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HYDRODYNAMIC FREE SHEAR LAYERS IN A ROTATING FLUID DUE TO AN APPLIED STRESS AT THE FREE SURFACE

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ABSTRACT: Vertical free shear layers, which arise due to an azimuthal stress at the free surface of a linear, steady, axisymmetric, rotating fluid bounded below by a rigid surface, are analyzed using a combination of Fourier transform and boundary layer techniques. It is found that the interior azimuthal velocity is equal to the applied stress itself and the vertical mass flux pumped by the bottom Ekman layer enters directly into the free surface Ekman layer at the top. The vertical free shear layers are the usual Stewartson boundary layers namely, the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers, which arise to provide a return path for the Ekman mass flux and adjust the azimuthal velocity to its suitable value at the vertical boundary /discontinuity. However, In contrast to top rigid surface case, the Ekman mass flux cannot enter directly into the $E^{\frac{1}{4}}$ layer at the top free surface.

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1. INTRODUCTION

The viscous boundary layer in a rotating fluid that appears on rigid or free horizontal surfaces perpendicular to the rotation vector is called Ekman layer. This layer is fundamental to all rotating flows. It is well known fact that Ekman layer satisfies the boundary conditions on tangential velocity field and provides a condition on the interior axial velocity. This condition is referred to as the Ekman compatibility condition on mass flux. The Ekman layer may be conveniently replaced by the Ekman compatibility condition in any vertical layer to make the mathematical analysis relatively simple provided the thickness of this layer/region is much greater than the Ekman layer.

Stewartson [4] considered steady, linear, axisymmetric flow in a rotating cylindrical container and found that the meridional circulation of the fluid driven by the Ekman layer is completed within two side-wall boundary layers of thickness '$E^{\frac{1}{3}}$' and '$E^{\frac{1}{4}}$' where $E\left(=\frac{\upsilon}{\Omega L^2}\right)$ is the Ekman number. Pedlosky [3] examined the Ekman layer at a free surface due to an applied azimuthal stress $\tau_\theta(r)$ and extended the analysis to vertical boundary layers in a closed container using general potential vorticity equation. He shows that the interior azimuthal velocity is equal to the applied stress itself and analyzes the role of $E^{\frac{1}{4}}$ layer in adjusting the azimuthal velocity to zero at the side wall. The present work on free shear layers using a different mathematical technique, namely the integral transform method not only helps to confirm his results but also extends his work to the analysis of $E^{\frac{1}{3}}$ layer and strives to throw more light on certain physical aspects of the problem, namely the closure of the meridional circulation of the fluid through the interior and $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers. Greenspan [2] illustrated how the integral transform analysis for the analysis of free shear layers can be simplified by replacing the Ekman boundary layers by the respective Ekman compatibility conditions. This method has been successfully used by several others [5, 7] in similar contexts. It may be mentioned here that a free layer situation amenable to Integral transform techniques yields directly the necessary information on shear layer structure and possible interior dynamics.

It can be expected that the essential dynamics of the vertical free shear layers, in a situation as described below in section 2, will be the same as the side wall boundary layers that arise when the side walls are rigid. Though this work itself may be treated as a supplement/complement to Pedlosky's work [3], it may also be treated as a contribution to mathematical and physical fluid dynamics. Dynamics of stress driven flows are important not only in ocean dynamics but also in other situations that may arise in astrogeophysical flows.

This paper is divided into four sections including the present introductory section. The second section deals with the mathematical formulation of the problem. In section 3 the problem is analyzed using Fourier transform technique, and solutions for the mass flux 'ψ' and azimuthal velocity 'v' are presented and discussed. Finally, in the Fourth section we present our conclusions.

2. STATEMENT OF THE PROBLEM AND GOVERNING EQUATIONS

The geometrical configuration of the problem is shown in Figure 1. We propose to analyze the problem in cylindrical geometry assuming axisymmetry. The free surface at a height $z = L$ experiences an azimuthal stress $\tau_\theta(r)$ only for $r \leq R$ while the remaining portion of the free surface and the bottom boundary are undisturbed. Fluid motions are driven by the applied stress. Free shear layers are generated at $x(= R - r) = 0$ to smoothen the discontinuity in the stress boundary condition and close the meridional circulation of mass flux. The governing equations for the present problem are the same as those given in Greenspan [3]; the boundary conditions are, however, different. In Greenspan's work, conditions on azimuthal velocity are prescribed at both the boundaries, which are rigid. In our work, stress is prescribed on the top free surface boundary and azimuthal velocity is prescribed on the bottom rigid boundary. In order to make the paper self-contained, we present below the mathematical formulation briefly. The equations governing the steady flow of an incompressible fluid and referred to a frame rotating with an angular velocity $\vec{\Omega}$ are

