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THE INFIMUM PROPERTY AND FIXED POINTS IN A G-METRIC SPACE

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ABSTRACT: Fixed points are obtained under some contractive type conditions in G-metric space through the well-known infimum property of non-negative real numbers.

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1. INTRODUCTION

Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ such that

(G1) $G(x, y, z) = 0$ whenever $x, y, z \in X$ are such that $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(\pi(x, y, z))$ for all $x, y, z \in X$, where $\pi(x, y, z)$ is a permutation on the set $\{x, y, z\}$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then G is called a G-metric on X, and the pair (X, G) denotes a G-metric space. Axiom (G5) is known as the rectangle inequality (of the G-metric G). This notion was introduced by Mustafa and Sims [5] in 2006.

In any G-metric space (X, G), we have

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Let (X, G) be a G-metric space. A G-ball in X is defined by

$$B_G(x, r) = \{y \in X : G(x, y, y) < r\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X. Also

$$\rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y \in X. \tag{1.2}$$

induces a metric on X, and the G-metric topology coincides with the metric topology induced by the metric ρ_G. This allows us to readily transform many concepts from a metric space into the setting of a G-metric space.

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-convergent, with limit $p \in X$, if it converges to p in the G-metric topology.

Lemma 1.1. *The following statements are equivalent in a G-metric space (X, G):*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$,

(b) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$,

(c) $\lim_{n\to\infty} G(x_n, p, p) = 0$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy, if $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

Definition 1.3. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in X converges in it.

Motivated by Joseph and Kwack's works [2] in a metric space, the first author and Kumara Swamy [7] employed the infimum property to obtain a unique fixed point of a Banach contraction in G-metric space. In this paper, fixed points are obtained under some contractive type conditions in G-metric space. The results are thus interesting applications of the well-known infimum property of nonnegative real numbers.

2. FIXED POINT THEOREMS

The infimum property of real numbers, is stated below by the following lemma.

Lemma 2.1. *Let $S \subset \mathbb{R}$ be nonempty and bounded below.*
Then $\alpha = \inf S$ exists.

Our first result is the following theorem.

Theorem 2.1. *Let (X, G) be a complete G-metric space and f be a self-map on X such that*

$$\begin{aligned} G(fx, fy, fz) \leq k \max \big\{ & G(x, fx, fx) + G(x, fy, fy) + G(x, fz, fz), \\ & G(y, fy, fy) + G(y, fx, fx) + G(y, fz, fz), \\ & G(z, fz, fz) + G(z, fx, fx) + G(z, fy, fy) \big\}, \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, where $0 \leq k < 1/5$.
Then f has a unique fixed point.

Proof. When $k = 0$, from (2.1) with $y = x, z = fx$, note that $G(fx, fx, f^2x) = 0$ for all $x \in X$, which implies that $f^2x = fx$. That is each fx is a fixed point of f. Therefore, we take $0 < k < 1/5$.

Let $S = \{G(x, fx, fx) : x \in X\}$. Each S is a nonempty set of nonnegative numbers which is bounded below. Hence by Lemma 2.1, it has the infimum, $a \geq 0$. If $a > 0$, then from (2.1) with $y = fx$ and $z = fx$ and (G5), we have

$$\begin{aligned} G(fx, f^2x, f^2x) \leq & k \max \big\{ G(x, fx, fx) + G(x, f^2x, f^2x) + G(x, f^2x, f^2x), \\ & G(fx, f^2x, f^2x) + G(fx, fx, fx) + G(fx, f^2x, f^2x), \\ & G(fx, f^2x, f^2x) + G(fx, fx, fx) + G(fx, f^2x, f^2x) \big\} \end{aligned}$$

$$\leq \left(\frac{3k}{1-2k}\right) G(x, fx, fx) < G(x, fx, fx) \leq a,$$

since $3k/(1-2k)$ is less than 1. Thus $G(fx, f^2x, f^2x) \in S$, and hence a cannot be a lower bound of S, which is a contradiction. Therefore, $a = \inf S = 0$.

