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EQUILIBRIUM CHARACTERISTICS OF SOVEREIGN DEFAULT IN A TWO COUNTRY CURRENCY AREA: A TWO PLAYER DYNAMIC HIERARCHICAL GAME

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ABSTRACT: This paper is aimed at developing a two player dynamic hierarchical game called the Sovereign Default Game in order to study the optimality of default in a two country currency area where the larger economy is the leader and the smaller economy acts as the follower. Our approach is a monetary general equilibrium model where the leader sets the monetary policy in the currency area but the follower experiences a real exchange rate over-valuation that may lead to default. We find two maxima of maxima, one of them "perverse" in the sense that it generates moral hazard.

AMS Subject Classification: 90A09, 91B28

Key Words: currency areas, sovereign default, monetary general equilibrium, purchasing power parity, Nash equilibria and Pareto optimality

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1. INTRODUCTION

Default can be optimal for a sovereign government under certain circumstances in a Currency Area (CA), but is it optimal for the CA itself? That is, can default be a cooperative equilibrium? Can it be a selfish (non-cooperative) equilibrium? Or worse, can it be an equilibrium at all? The existing literature cannot answer the question. Mendoza and Yue (2012) prove that default can be the optimal decision of a sovereign government—formally a *benevolent planner*—for whom financial autarky has a higher payoff than debt repayment. We generalize this result by building a two player dynamic game. We find two equilibria which are cooperative and non-cooperative, that is, Pareto optimal and Nash equilibria; maxima of maxima.

In our proposal, time is discrete and horizon is infinite. Two countries form a CA: country j, the largest of the two economies, is the *leader*; and country i is the *follower*. Both countries have two agents: a sovereign government and continuum of households. This is an exchange economy with a cash-in-advance structure (CIA). Forming a CA means that at some time $t = 0$ both countries relinquished their nominal currencies and adopted a joint currency, the euro. The initial exchange rate of country i at time $t = 0$ is denoted by $e^0_{i/€}$. For example, Greece substituted the drachma for the euro on January 1st, 2002, at a initial rate of $e^0_{\Delta\rho/€} = 340.750$ drachmas for 1 euro.

Forming a CA also implies renouncing to a national monetary policy and adopting a joint policy. Let M^i_t denote country i's stock of currency in period t, and let $\mu^i_t = M^i_t/M^i_{t-1}$ denote the growth rate of this stock—M^j_t denotes country j's stock of currency in period t and μ^j_t denotes the growth rate of this stock—. Similarly, let $\mu^*_t = M^*_t/M^*_{t-1}$ denote the growth rate of stock M^*_t of euros. As usual, M^i_t and M^j_t are controlled by country i's and j's central banks respectively, and M^*_t is controlled by the union's central bank, but the monetary policy in the CA is determined by the leader j. Country i's monetary policy is a scaled version of j's: $\mu^i_t = \kappa\mu^j_t, \kappa \in (0, 1]$.

The real value of a euro at country i at time t is given by the real exchange rate. We assume that purchasing power parity, PPP, holds, i.e., to determine the real exchange rate, we use PPP defined as

$$e^{t,PPP}_{i/€} = \frac{P^t_i}{P^t_€},$$

where P_i^t is the Harmonized Index of Consumer Prices (HICP) in country i at time t; and $P_€^t$ is the HICP in the CA at time t. $P_€^t$ is given by

$$P_€^t = \alpha^i P_i^t + \alpha^j P_j^t, \;\; \alpha^i + \alpha^j = 1.$$

α^i and α^j are the weights of countries i and j relative to consumption expenditure in the total CA. Price equilibrium at time t means that $P_i^t / P_€^t = 1$, equivalently, $P_i^t = P_j^t$. Furthermore, over-valuation of country i in the CA implies under-valuation of country j: $P_i^t / P_€^t > 1$ implies $P_i^t > P_j^t$.

We assume that there are persistent diverging inflations in our CA [see Lane (2006)], in particular, i is over-valued. Trade in this economy in periods $t \geq 1$ occurs in three separate locations: 1) an asset market available to both country governments, households, and the CA government; 2) a goods market in country i; and 3) a goods market in country j. In the asset market the two country governments sell and buy euro bonds via open market operations. The euro bonds promise delivery of an amount of euros in the asset market in the next period.

In country i's goods market the households use euros to sell and purchase a single home good and to consume a single imported good from country j; country j's goods market functions in an analogous manner. We assume that both goods are tradable. The asset market allows CA central banks to execute their monetary policy and it permits households in both countries to hold their money in interest-bearing instruments which strictly dominate cash. In order to move an amount of cash x_t from the asset market to the goods market, households in each country must pay a real fixed cost γ. As in Alvarez, Atkeson and Kehoe (2008), households that pay γ at time t are called active households and those that do not are inactive. We assume that there is always a positive amount of active households. The CIA constraint for active households in country i at time t is

$$c_t^i = n_t^i + x_t^i$$

where c_t^i is country i's consumption at time t, and n_t^i is country i's households current real balances. The CIA constraint for inactive households is simply $c_t^i = n_t^i$. In this context, the timing of the Default stage game is as follows:

1. At each time t country j, the leader, decides the monetary policy to be applied in his own country μ_t^j and in the CA as a whole.

2. Country i follows by imitating the direction of the leader's monetary policy $\mu_t^i = \kappa \mu_t^j$, $\kappa \in (0, 1]$, but possibly not the same amount. Recall that i is a smaller economy than j.