$$\vec{q}.\nabla\vec{q} + 2\vec{\Omega}\times\vec{q} = -\frac{1}{\rho}\nabla p + \upsilon\nabla^2\vec{q}, \tag{1}$$

$$\nabla.\vec{q} = 0. \tag{2}$$

Here $\vec{q}\left(= u\,\widehat{i} + v\widehat{\theta} + w\widehat{k}\right)$ is the velocity field measured in a coordinate

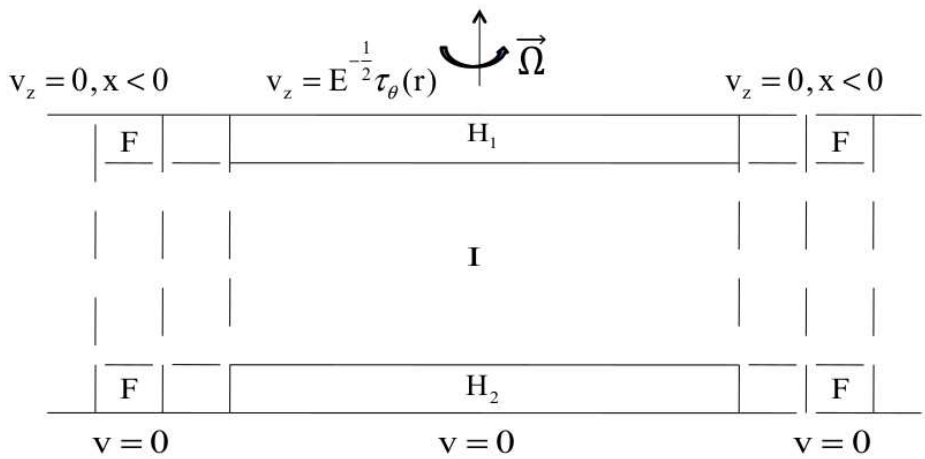

Figure 1: A diagram showing the flow configuration. I=Interior region, H_1=Free surface Ekman layer, H_2=Rigid surface Ekman layer, F=Ekman extension regions, E = Vertical shear layer regions.

system rotating with constant angular velocity $\vec{\Omega}$, and p, ρ, ν are respectively reduced pressure, density and kinematic viscosity. The boundary conditions will be introduced directly in non-dimensional form after non-dimensionalizing the equations. Since the flow is axisymmetric, we may introduce a stream function ψ such that

$$u = \frac{\partial \psi}{\partial z},\ w = -\frac{1}{r}\frac{\partial (r\psi)}{\partial r}$$

Let us now introduce dimensionless variables as follows. Let L and $\varepsilon\ \Omega\ L$ represent the characteristic length scale and characteristic velocity scale U, where L is the distance between the horizontal boundaries and $\varepsilon \left(= \frac{U}{L\Omega}\right)$ is the Rossby number that represents the ratio between inertial and Coriolis force. Let

$$\begin{aligned} & r = Lr^*, z = Lz^*, \\ & \vec{q} = \in L\Omega\left[\psi_z^*(r^*, z^*) + v^*(r^*, z^*) - {}^1/_r (r\psi)_r^*\right], \\ & p = \rho \in \Omega^2 L^2 p^* \end{aligned} \tag{3}$$

where a subscript denotes differentiation. Assuming that the Rossby number ε is sufficiently small, the nonlinear inertial term may be dropped from the equations. The nondimensional equations (after dropping asterisks) in cylindrical coordinates will then become after some simplification (see Greenspan[2])

$$2v_z = -E\left(\nabla^2 - \frac{1}{r^2}\right)^2 \psi, \tag{4}$$

$$2\psi_z = E\left(\nabla^2 - \frac{1}{r^2}\right)^2 v, \tag{5}$$

where E, the Ekman number $\left(= \frac{\upsilon}{\Omega L^2}\right)$ and $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$. Though $E << 1$, the derivatives can be large in viscous boundary layers, and therefore, the viscous terms are retained in the equations. The non-dimensional boundary conditions on velocity field may be written as

$$\begin{aligned} &v = \psi = \psi_z = 0 \text{ at } z = 0, \\ &\frac{\partial v}{\partial z} = E^{-\frac{1}{2}}\tau_\theta(r)\,\delta\left(1 - \frac{r}{a}\right), \psi = \psi_z = 0 \text{ at } z = 1, \end{aligned} \tag{6}$$

where δ is unit step function defined as, $\delta(x) = \begin{cases} 1, \; x > 0, \\ 0, \; x < 0, \end{cases}$ and $a \left(= \frac{R}{L}\right)$ is the aspect ratio. It will be assumed that $a >> 1$.