Note that any number in S which exceeds its infimum cannot be a lower bound of S. Thus for each $n \geq 1$, $1/n$ is not a lower bound of S. Choose points $x_1, x_2, ..., x_n, ...$ in X such that $0 \leq G(x_n, fx_n, fx_n) < 1/n$ for each n. As $n \to \infty$ this gives

$$G(x_n, fx_n, fx_n) \in S,\ n = 1, 2, 3, ...; \quad \lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{2.2}$$

Repeatedly using the rectangle inequality (G5), and (1.1), we get

$$\begin{aligned} G(x_n, x_m, x_m) \leq& G(x_n, fx_n, fx_n) + G(fx_n, fx_m, fx_m) \\ &+ 2G(x_m, fx_m, fx_m). \end{aligned} \tag{2.3}$$

But by (2.1) with $x = x_n$, $y = z = x_m$, (G5) and (1.1), we get

$$\begin{aligned} G(fx_n, fx_m, fx_m) \leq& k[G(x_n, fx_n, fx_n) + 2G(x_m, fx_m, fx_m) \\ &+ 2G(x_n, x_m, x_m)]. \end{aligned} \tag{2.4}$$

Substitute (2.4) in (2.3) and simplifying, we get

$$G(x_n, x_m, x_m) \leq \left(\frac{1+k}{1-2k}\right) G(x_n, fx_n, fx_n) + \left(\frac{2+2k}{1-2k}\right) G(x_m, fx_m, fx_m).$$

As $m, n \to \infty$, this and (2.2) give $G(x_n, x_m, x_m) \to 0$, proving that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{2.5}$$

Again by repeated application of (G5), and (1.1), we have

$$G(p, fp, fp) \leq [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp). \tag{2.6}$$

Now, by (2.1) with $x = x_n$, $y = z = p$ and (G5), we receive

$$G(fx_n, fp, fp) \leq k \max\{G(x_n, fx_n, fx_n) + 2G(x_n, fp, fp),$$

$$2G(p, fp, fp) + [G(x_n, fx_n, fx_n) + G(p, x_n, x_n)]\}. \quad (2.7)$$

Substituting (2.7) in (2.6), we obtain

$$\begin{aligned} G(p, fp, fp) \leq & G(p, x_n, x_n) + G(x_n, fx_n, fx_n) \\ & + k \max\big\{G(x_n, fx_n, fx_n) + 2G(x_n, fp, fp), \\ & 2G(p, fp, fp) + G(x_n, fx_n, fx_n) + G(p, x_n, x_n)\big\} \end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, and using (2.2) and (2.5),

$$G(p, fp, fp) \leq [0 + 0 + k \max\{0 + 2G(p, fp, fp), 2G(p, fp, fp) + 0 + 0\}],$$

which implies that $(1 - 2k)G(p, fp, fp) \leq 0$ or $fp = p$. That is p is a fixed point of f.

Suppose q is another fixed point of f. That is $fq = q$. From (2.1) with $x = p$ and $y = z = q$ and (1.1), we have

$$\begin{aligned} G(p, q, q) = & G(fp, fq, fq) \\ \leq & k \max\{G(p, fp, fp) + G(p, fq, fq) + G(p, fq, fq), \\ & G(q, fq, fq) + G(q, fp, fp) + G(q, fq, fq), \\ & G(q, fq, fq) + G(q, fp, fp), G(q, fq, fq)\} \\ = & k \max\{2G(p, q, q), 2G(p, q, q)\} \\ = & 2kG(p, q, q). \end{aligned}$$

so that $(1 - 2k)G(p, q, q) \leq 0$ or $p = q$. That is, p is unique fixed point of f. □

Taking $z = y$ in Theorem 2.1, we have

Corollary 2.1. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fy) \leq k \max\big\{ & G(x, fx, fx) \\ & + 2G(x, fy, fy), G(y, fx, fx) + 2G(y, fy, fy)\big\}, \quad (2.8) \end{aligned}$$

for all $x, y \in X$, where $0 < k < 1/5$.

Then f has a unique fixed point.

