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ON EXACT SOLUTIONS FOR A GENERALIZED BURGERS-SHARMA-TASSO-OLVER EQUATION WITH FORCING TERM

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ABSTRACT: We obtain exact traveling solutions for a combination between the Burgers equation and the Sharma-Tasso-olver equation (B-STO) with variable coefficients and forcing term. Each one of this equations have been studied in a independent way, however, with the use of a forcing term the results presented here are new. The improved tanh-coth method is used to obtain the solutions of the generalized model. We show that from the solutions of the new model, we can obtain solutions for the Burgers equation with forcing term and for the Sharma-Tasso-Olver equation with forcing term.

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Key Words: improved tanh-coth method, Burgers equation, Sharma-Tasso-Olver equation, combined Burgers-Sharma-Tasso-Olver equation (B-STO)

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1. INTRODUCTION

The KdV equation $u_t + 6uu_x + u_{xxx} = 0$ was obtained many years ago and since, the developed of the soliton theory has taken great relevance due to the great number of applications in several branches of the pure and applied mathematics. Some generalizations of that equation, as the equation $u_t + k_1 t^n u u_x + k_2 t^m u_{xxx} = 0$, have been analyzed by several authors because they are used in some models of the applied mathematics (see [1] and references therein). In other words, the use of NLPDE with variable coefficients(depending on the time) have taken great relevance on the work of the many mathematicians. The following two models can be considered as models of great relevance in the solitons theory: The Burgers equation with variable coefficients and forcing term

$$u_t(x,t) - 2\rho(t)u(x,t)u_x(x,t) - \rho(t)u_{xx}(x,t) = F(t), \tag{1}$$

from which, in the case $\rho(t)$ =constant and $F(t) = 0$ the classical Burgers equations is obtained [2] [3]. The other model is the Sharma-Tasso-Olver equation with variable coefficients and forcing term

$$u_t(x,t) - \delta(t)[u(x,t)^3 + u(x,t)u_x(x,t) + u_{xx}(x,t)]_x = F(t), \tag{2}$$

from which, as in the previous model, if $\delta(t)$ =constant, and $F(t) = 0$, the standard Sharma-Tasso-Olver equations is obtained [2]. Solutions to equation (1) in the case $F(t) = 0$ have been derived in [2]. In the same way, solutions to (2) in the case $\delta(t)$ =constant, have been obtained in [4]. In this work, we consider the following combination of the Eqs. (1) and (2) which is given by

$$\begin{cases} u_t(x,t) + \delta(t)[u(x,t)^3 + u(x,t)u_x(x,t) + u_{xx}(x,t)]_x, \\ \\ +\rho(t)[u(x,t)u_x(x,t) + u_{xx}(x,t)] = F(t). \end{cases} \tag{3}$$

Here, $u = u(x,t)$ depend on x and t, the coefficients $\delta(t)$, $\rho(t)$ are arbitrary functions depending on the time t and $F(t)$ is an external force. The new model can be called Burgers-Sharma-Tasso-Olver (B-STO) equation with variable coefficients and forcing term. Note that, in the case $\delta(t) = 1$, $\rho(t) = 0$ and $F(t) = 0$ we obtain the Sharma-Tasso-Olver equation, and in the case $\delta(t) = 0$, $\rho(t) = 1$ and $F(t) = 0$ the Burgers equation is derived. In section 2., we use the improved tanh-coth method [5] for obtain exact traveling wave to equation (3). We show that, solutions to (1) and (2) can be obtained, and therefore, solutions for the classical models ($F(t) = 0$) can be obtained. Finally, some conclusion are given.

