1 p(s)|G(t_2,s) - G(t_1,s)|\,ds$$
$$+ \frac{Q}{\lambda - e^{\int_0^1 r(\eta)\,d\eta}}|e^{\int_0^{t_2} r(\eta)\,d\eta} - e^{\int_0^{t_1} r(\eta)\,d\eta}|.$$

Since $G(t,s)$ is a continuous function of t,s and $e^{\int_0^t r(\eta)\,d\eta}$ is a continuous function of t, then the right hand side of the above inequality tends to zero as $t_2 \to t_1$. Hence, by Arzela-Ascoli theorem, A is completely continuous.

In order to use Theorem 2.1, we consider $x \in K$ with $x(t) = \mu(Ax)(t)$, $0 < \mu < 1$. Then, using (26), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \le B,$$

which implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 2.1, the operator A has a fixed point in X, which is a positive solution of the BVP (1)–(2). This completes the proof of the theorem.

Proof of Theorem 1.2: Let $X = C[0,1]$; then X is a Banach space endowed with the sup. norm. From (12), there exist positive constants ϵ and N such that

$$f_i(t, x(t)) < \epsilon x(t) \ for\ x(t) \ge N,\ 1 \le i \le m,\ 0 \le t \le 1 \text{ and } x \in X$$

where ϵ is chosen such that (20) is satisfied. Let

$$\gamma = \max_{0\le t\le 1, 0\le x\le N, 1\le i\le m} f_i(t, x(t)).$$

Then

$$f_i(t, x(t)) < \epsilon x(t) + \gamma \ for\ x(t) \ge 0,\ 0 \le t \le 1, \text{ and } 1 \le i \le m,\ x \in X. \quad (27)$$

For the above choice of ϵ and γ, we consider a constant B by

$$B \ge \frac{\lambda\gamma m \int_0^1 p(s) e^{\int_s^1 r(\eta)\,d\eta}\,ds}{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) e^{\int_0^1 r(\eta)\,d\eta} - \lambda\epsilon m \int_0^1 p(s) e^{\int_s^1 r(\eta)\,d\eta}\,ds}. \quad (28)$$

Let $0 < x \le B$. Since $0 \le \alpha_i(t) \le t \le 1$ on $0 \le t \le 1$, then from $0 < x(t) \le B$, it follows that $0 \le x(\alpha_i(t)) \le B$ on $0 \le t \le 1$. Hence from (27) we have

$$f_i(t, x(\alpha_i(t))) < \epsilon B + \gamma \ for\ x(t) \ge 0,\ 0 \le t \le 1, \text{ and } 1 \le i \le m,\ x \in X. \quad (29)$$

Now, we define a convex set K on X by (22) and an operator $A : K \to X$ by (23), where $G(t, s)$ is the Green's Kernel given in (18). One may verify that A is completely continuous. Proceeding as in Theorem 1.1, we can prove that a fixed point of the operator A in the cone K is equivalent to the existence of a positive solution of the BVP (1)–(2), $(Ax)(t) \geq 0$ and Ax is nondecreasing for $0 \leq t \leq 1$. Now, we show that $Ax \leq B$ for $0 \leq t \leq 1$, where B is defined in (28). Hence from (25), using (29), we have

$$\begin{aligned}(Ax)(t) \leq & \frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\,d\eta})}\left[\lambda \sum_{i=1}^{m} \int_0^1 p(s) e^{-\int_0^s r(\eta)\,d\eta}(\epsilon B + \gamma)\,ds + \|x\| \sum_{j=1}^{n} \beta_j\right] \\ \leq & \frac{e^{\int_0^1 r(\eta)\,d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\,d\eta})}\left[\lambda m(\epsilon B + \gamma) \int_0^1 p(s) e^{-\int_0^s r(\eta)\,d\eta}\,ds + B \sum_{j=1}^{n} \beta_j\right] \\ \leq & B \qquad (30)\end{aligned}$$

for $0 < x \leq B$. This proves that $A(K) \subset K$.

Next, suppose that $x \in K$ with $x(t) = \mu(Ax)(t)$, $0 < \mu < 1$. Then, using (30), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \leq B,$$

which, in turn, implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 2.1, the operator A has a fixed point in X, which is a positive solution of the BVP (1)–(2). This completes the proof of the theorem.