Proof. Let us set

$$\rho_G(x,y) = \max\{(G(x,y,y), G(x,x,y)\} \text{ for all } x,y \in X, \tag{2.9}$$

where the exchange of x and y yields the symmetry of ρ_G. It can be seen that ρ_G defines a metric on X. Now, (2.8) implies that

$$\begin{aligned} G(fx,fy,fy) \leq& 2k \max\big\{G(x,fx,fx) + G(x,fy,fy), \\ & G(y,fx,fx) + G(y,fy,fy)\big\} \\ =& 4k \max\big\{\frac{1}{2}[G(x,fx,fx) + G(x,fy,fy)], \\ & \frac{1}{2}[G(y,fx,fx) + G(y,fy,fy)]\big\} \\ \leq& 4k \max\big\{G(x,fx,fx), G(x,fy,fy), \\ & G(y,fx,fx), G(y,fy,fy)\big\}. \end{aligned} \tag{2.10}$$

Interchanging x and y in (2.11), we have

$$\begin{aligned} G(fy,fx,fx) \leq 4k \max\big\{&G(y,fy,fy), G(y,fx,fx), \\ & G(x,fy,fy), G(x,fx,fx)\big\}. \end{aligned} \tag{2.11}$$

Using (2.10) and (2.11) in (2.9), it follows that

$$\begin{aligned} \rho_G(fx,fy) \leq& 4k \max\big\{G(y,fy,fy), G(y,fx,fx), \\ & G(x,fy,fy), G(x,fx,fx)\big\} \\ \leq& 4k \max\big\{\rho_G(x,fx), \rho_G(x,fy), \rho_G(y,fx), \\ & \rho_G(y,fy)\big\}, \quad \text{for all } \ x,y \in X, \end{aligned}$$

which is a special case of Ciric's quasi-contraction, and a unique fixed point follows from [1] for a complete metric space. In general, if any two of the three variables in the the contraction type condition (2.8) are the same, a unique fixed point can be obtained from Ciric's theorem. □

Letting $z = fy$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx,fy,f^2y) \leq k \max\big\{G(x,fx,fx) + G(x,fy,fy) + G(x,f^2y,f^2y),$$

$$G(y, fy, fy) + G(y, fx, fx) + G(y, f^2y, f^2y),$$
$$G(fy, f^2y, f^2y) + G(fy, fx, fx)\big\}, \tag{2.12}$$

for all $x, y, z \in X$*, where* $0 < k < 1/5$*.*

Then f *has a unique fixed point.*

Remark 2.1. Write

$$\rho_G(x, y) = \max\{(G(x, fx, f^2y), G(y, fy, f^2x)\} \text{ for all } x, y \in X. \tag{2.13}$$

It is interesting to remark that $x = y$ does not imply that $\rho_G(x, y) = 0$. In fact, from (2.13) with $y = x$, we get $\rho_G(x, x) = G(x, fx, f^2x)$, which is positive for each $x \in X$ with $fx \neq x$. That is, ρ_G is not a metric on X. In other words, Corollary 2.2 cannot be characterized in terms of metric, given in (2.13) to determine a fixed point of f.

By restricting the the terms the inequality 2.1 and extending the choice of k as $\alpha \in (0, 1)$, we have another result

Corollary 2.3. *Let* (X, G) *is a complete G-metric space satisfying,*

$$G(fx, fy, fz) \leq \alpha[G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)],$$
$$\textit{for all } x, y, z \in X \tag{2.14}$$

Then f *has a unique fixed point.*

Our second result is

Theorem 2.2 (Mustafa and Obiedat, [6])**.** *Let* (X, G) *be a complete* G*-metric space and f, a self-map on* G *such that*

$$G(fx, fy, fz) \leq \alpha G(x, y, z) + \beta \max\big\{G(x, fx, fx), G(y, fy, fy),$$
$$G(z, fz, fz)\big\} \textit{ for all } x, y, z \in X, \tag{2.15}$$

where $\alpha \geq 0$ *and* $\beta \geq 0$ *such that* $\alpha + \beta < 1$*. Then* f *has a unique fixed point* p *and* f *is* G*-continuous at* p*.*