Since the Ekman thickness is $O(E^{\frac{1}{2}})$, the assumed boundary condition (6) makes the applied stress equal to $O(1)$. As mentioned earlier, there is a discontinuity in the stress boundary condition at $r = a$ and we expect vertical free shear layers at $x = a - r = 0$. We should make sure in our analysis that the field variables are continuous at $x = 0\,(0 < z < 1)$. It is well known that the Ekman layers impose compatibility conditions (referred to as Ekman compatibility conditions) on axial flow at their outer edge. The mathematical analysis of the problem can be very much simplified if we replace the Ekman layers by these compatibility conditions. Now that the problem no longer involves horizontal boundary layers, and radial variations are large in vertical layers, we have $\frac{\partial}{\partial r} >> \frac{\partial}{\partial z}$ for $x(= a - r) >> E^{\frac{1}{2}}$. Also, since the vertical boundary layers are thin and boundary layer contributions vanish at the edge of the boundary layers, we may safely neglect the curvature terms multiplied by Ekman number and simplify the equations [2, 5]. The governing equations, which are valid in the vertical shear layer regions and in the interior very near to $r = a$ may now be written as,

$$Ev_{xx} - 2\psi_z = 0, \tag{7}$$

$$E\psi_{xxxx} + 2v_z = 0. \tag{8}$$

In view of conditions (6) the Ekman compatibility conditions become [1, 2, 3]

$$\psi_i = -\frac{E^{\frac{1}{2}}}{2} v : z = 0, \tag{9}$$

$$\psi_i = -\frac{E^{\frac{1}{2}}}{2} \tau_\theta(a)\delta(x) : z = 1. \tag{10}$$

It may be noted that since $a - r = x << 1$ in the shear layer region, $\tau_\theta(r)$ is replaced by $\tau_\theta(a)$. Also, since the boundary layer contributions vanish at the edge of boundary layers, the unstretched boundary layer coordinate maybe allowed to range between plus and minus infinity. The problem is now amenable to Fourier transformation in variable x. We define the Fourier transformation pair as

$$\bar{g}(\xi) = F\left(g(x)\right) = \int_{-\infty}^{\infty} g(x)e^{-i\xi x} dx \text{ and } g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(\xi)e^{i\xi x} d\xi. \tag{11}$$

3. SOLUTION OF THE PROBLEM

The equations (2.7)-(2.8) along with boundary conditions (2.9)-(2.10) may now be written after Fourier transformation (2.11) as

$$-E\xi^2\bar{v} = 2\frac{\partial\bar{\psi}}{\partial z}, \tag{12}$$

$$E\xi^4\bar{\psi} = -2\frac{\partial\bar{v}}{\partial z}, \tag{13}$$

with boundary conditions

$$\bar{\psi}_i = -\frac{E^{\frac{1}{2}}}{2}\bar{v} : z = 0, \tag{14}$$

$$\bar{\psi}_i = -\frac{E^{\frac{1}{2}}}{2i\xi}\tau_\theta(a) : z = 1. \tag{15}$$

Here, the Fourier transformation of the constant $\tau_\theta(a)$ is written as $\frac{\tau_\theta(a)}{i\xi}$. (for example see [2]). The solutions for the mass flux ψ and the azimuthal velocity v may be obtained from equations (3.1)-(3.2) subject to the boundary conditions (3.3) and (3.4). From (3.1)-(3.2) we may write $\bar{\psi}$ as,

$$\bar{\psi} = C_1(\xi)\cosh\frac{E\xi^3}{2}z + C_2(\xi)\sinh\frac{E\xi^3}{2}z \tag{16}$$

Using the boundary conditions (3.3) and (3.4) in equation (3.5), we get $\bar{\psi}$ as,

$$\bar{\psi} = \frac{E^{\frac{1}{2}}\bar{f}(\xi)\left[\sinh(z-1)\frac{E\xi^3}{2}\right]}{2\sinh\frac{E\xi^3}{2}} - \frac{E^{\frac{1}{2}}\tau_\theta\sinh\frac{E\xi^3}{2}z}{2i\xi\sinh\frac{E\xi^3}{2}} \tag{17}$$

where $\bar{f}(\xi) = (\bar{v})_{z=0}$.