2. EXACT SOLUTIONS TO B-STO EQUATION WITH FORCING TERM (3)

As we mentioned early, with the aim to solve (3) we will use the improved tanh-coth method [5]. In this order of ideas, we begin using the transformation

$$\begin{cases} u(x,t) = v(\xi) + \displaystyle\int F(t)dt, \\ \xi = x + \displaystyle\int h(t)dt, \end{cases} \tag{4}$$

being $h(t)$ a function to determinate later. Applying the transformation (4) to (3) we have the following ordinary differential equation

$$\begin{aligned} &[h(t) + 3\delta(t)(\int F(t)dt)^2 + 2\rho(t)\int F(t)dt]v'(\xi) \\ &\quad + [6\delta(t)\int F(t)dt + 2\rho(t)]v(\xi)v'(\xi) + [3\delta(t)\int F(t)dt + \rho(t)]v''(\xi) \\ &\quad + \delta(t)v'''(\xi) + 3\delta(t)v^2(\xi)v'(\xi) + 3\delta(t)(v'(\xi))^2 + 3\delta(t)v(\xi)v''(\xi) = 0. \end{aligned} \tag{5}$$

Now, we seek a solution to (5) by means of the expansion

$$v(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \tag{6}$$

where M is a positive integer determined by balancing method. The new variable $\phi(\xi)$ is solution of the generalized Riccati equation

$$\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2. \tag{7}$$

The $a_i(t)$, $i = 1, 2, \ldots, 2M$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are functions depending only of the variable t to be determined later. The solution of (7) in the case $\beta(t)^2 - 4\alpha(t)\gamma(t) \neq 0$ is given by (see [6]):

$$\phi(\xi) = \frac{-\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\xi + \xi_0] - \beta(t)}{2\gamma(t)}. \tag{8}$$

Substituting (6) into (5), and balancing $v'''(\xi)$ with $v(\xi)^2v'(\xi)$ we obtain

$$M + 3 = 3M + 1,$$

so that

$$M = 1.$$

Therefore, (6) reduces to

$$v(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^{-1}. \tag{9}$$

The substitution of (9) into (5) and taking into account (7) leads to following algebraic system in the unknowns $a_0(t)$, $a_1(t)$, $a_2(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $h(t)$,

$$\begin{aligned}
&12a_1(t)\delta(t)\beta(t)\gamma^2(t) + 15a_1^2(t)\delta(7)\beta(t)\gamma(t) + 3a_1^3(t)\delta(t)\beta(t)\\
&\quad + 6a_0(t)a_1(t)\delta(t)\gamma^2(t) + 6a_0(t)a_1^2(t)\delta(t)\gamma(t) + 6a_1(t)\delta(t)\gamma^2(t)F\\
&\qquad + 6a_1^2(t)\delta(t)\gamma(t)F + 2a_1(t)\rho(t)\gamma^2(t) + 2a_1^2(t)\rho(t)\gamma(t) = 0,
\end{aligned}$$

$$6a_1(t)\delta(t)\gamma^3(t) + 9a_1^2(t)\delta(t)\gamma^2(t) + 3a_1^3(t)\delta(t)\gamma(t) = 0,$$

$$\begin{aligned}
&8\alpha(t)a_1(t)\delta(t)\beta(t)\gamma(t) + 9\alpha(t)a_1^2(t)\delta(t)\beta(t)\\
&\quad + 6\alpha(t)a_0(t)a_1(t)\delta(t)\gamma(t) + 6\alpha(t)a_0(t)a_1^2(t)\delta(t) + a_1(t)\delta(t)\beta^3(t)\\
&\quad + 3a_0(t)a_1(t)\delta(t)\beta^2(t) + 3a_0^2(t)a_1(t)\delta(t)\beta(t) + 3a_1^2(t)a_2(t)\delta(t)\beta(t)\\
&\quad + 3a_1(t)\delta(t)\beta(t)F^2 + 6\alpha(t)a_1(t)\delta(t)\gamma(t)F + 6\alpha(t)a_1^2(t)\delta(t)F\\
&\quad + 3a_1(t)\delta(t)\beta^2(t)F + 6a_0(t)a_1(t)\delta(t)\beta(t)F + 2\alpha(t)a_1(t)\rho(t)\gamma(t)\\
&\quad + 2\alpha(t)a_1^2(t)\rho(t) + a_1(t)\beta^2(t)\rho(t) + 2a_0(t)a_1(t)\beta(t)\rho(t)\\
&\qquad + 2a_1(t)\beta(t)\rho(t)F + a_1(t)\beta(t)h(t) = 0,
\end{aligned}$$