Proof. Define

$$S = \{G(x, fx, fx) : x \in X\}.$$

Each S is a nonempty set of nonnegative numbers which is bounded below. Hence by the Lemma 2.1, $\inf S = a$ exists. We prove that $a = 0$. If it is possible, Let $a > 0$. Now $G(fx, f^2x, f^2x) \in S$. Writing $y = z = fx$ in (2.15), we have

$$\begin{aligned} G(fx, f^2x, f^2x) \leq & \alpha G(x, fx, fx) \\ & + \beta \max\{G(x, fx, fx), G(fx, f^2x, f^2x), G(fx, f^2x, f^2x)\} \\ \leq & \alpha G(x, fx, fx) + \beta M, \end{aligned} \tag{2.16}$$

where $M = \max\{G(x, fx, fx), G(fx, f^2x, f^2x)\}$.

Case(a). If $M = G(fx, f^2x, f^2x)$, from (2.16) and Lemma 2.1, we have

$$G(fx, f^2x, f^2x) \leq \alpha G(x, fx, fx) + \beta G(fx, f^2x, f^2x)$$

or

$$G(fx, f^2x, f^2x) \leq \left(\frac{\alpha}{1-\beta}\right) G(x, fx, fx) < a,$$

which is a contradiction.

Case(b). If $M = G(x, fx, fx)$, from again (2.16) and Lemma 2.1,

$$G(fx, f^2x, f^2x) \leq \alpha G(x, fx, fx) + \beta G(x, fx, fx) \leq (\alpha + \beta) G(x, fx, fx) < a.$$

This shows that a is not a lower bound of S, which is again a contradiction.

These two contradictions prove that $a = 0$. Hence, there exists a $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$G(x_n, fx_n, fx_n) \in S \text{ for all } n = 1, 2, 3, \ldots \text{ and } \lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{2.17}$$

By the rectangle inequality of G and (1.1), we have

$$\begin{aligned} G(x_n, x_m, x_m) \leq & G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ \leq & G(x_n, fx_n, fx_n) \\ & + G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m). \end{aligned} \tag{2.18}$$

Now, with $x = x_n$ and $y = z = x_m$, (2.15) gives,

$$G(fx_n, fx_m, fx_m) \leq \alpha G(x_n, x_m, x_m) + \beta \max\{G(x_n, fx_n, fx_n),$$

$$G(x_m, fx_m, fx_m), G(x_m, fx_m, fx_m)\}$$
$$\leq \alpha G(x_n, x_m, x_m)$$
$$+ \beta[G(x_n, fx_n, fx_n) + G(x_m, fx_m, fx_m)].$$

Inserting this in (2.18) and then simplifying, we get

$$G(x_n, x_m, x_m) \leq \left(\frac{1+\beta}{1-\alpha}\right) G(x_n, fx_n, fx_n) + \left(\frac{2+\beta}{1-\alpha}\right) G(x_m, fx_m, fx_m).$$

Applying the limit as $m, n \to \infty$ in this and using (2.17), we obtain that $\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in X.

Since, X is G-complete, we find the point p in X such that

$$\lim_{n\to\infty} x_n = p. \tag{2.19}$$

Again repeatedly using (G5),

$$\begin{aligned} G(p.fp, fp) \leq & G(p, fx_n, fx_n) + G(fx_n, fp, fp) \\ \leq & [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp). \end{aligned} \tag{2.20}$$

Now, from (2.15) with $x = x_n$ and $y = z = p$, it follows that

$$\begin{aligned} G(fx_n, fp, fp) \leq & \alpha G(x_n, p, p) + \beta \max\big\{G(x_n, fx_n, fx_n), \\ & G(p, fp, fp), G(p, fp, fp)\} \\ \leq & \alpha G(x_n, p, p) + \beta[G(x_n, fx_n, fx_n) + G(p, fp, fp)]. \end{aligned} \tag{2.21}$$

Substituting (2.21) in (2.20) and then using (G5), we get

$$(1-\beta)G(p.fp, fp) \leq (1+\beta)G(x_n, fx_n, fx_n) + G(p, x_n, x_n) + \alpha G(x_n, p, p).$$

In the limiting case as $n \to \infty$, this in view of (2.17), (2.19) and Lemma 1.1, implies that $G(p, fp, fp) = 0$ or $fp = p$. Thus p is a fixed point.