Substituting (3.6) in (3.1) we get the expression for azimuthal velocity $\bar{v}$ and hence we get

$$\bar{f}(\xi) = \bar{v} = \frac{E^{\frac{1}{2}}\tau_\theta}{i\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]} \tag{18}$$

Thus, we obtain

$$\bar{\psi} = -\frac{E^{\frac{1}{2}}\tau_\theta\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2}z + 2\sinh\frac{E\xi^3}{2}z\right]}{2i\xi\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]} \tag{19}$$

and

$$\bar{v} = \frac{E^{\frac{1}{2}}\tau_\theta\left[2\cosh\frac{E\xi^3}{2}z + E^{\frac{1}{2}}\xi\sinh\frac{E\xi^3}{2}z\right]}{2i\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]} \tag{20}$$

Taking inverse Fourier transform for $\bar{\psi}$ and $\bar{v}$, we get

$$\begin{aligned}\psi = &-\frac{E^{\frac{1}{2}}\tau_\theta}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}E^{\frac{1}{2}}\cosh\frac{E\xi^3}{2}z}{\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]}d\xi \\ &-\frac{E^{\frac{1}{2}}\tau_\theta}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}\sinh\frac{E\xi^3}{2}z}{\xi\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]}d\xi\end{aligned} \tag{21}$$

and

$$v = \frac{E^{\frac{1}{2}}\tau_\theta}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}\cosh\frac{E\xi^3}{2}z}{\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2} + 2\sinh\frac{E\xi^3}{2}\right]}d\xi$$

$$+\frac{E^{\frac{1}{2}}\tau_\theta}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}E^{\frac{1}{2}}\xi\sinh\frac{E\xi^3}{2}z}{\left[E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2}+2\sinh\frac{E\xi^3}{2}\right]}d\xi \tag{22}$$

The above expression (3.10)-(3.11) are expressed in a form appropriate for the boundary layer region near $r = a$. Realizing the fact that the pole at $\xi = 0$ corresponds to the interior solution at the edge of the boundary layers and this is zero for $x < 0$ in view of the boundary conditions (2.6), the integration contour should be taken beneath the origin. Now we shall evaluate the integrals (3.10)-(3.11) by contour integration subject to our earlier mentioned assumption $E^{\frac{1}{2}}\xi << 1$. This assumption physically means that only the region $x >> E^{\frac{1}{2}}$ is important to the overall dynamics of the problem.

For $E\xi^3 = O(1)$, we have $\sinh\frac{E\xi^3}{2} >> E^{\frac{1}{2}}\xi\cosh\frac{E\xi^3}{2}$. As a result the singularities of the integrals for ψ and v are located at approximate positions given by $\sinh\frac{E\xi^3}{2} = 0$ leading to poles at $\xi_n = \left[\frac{2n\pi i}{E}\right]^{\frac{1}{3}}$. These poles correspond to $E^{\frac{1}{3}}$ layer.

In a much thicker layer where $E\xi^3 < O(1)$ both terms in the denominators of (3.10) and (3.11) balance each other to give rise to distinct poles at $\xi = 0$ and $\xi = i\left[\frac{1}{E^{\frac{1}{2}}}\right]^{\frac{1}{2}}$. The pole $\xi = 0$ give rise to the Interior solution while the other pole corresponds to the $E^{\frac{1}{4}}$ layer. The evaluation of the integrals of ψ and v by method of residues yields the following results, which are correct to the lowest order:

$$\psi = -\frac{E^{\frac{1}{2}}\tau_\theta}{2}\delta\left(1-\frac{r}{a}\right)+\left(\frac{a-r}{|a-r|}\right)\frac{E^{\frac{1}{2}}\tau_\theta}{4}e^{-E^{-\frac{1}{4}}|x|}(1-z)$$
$$-\left(\frac{a-r}{|a-r|}\right)\frac{E^{\frac{1}{6}}\tau_\theta}{6\pi}\sum_{n=1}^{\infty}\left(\frac{\text{sinn}\,\pi z}{(-1)^n}.E_n\right) \tag{23}$$