$$\begin{aligned}
&8\alpha(t)a_1(t)\delta(t)\gamma^2(t) + 12\alpha(t)a_1^2(t)\delta(t)\gamma(t) + 3\alpha(t)a_1^3(t)\delta(t)\\
&\quad + 7a_1(t)\delta(t)\beta^2(t)\gamma(t) + 6a_1^2(t)\delta(t)\beta^2(t) + 9a_0(t)a_1(t)\delta(t)\beta(t)\gamma(t)\\
&\quad + 6a_0(t)a_1^2(t)\delta(t)\beta(t) + 3a_2(t)a_1^2(t)\delta(t)\gamma(t) + 3a_0^2(t)a_1(t)\delta(t)\gamma(t)\\
&\quad + 3a_1(t)\delta(t)\gamma(t)F^2 + 9a_1(t)\delta(t)\beta(t)\gamma(t)F + 6a_1^2(t)\delta(t)\beta(t)F\\
&\quad + 6a_0(t)a_1(t)\delta(t)\gamma(t)F + 3a_1(t)\beta(t)\rho(t)\gamma(t) + 2a_1^2(t)\beta(t)\rho(t)\\
&\qquad + 2a_0(t)a_1(t)\rho(t)\gamma(t) + 2a_1(t)\rho(t)\gamma(t)F + a_1(t)\gamma(t)h(t) = 0,
\end{aligned}$$

$$\begin{aligned}
&- 8\alpha(t)a_2(t)\delta(t)\beta(t)\gamma(t) + 6\alpha(t)a_0(t)a_2(t)\delta(t)\gamma(t)\\
&\quad + a_2(t)(-\delta(t))\beta^3(t) + 3a_0(t)a_2(t)\delta(t)\beta^2(t) + 9a_2^2(t)\delta(t)\beta(t)\gamma(t)
\end{aligned}$$

$$
\begin{aligned}
&- 3a_1(t)a_2^2(t)\delta(t)\beta(t) - 3a_0^2(t)a_2(t)\delta(t)\beta(t) - 6a_0(t)a_2^2(t)\delta(t)\gamma(t)\\
&- 3a_2(t)\delta(t)\beta(t)F^2 + 6\alpha(t)a_2(t)\delta(t)\gamma(t)F + 3a_2(t)\delta(t)\beta^2(t)F\\
&- 6a_0(t)a_2(t)\delta(t)\beta(t)F - 6a_2^2(t)\delta(t)\gamma(t)F + 2\alpha(t)a_2(t)\rho(t)\gamma(t)\\
&+ a_2(t)\beta^2(t)\rho(t) - 2a_0(t)a_2(t)\beta(t)\rho(t) - 2a_2^2(t)\rho(t)\gamma(t) - 2a_2(t)\beta(t)\rho(t)F\\
&- a_2(t)\beta(t)h(t) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2\alpha^2(t)a_1(t)\delta(t)\gamma(t) + 3\alpha^2(t)a_1^2(t)\delta(t) + \alpha(t)a_1(t)\delta(t)\beta^2(t)\\
&+ 3\alpha(t)a_0(t)a_1(t)\delta(t)\beta(t) - 2\alpha(t)a_2(t)\delta(t)\gamma^2(t) + 3\alpha(t)a_0^2(t)a_1(t)\delta(t)\\
&+ 3\alpha(t)a_1^2(t)a_2(t)\delta(t) - a_2(t)\delta(t)\beta^2(t)\gamma(t) + 3a_0(t)a_2(t)\delta(t)\beta(t)\gamma(t)\\
&+ 3a_2^2(t)\delta(t)\gamma^2(t) - 3a_1(t)a_2^2(t)\delta(t)\gamma(t) - 3a_0^2(t)a_2(t)\delta(t)\gamma(t)\\
&+ 3\alpha(t)a_1(t)\delta(t)F^2 - 3a_2(t)\delta(t)\gamma(t)F^2 + 3\alpha(t)a_1(t)\delta(t)\beta(t)F\\
&+ 6\alpha(t)a_0(t)a_1(t)\delta(t)F + 3a_2(t)\delta(t)\beta(t)\gamma(t)F - 6a_0(t)a_2(t)\delta(t)\gamma(t)F\\
&+ \alpha(t)a_1(t)\beta(t)\rho(t) + 2\alpha(t)a_0(t)a_1(t)\rho(t) + a_2(t)\beta(t)\rho(t)\gamma(t)\\
&- 2a_0(t)a_2(t)\rho(t)\gamma(t) + 2\alpha(t)a_1(t)\rho(t)F - 2a_2(t)\rho(t)\gamma(t)F\\
&+ \alpha(t)a_1(t)h(t) - a_2(t)\gamma(t)h(t) = 0,
\end{aligned}
$$