To prove the uniqueness, let q be another fixed point of f so that $fq = q$. Then, writing $x = p$ and $y = z = q$ in (2.15),

$$\begin{aligned} G(p, q, q) &= G(fp, fq, fq) \\ &\leq \alpha G(p, q, q) + \beta \max\big\{G(p, fp, fp), G(q, fq, fq), G(q, fq, fq)\} \\ &= \alpha G(p, q, q) + \beta \max\big\{G(p, p, p), G(q, q, q)\} \end{aligned}$$

or $(1-\alpha)G(p, q, q) \leq 0$ so that $p = q$. That is p is the unique fixed point of f. □

In away similar to the previous two proofs, one can obtain a uniue fixed point under each of the following contraction-type conditions:

1. (Mustafa et al, [4]). There exists a number $0 \leq k < 1$ such that for all $x, y, z \in X$,

$$\begin{aligned} G(fx, fy, fz) \leq & k \max\{G(x, fy, fy), G(x, fz, fz), G(y, fx, fx), \\ & G(y, fz, fz), G(z, fx, fx), G(z, fy, fy)\}. \end{aligned}$$

2. (Mustafa et al, [4]). There exists a number $0 \leq k < 1$ such that for all $x, y, z \in X$,

$$\begin{aligned} G(fx, fy, fz) \leq & k \max\{G(x, x, fy), G(x, x, fz), G(y, y, fx), \\ & G(y, y, fz), G(z, z, fx), G(z, z, fy)\}. \end{aligned}$$

3. (Mohanta, [3]). There exist nonnegative numbers $a.b, c, d$ and e such that $2a + 2b + 2c + d + 2e < 1$, and for all $x, y, z \in X$,

$$\begin{aligned} G(fx, fy, fz) \leq & a[G(x, fy, fy) + G(y, fx, fx)] \\ & + b[G(y, fz, fz) + G(z, fy, fy)] \\ & + c[G(x, fz, fz) + G(z, fx, fx)] \\ & + dG(x, y, z) + e \max\big\{G(x, fx, fx), \\ & G(y, fy, fy), G(z, fz, fz)\}. \end{aligned}$$

4. (Vats et al, [8]). There exists a number $0 \leq k < 1/2$ such that for all $x, y, z \in X$,

$$\begin{aligned} G(fx, fy, fz) \leq & k \max\big\{G(x, fx, fx), G(x, fy, fy), G(x, fz, fz), \\ & G(y, fy, fy), G(y, fx, fx), G(y, fz, fz), , \\ & G(z, fz, fz), G(z, fx, fx), G(z, fy, fy)\big\}. \end{aligned}$$

5. (Vats et al, [8]). There exists a number $0 \leq k < 1/2$ such that for all $x, y, z \in X$,

$$\begin{aligned} G(fx, fy, fz) \leq & k \max\big\{G(x, x, fx), G(x, x, fy), G(x, x, fz), \\ & G(y, y, fy), G(y, y, fx), G(y, y, fz), \\ & G(z, z, fz), G(z, z, fx), G(z, z, fy)\big\}. \end{aligned}$$

6. (Vats et al, [8]). There exists a number $0 \leq k < 1/4$ such that for all $x, y, z \in X$,

$$\begin{aligned}G(fx, fy, fz) \leq & k \max \big\{ G(x, fx, fx) + G(x, fy, fy) + G(x, fz, fz), \\ & G(y, fy, fy) + G(y, fx, fx) + G(y, fz, fz), \\ & G(z, fz, fz) + G(z, fx, fx) + G(z, fy, fy) \big\}.\end{aligned}$$

7. (Vats et al, [8]). There exists a number $0 \leq k < 1/4$ such that for all $x, y, z \in X$,

$$\begin{aligned}G(fx, fy, fz) \leq & k \max \big\{ G(x, x, fx) + G(x, x, fy) + G(x, x, fz), \\ & G(y, y, fy) + G(y, y, fx) + G(y, y, fz), \\ & G(z, z, fz) + G(z, z, fx) + G(z, z, fy) \big\}.\end{aligned}$$

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