The solution of azimuthal velocity 'v' becomes,

$$v = \tau_\theta\,\delta\left(1-\frac{r}{a}\right)-\left(\frac{a-r}{2\,|a-r|}\right)\tau_\theta e^{-E^{-\frac{1}{4}}|x|}(1-E^{\frac{1}{2}}z)$$
$$+\left(\frac{a-r}{|a-r|}\right)\frac{E^{\frac{1}{6}}\tau_\theta}{3\pi^{2/3}}\sum_{n=1}^{\infty}\left(\frac{cosn\pi z}{(-1)^n(2n)^{2/3}}.D_n\right) \tag{24}$$

where

$$D_n = -e^{-\gamma_n|x|}+2e^{-\frac{\gamma_n}{2}|x|}\cos\left(\frac{\sqrt{3}}{2}\gamma_n\,|x|-\frac{\pi}{3}\right) \tag{25}$$

$$E_n = e^{-\gamma_n|x|} + 2e^{-\frac{\gamma_n}{2}|x|} \cos \gamma_n \frac{\sqrt{3}}{2}\,|x| \ \text{ and } \gamma_n = \left[\frac{2n\pi}{E}\right]^{\frac{1}{3}} \qquad (26)$$

The above analysis shows two vertical shear layers, namely the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ Stewartson layers. The dynamics and functions of the two layers are the same as those illustrated by Greenspan [2]. These layers arise to satisfy the boundary conditions on ψ and v at $x = 0$ respectively.

It is seen from (3.12) and (3.13) that v and ψ are z-independent in the interior region as expected from the equations (2.7)-(2.8). The interior azimuthal velocity (non-dimensional value) becomes equal to $\tau_\theta(r)$ itself. Equation (3.12) shows that the flow in the interior region is vertically upwards. Hence, the net flow in the free surface Ekman layer is radially outwards and it agrees with the fact that the net mass flux in the layer should be perpendicular to the applied stress. Thus, in the hydrodynamic case, the mass flux sucked by the free surface Ekman layer is exactly equal to that pumped by the bottom Ekman layer.

Equation (3.13) says that the purpose of the thicker $E^{\frac{1}{4}}$ layer is to adjust the interior azimuthal velocity to its suitable value at the discontinuity (or boundary) at $x = 0$. This layer is hydrostatic in nature, and the boundary layer correction term to azimuthal velocity to lowest order is z-independent. However, higher order correction terms can be z-dependent as revealed in (3.13) since viscous term can become important in radial momentum equation to higher order. The correction term to the azimuthal velocity in $E^{\frac{1}{3}}$ layer is of higher order ($=O(E^{\frac{1}{6}})$) and its z-average is zero. This layer, as Barcilon [1] has noted, can also give rise to $O(1)$ correction term if necessary (if the interior azimuthal velocity has a zero z-average component), since it adjusts that part of the interior azimuthal velocity, which has zero z-average.

It is seen from (3.12) that the mass flux of $O(E^{\frac{1}{2}})$ generated in the interior region is closed through the Ekman layers and the two vertical layers. In contrast to the case where the top surface is also rigid, the mass flux in our problem cannot directly enter into the $E^{\frac{1}{4}}$ layer at the top surface, since $\psi = 0$ at $z = 1$ (see equation (3.12)). The physical reason is that the free surface Ekman layer is weak. The fluid is, however, directly sucked from $E^{\frac{1}{4}}$ layer into the bottom Ekman layer due to the vorticity difference between the bottom boundary and $E^{\frac{1}{4}}$ layer region (see equation (3.13)). Figure 2 shows the meridional circulation of fluid schematically.