$$
-6a_2(t)\alpha^3(t)\delta(t) + 9a_2^2(t)\alpha^2(t)\delta(t) - 3a_2^3(t)\alpha(t)\delta(t) = 0,
$$

$$
\begin{aligned}
&- 12\alpha^2(t)a_2(t)\delta(t)\beta(t) + 6\alpha^2(t)a_0(t)a_2(t)\delta(t) + 15\alpha(t)a_2^2(t)\delta(t)\beta(t)\\
&- 6\alpha(t)a_0(t)a_2^2(t)\delta(t) - 3a_2^3(t)\delta(t)\beta(t) + 6\alpha^2(t)a_2(t)\delta(t)F\\
&- 6\alpha(t)a_2^2(t)\delta(t)F + 2\alpha^2(t)a_2(t)\rho(t) - 2\alpha(t)a_2^2(t)\rho(t) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&- 8\alpha^2(t)a_2(t)\delta(t)\gamma(t) - 7\alpha(t)a_2(t)\delta(t)\beta^2(t) + 9\alpha(t)a_0(t)a_2(t)\delta(t)\beta(t)\\
&+ 12\alpha(t)a_2^2(t)\delta(t)\gamma(t) - 3\alpha(t)a_1(t)a_2^2(t)\delta(t) - 3\alpha(t)a_0^2(t)a_2(t)\delta(t)\\
&+ 6a_2^2(t)\delta(t)\beta^2(t) - 6a_0(t)a_2^2(t)\delta(t)\beta(t) - 3a_2^3(t)\delta(t)\gamma(t)\\
&- 3\alpha(t)a_2(t)\delta(t)F^2 + 9\alpha(t)a_2(t)\delta(t)\beta(t)F - 6\alpha(t)a_0(t)a_2(t)\delta(t)F\\
&- 6a_2^2(t)\delta(t)\beta(t)F + 3\alpha(t)a_2(t)\beta(t)\rho(t) - 2\alpha(t)a_0(t)a_2(t)\rho(t)\\
&- 2a_2^2(t)\beta(t)\rho(t) - 2\alpha(t)a_2(t)\rho(t)F - \alpha(t)a_2(t)h(t) = 0.
\end{aligned}
$$

In all equations, $F = \displaystyle\int F(t)dt$. The following, are some solutions of the system and the respective solutions to (3), for which we have taken into account (9), (8) and (4):