Proof of Theorem 2.2: Let $X = C[0, 1]$ be a Banach space endowed with the sup. norm. On the space X, we define a cone K by

$$K = \{x \in X; x(t) \geq 0, x(t)\,\text{nondecreasing}, \text{t} \in [0, 1]\}$$

and an operator $A : K \to X$ by (23), where $G(t, s)$ is the Green's Kernel given in (18). Proceeding as in the lines of Theorem 1.1, we can show that $A(K) \subset K$. Further, it is easy to see that $A : K \to K$ is completely continuous.

First, we consider (12). Then there exist constants $\epsilon > 0$ and $N > 0$ such that

$$f_i(t, x(t)) < \epsilon x(t) \ for \ x \geq N, \ 1 \leq i \leq m, \ 0 \leq t \leq 1 \text{ and } x \in X$$

where $\epsilon > 0$ is chosen so that it satisfies the property (20). Let

$$\gamma = \max_{0 \leq t \leq 1, 0 \leq x \leq N, 1 \leq i \leq m} f_i(t, x).$$

Then

$$f_i(t, x(t)) < \epsilon x(t) + \gamma \ for \ x \geq 0, \ 1 \leq i \leq m, \ 0 \leq t \leq 1 \text{ and } x \in X. \quad (31)$$

Choose a constant $c_4 > 0$ such that

$$c_4 \geq \left\{ \lambda c_2, \frac{\lambda \gamma m \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds}{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) e^{\int_0^1 r(\eta)\, d\eta} - \lambda \epsilon m \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds} \right\}.$$

Let $x \in \overline{K}_{c_4}$. Then $x \in K$ with $0 \leq x(t) \leq c_4$. Since $0 \leq \alpha_i(t) \leq t \leq 1$, then $0 \leq x(t) \leq c_4$ implies that $0 \leq x(\alpha_i(t)) \leq c_4$ on $0 \leq t \leq 1$. Hence, for $0 \leq x(t) \leq c_4$, we have from (31)

$$f_i(t, x(\alpha_i(t))) < \epsilon c_4 + \gamma \ for \ x \geq 0, \ 1 \leq i \leq m, \ 0 \leq t \leq 1 \text{ and } x \in X.$$

Thus, for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}
(Ax)(t) \leq ||Ax|| = Ax(1) &= \sum_{i=1}^m \int_0^1 G(1,s) p(s) f_i(s, x(\alpha_i(s)))\, ds \\
&+ \frac{e^{\int_0^1 r(\eta)\, d\eta} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \\
\leq & \frac{e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda \sum_{i=1}^m \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta} f_i(s, x(\alpha_i(s)))\, ds + x(t) \sum_{j=1}^n \beta_j \right] \\
\leq & \frac{e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda \sum_{i=1}^m \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta} (\epsilon c_4 + \gamma)\, ds + \|x\| \sum_{j=1}^n \beta_j \right] \\
\leq & \frac{e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda m (\epsilon c_4 + \gamma) \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta}\, ds + c_4 \sum_{j=1}^n \beta_j \right]
\end{aligned}$$

$$\leq c_4,$$

that is, $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$.

Next, we consider (11). Then there exist constants ϵ and $c_1 \in (0, c_2)$ such that

$$f_i(t, x(t)) < \epsilon x(t) \text{ for } 0 < x(t) \leq c_1,\ 1 \leq i \leq m,\ 0 \leq t \leq 1 \text{ and } x \in X, \quad (32)$$

where ϵ satisfies the property

$$0 < \epsilon < \frac{\lambda - \left(1 + \sum_{j=1}^{n} \beta_j\right) e^{\int_0^1 r(\eta)\, d\eta}}{\lambda m \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta}\, ds)}.$$

Let $x \in \overline{K}_{c_1}$, that is, $0 < x(t) \leq c_1$. Since $0 \leq \alpha_i(t) \leq t \leq 1$, then $0 < x(t) \leq c_1$ implies that $0 < x(\alpha_i(t)) \leq c_1$ on $0 \leq t \leq 1$. Hence, from (32), we have that

$$f_i(t, x(\alpha_i(t))) < \epsilon x(\alpha_i(t)) < \epsilon c_1$$
$$\text{for } 0 < x(t) \leq c_1,\ 1 \leq i \leq m,\ 0 \leq t \leq 1 \text{ and } x \in X.$$

Consequently, we have

$$\begin{aligned}
(Ax)(t) \leq ||Ax|| = Ax(1) = \sum_{i=1}^{m} \int_0^1 G(1, s) p(s) f_i(s, x(\alpha_i(s)))\, ds \\
+ \frac{e^{\int_0^1 r(\eta)\, d\eta} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \\
\leq \frac{e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda \sum_{i=1}^{m} \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta} f_i(s, x(\alpha_i(s)))\, ds + x(t) \sum_{j=1}^{n} \beta_j \right] \\
\leq \frac{e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda \epsilon \sum_{i=1}^{m} \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta} c_1\, ds + \|x\| \sum_{j=1}^{n} \beta_j \right] \\
\leq \frac{c_1 e^{\int_0^1 r(\eta)\, d\eta}}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\lambda \epsilon m \int_0^1 p(s) e^{-\int_0^s r(\eta)\, d\eta}\, ds + \sum_{j=1}^{n} \beta_j \right] \\
< c_1.
\end{aligned}$$

This proves the condition (ii) of Theorem 2.2.