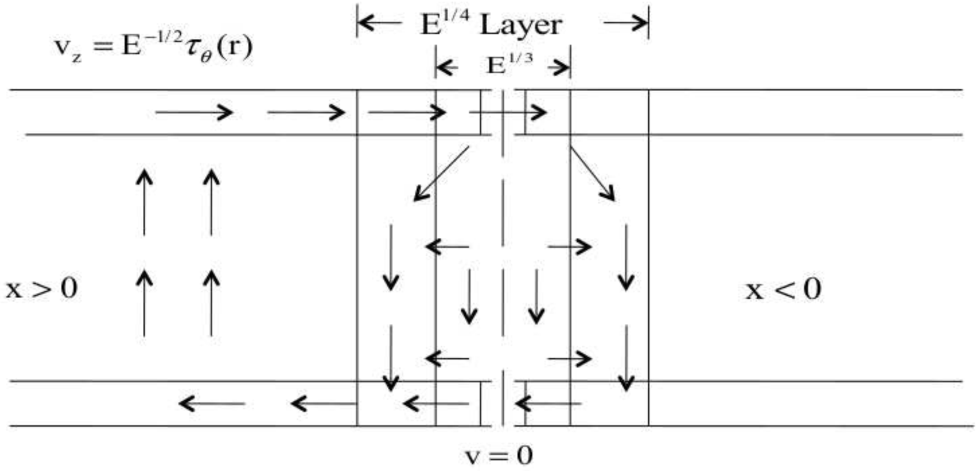

Figure 2: Azimuthal stress is applied for $x > 0$. The schematic diagram (Rotation vector not shown) shows the meridional circulation of $O(E^{\frac{1}{2}})$ in the $x - z$ plane through the interior, and shear layers at the right hand side discontinuity. (Same kind of flow pattern exists at the left discontinuity.)

Finally, it may be seen from (3.12) and (3.13) that,

$$[v]_{x=0+} = [v]_{x=0-} = \frac{1}{2}$$

and in view of the identity,

$$-z = \sum_{n=1}^{\infty} \frac{2(-1)^n \sin n\pi z}{n\pi}, \tag{27}$$

$$[\psi]_{x=0+} = [\psi]_{x=0-} = -\frac{1}{4}$$

Thus the continuity conditions on v and ψ are satisfied at $x = 0$ as demanded by our mathematical technique. It acts as a check on our solutions. Since interior ψ is $-\frac{1}{2}$ and ψ at $x = 0$ is $-\frac{1}{4}$, and $\psi_x = w$ the fluid flow in the boundary layer region is downwards, as expected. Further the solution of ψ in $E^{\frac{1}{4}}$ layer shows that there is a cross flow from $E^{\frac{1}{3}}$ layer to $E^{\frac{1}{4}}$ layer.

4. CONCLUSIONS

In section 3 we have presented the mathematical analysis for the flow field in the viscous vertical hydrodynamic shear layers and inviscid interior region of a rotating fluid, when the motions are driven by an azimuthal stress at the top free surface. Since the net mass flux in the hydrodynamic Ekman layer is perpendicular to the applied stress, azimuthal stress alone can produce radial flux in the boundary layer and hence a meridional circulation through interior and vertical shear layers. We may mention here that in the presence of magnetic field, both radial stress and azimuthal stress can cause meridional circulations, and also make them z-dependent in the interior region, because of the asymmetry in the boundary conditions. It may be noted that while the $E^{\frac{1}{4}}$ layer is hydrostatic, the thinner $E^{\frac{1}{3}}$ layer is non-hydrostatic, and also non-geostrophic. This thin layer, which may be called up welling layer assumes importance in dynamical oceanography.

REFERENCES

[1] V. Barcilon, Stewartson layers in transient rotating fluid flows, *J. Fluid Mech.*, **33** (1968), 815-825.

[2] H.P. Greenspan, *The Theory of Rotating Fluids*, Cambridge University Press, Cambridge, 1968.

[3] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, 1979.

[4] K. Stewartson, On almost rigid rotations, *J. Fluid Mech.*, **3** (1957), 17-26.

[5] S. Vempaty and D.E Loper, Hydromagnetic Free Shear Layers in a rotating flow, *Z. Angew. Math. Phys.*, **29** (1978), 450-461.

[6] S. Vempaty and R. Balasubramanian, Ekman-Hartmann Layer on a free surface, *Indian J. pure appl. Math.*, **18** (1987), 442-450.

[7] S. Vempaty and P. Satyamurthy, Hydromagnetic free shear layers in rotating flows with magnetic field perpendicular to rotation vector and shear layers, *Z. Angew. Math. Mech.*, **83** (2003), 321-332.

[8] P. W. Livermore, L.M. Bailey and R. Hollerbach, A comparison of no-slip, stress-free and inviscid models of rapidly rotating fluid in a spherical shell, *Scientific Reports*, **6**, (2016), Article number: 22812.

[9] Kwan Y. Kim, Jae M. Hyun, Solution for spin-up from rest of liquid with a free surface, *AIAA Journal*, **34**, No. 7 (1996), 1441-1446.