First case:

$$
\begin{cases}
\alpha(t) = -\dfrac{2\beta^2(t)}{\gamma(t)}, \\
h(t) = \frac{\rho^2(t)-63\delta^2(t)\beta^2(t)}{3\delta(t)}, \\
a_0(t) = \dfrac{-3\delta(t)\beta(t) - 3\delta(t)(\int F(t)dt) - \rho(t)}{3\delta(t)}, \;\; a_1(t) = -2\gamma(t), \\
a_2(t) = -\frac{2\beta^2(t)}{\gamma(t)}, \\
u(x,t) = v(\xi) + \displaystyle\int F(t)dt = \dfrac{-3\delta(t)\beta(t) - 3\delta(t)(\int F(t)dt) - \rho(t)}{3\delta(t)} \\
+\beta(t) + \dfrac{4\beta^2(t)}{\beta(t) + 3\beta(t)\tanh\left(\frac{3\beta(t)\xi}{2}\right)} + 3\beta(t)\tanh\left(\dfrac{3\beta(t)\xi}{2}\right) \\
+\int F(t)dt,
\end{cases} \tag{10}
$$

where, $\gamma(t)$, $\beta(t)$ are arbitrary functions depending on the variable t and

$$
\xi = x + \int (\frac{\rho^2(t) - 63\delta^2(t)\beta^2(t)}{3\delta(t)})dt.
$$

As we mentioned early, if we take $\rho(t) = 0$ in equation (3) we have equation (2). Now, if we assume that $\rho(t) = 0$ in all expressions of (10), we obtain one solution for equation (2), which, in this case, is given by

$$
\begin{cases}
u(x,t) = v(\xi) + \displaystyle\int F(t)dt = \dfrac{-3\delta(t)\beta(t) - 3\delta(t)(\int F(t)dt)}{3\delta(t)} \\
+\beta(t) + \dfrac{4\beta^2(t)}{\beta(t) + 3\beta(t)\tanh\left(\frac{3\beta(t)\xi}{2}\right)} + 3\beta(t)\tanh\left(\dfrac{3\beta(t)\xi}{2}\right) \\
+\int F(t)dt,
\end{cases} \tag{11}
$$

where, as previously, $\gamma(t)$, $\beta(t)$ are arbitrary functions, and

$$
\xi = x + \int (\frac{-63\delta^2(t)\beta^2(t)}{3\delta(t)})dt.
$$

Second case:

$$\begin{cases} h(t) = -6a_0(t)\delta(t)(\int F(t)dt) - 3a_0^2(t)\delta(t) - 2a_0(t)\rho(t) \\ +4\alpha(t)\delta(t)\gamma(t) - 3\delta(t)(\int F(t)dt)^2 - 2\rho(t)(\int F(t)dt), \\ \beta(t) = 0, \;\; a_1(t) = -\gamma(t), \;\; a_2(t) = \alpha(t), \\ u(x,t) = v(\xi) + \int F(t)dt = a_0(t) \\ +\sqrt{-\alpha(t)\gamma(t)} \tanh\left(\sqrt{-\alpha(t)\gamma(t)}\xi\right) - \dfrac{\alpha(t)\gamma(t)\coth\left(\sqrt{-\alpha(t)\gamma(t)}\xi\right)}{\sqrt{-\alpha(t)\gamma(t)}} \\ + \int F(t)dt. \end{cases}$$

Here, $\alpha(t)$ and $\gamma(t)$ are arbitrary functions (depending on t), and

$$\xi = x + \int [-6a_0(t)\delta(t)(\int F(t)dt) - 3a_0^2(t)\delta(t) - 2a_0(t)\rho(t) \\ + 4\alpha(t)\delta(t)\gamma(t) - 3\delta(t)(\int F(t)dt)^2 - 2\rho(t)(\int F(t)dt)]dt.$$

In this last case, can be verified that the obtained solution to equation (3), also is a solution of equation (2) and equation (1).

Similar solutions can be constructed with other solutions of the system, however, we have put here the most general solutions. The other solutions are particular cases of the obtained here. For sake of simplicity we omit them.

3. CONCLUSIONS

We have studied a new mathematical model from the point of view of its traveling wave solutions. Clearly, from the Burgers-Sharma-Tasso-Olver equation with forcing term and variable coefficients studied here, can be derived the Burger equation and the Sharma-Tasso-Olver equation. We showed that from the solutions of this generalized model, we can obtain solutions for the classical Burgers equation as well as for standard Sharma-Tasso-Olver equation. We note that the solution obtained in the second case, satisfies equation (1) and equation (2).

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