Set $c_3 = \lambda c_2$. In order to verify the condition (i) of Theorem 2.2, we set $\theta(t) = \lambda c_2$ for $t \in [0,1]$. Let $\psi(t) = \min_{t\in[0,1]} x(t)$ be a nonnegative concave functional on K. Since $\psi(\theta(t)) = \min_{t\in[0,1]} \theta(t) = \lambda c_2 > c_2; c_2 \leq \psi(x), \|x\| = \lambda c_2$, then the set $\{x \in K; c_2 \leq \psi(x), \|x\| \leq \lambda c_2\}$ is nonempty. Let $x \in K(\psi, c_2, c_3)$; then $c_2 \leq \psi(x) \leq x \leq \|x\| = x(1) = \lambda c_2 = c_3$, and hence

$$\psi(Ax)(t) = Ax(0) = \sum_{i=1}^{m} \int_0^1 G(0,s)p(s)f_i(s, x(\alpha_i(s)))\, ds + \frac{\sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})}$$
$$= \frac{1}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \left[\sum_{i=1}^{m} \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta} f_i(s, x(\alpha_i(s)))\, ds + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)) \right]$$
$$\geq \frac{1}{(\lambda - e^{\int_0^1 r(\eta)\, d\eta})} \sum_{i=1}^{m} \int_0^1 p(s) e^{\int_s^1 r(\eta)\, d\eta} f_i(s, x(\alpha_i(s)))\, ds$$
$$> c_2 \ \text{(using(14))}$$

holds. Thus, the condition (i) of Theorem 2.2 is satisfied.

Finally, suppose that $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3 = \lambda c_2$. Then

$$\psi(Ax) = (Ax)(0) \geq \frac{1}{\lambda}(Ax)(1) = \frac{\|Ax\|}{\lambda} > \frac{\lambda c_2}{\lambda} = c_2$$

implies that the condition (iii) of Theorem 2.2 is satisfied. Hence the BVP (1)–(2) has at least three solutions. Since $f_{i0} = 0$ implies that $f_i(0) = 0$ for each $i = 1, 2, \cdots, m$, then the BVP (1)–(2) has at least two positive solutions. This completes the proof of the theorem.

4. ACKNOWLEDGMENTS

The authors are thankful to the referee for his/her helpful comments in revising the manuscript to the present form.

REFERENCES

[1] D.R. Anderson, Existence of three solutions for a first order problem with nonlinear nonlocal boundary conditions, *J. Math. Anal. Appl.*, **408** (2013), 318-323.

[2] E. Cetin and F.S.Topal, Existence of solutions for a first-order nonlocal boundary value problem with changing-sign nonlinearity, *Turkish Journal of Mathematics*, **39** (2015), 556-563.

[3] Julio G Dix, Seshadev Padhi and Shilpee Srivastava, Existence of three nonnegative periodic solutions for functional differential equations and applications to Hematopoiesis, *Pan American Mathematical Journal*, **19**, No. 1 (2009), 27-36.

[4] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[5] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.

[6] R.W. Leggett and L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana University Mathematical Journal*, **28** (1979), 673-688.

[7] M.C. Mackey and L. Glass, Oscillations and chaos in physiological control systems, *Science*, **197** (1997), 287-289.

[8] Seshadev padhi, John R. Graef, and P.D.N. Srinivasu, *Periodic Solutions of First order Functional Differential Equations in Population Dynamics*, Springer, India, 2014.

[9] Seshadev Padhi and Smita Pati, Multiple periodic solutions for a nonlinear first order functional difference equation with applications to Hematopoiesis model, *Communications in Applied Analysis*, **19** (2015), 319-328, see http://www.acadsol.eu/caa/19/3.

[10] Seshadev Padhi, Smita Pati, and D.K. Hota, Positive solutions of boundary value problems with nonlinear nonlocal boundary conditions, *Opuscula Mathematica*, **36**, No. 1 (2016), 69-79.