3. After the monetary policies are determined and executed, country i's government evaluates if default has a greater value than continuing in the asset markets. Let v^d represent the value of default and v^{nd} denote the value of not-defaulting, then:

$$\begin{array}{ll} \text{If} \quad v^d > v^{nd} & \text{then country } i \text{ will default; or} \\ \text{If} \quad v^d \leq v^{nd} & \text{country } i \text{ does not default; payoffs} \\ & \text{are made and the stage game ends.} \end{array}$$

If country i decides to continue in the asset market (i.e., decides not to default) then the stage game is repeated as above, steps 1-3.

4. If on the other hand country i's government decides to default then the leader will face the decision: bailout or not; expel i from the CA or not.

Finally, payoffs are made to i and j and the stage ends. If country i's government defaults at time t—due to over-valuation—, the following events will happen:

- Country i will stop paying its obligations: households in country i and j will not receive government i's next period's payoff B_{t+1}^i.

- η will shoot up, the relative risk aversion coefficient of the whole CA will increase significantly. It is important to note that η is assumed constant. We shall assume that is varies exogenously.

- We also assume exports between CA members are luxury or superior goods. Hence, a greater perceived risk will imply lower income expectations; and exports will plunge. For the sake of simplicity, we assume that exports are reduced to zero.

- Country i's government and households are excluded from the asset market. The CIA constraint is reduced to $c_t^i = n_t^i$ for all households in i.

- The CA's central bank takes over money expansion in country i and limits μ_{t+1}^{*i} to 2% increase per annum. The asterisk indicates that money growth in country i is controlled by the CA's monetary authority.

If country i's government defaults at time t and country j decides to apply a bailout, then country i will reenter the asset market in the next period, $t+1$, with a fresh record and zero debt. Relative risk aversion does not necessarily fall back to pre-default levels since the problem that generated the default may not have been resolved: real exchange rate over-valuation.

There are two sources of uncertainty in this economy: i) default of i's government and j's government actions in this scenario; and ii) shocks to money growth in the two countries and in the CA central bank. Let $s_t = (\mu_t^i, \mu_t^j, \mu_t^*, I^i, I^j)$ be the aggregate event in period t, the *state*; I^i and I^j are indicator functions:

$$I_t^i = \begin{cases} 0 & \text{if country } i \text{ defaults;} \\ 1 & \text{otherwise.} \end{cases}$$

Variable I^j is defined as follows

$$I_t^j = \begin{cases} 1 & \text{if government } j \text{ decides in favor of a bailout of } i; \\ 0 & \text{otherwise.} \end{cases}$$

Let $g(s_t)$ represent the density of the probability distribution over each state. Let $s^t = (s_1, \ldots, s_t)$ denote the history of aggregate events through period t. In this hierarchical game we find the existence of Stackelberg equilibria, hence Nash equilibria. Furthermore, both equilibria are also Pareto optimal. This means that the social optimum coincides with individual optima. Both equilibria are 1) no-default/no-bailout; and 2) default/bailout. The second equilibrium has the disadvantage of generating *moral hazard*. Once country i's defaults, it has no incentive to repay debt nor to control inflation; hence, equilibrium 2 will lead to further risk taking of sovereign i. Interestingly, moral hazard occurs under *asymmetric information*; after default we find out that the risk-taking party knows more about its intentions than the party who pays the consequences. In order to avoid moral hazard, country j has the option to expel i from the CA. This decision could be optimal for j, but it means the end of the CA; hence it is neither Nash nor Pareto optimal.

The paper is organized as follows: Section 2 explains the goods markets; Section 3 describes the asset market; Section 4 introduces our general equilibrium monetary model; Section 5 presents our results on equilibria; and Section 6 provides our conclusions.

2. THE GOODS MARKETS

Goods markets in countries i and j perform in exactly the same way. We explain the goods market in country i as a reference.

At time t, real euros buy $\left(e^{t,PPP}_{i/€}\right)^{-1}$ goods in country i. However, households use nominal euros $\left(e^{0}_{i/€}\right)^{-1}$ to sell and buy the local good; and to buy the imported good subject to the CIA constraint. More explicitly, at the beginning of each stage t, households act as sellers; and at the end of the stage they act as consumers.

Households start at each stage t with an endowment of the local good y^i_{t-1}. This endowment is worth $P^i(s_{t-1})y^i_{t-1}\left(e^{0}_{j/€}\right)^{-1}$ euros. Before goods markets open, the central banks execute their monetary policy. Thus, by the time the goods markets open i's households endowment is worth in nominal euros $P^i(s_t)y^i_{t-1}\left(e^{0}_{j/€}\right)^{-1}$; and worth in real euros

$$\left(\frac{P_i(s_{t-1})\left(e^{t-1,PPP}_{i/€}\right)^{-1}}{P_i(s_t)\left(e^{t,PPP}_{i/€}\right)^{-1}}\right) y^i_{t-1} \quad .$$

Households may sell their endowment domestically and abroad

$$P^i(s_t)y^i_{t-1}\left(e^{0}_{i/€}\right)^{-1} = \nu^i P^i(s_t)\pi^{i,i}\left(e^{0}_{i/€}\right)^{-1} + (1-\nu^j)P^j(s_t)\pi^{i,j}_t\left(e^{0}_{j/€}\right)^{-1},$$

where $\pi^{i,i}_t$ is households' profits obtained from selling the endowment domestically at time t; $\pi^{i,j}_t$ is the profit obtained from selling i's endowment in country j; ν^i is the Armington weight of the domestic input in country i, and $1-\nu^i$ is the weight of the imported input in country i—imported from j—. Observe now that

$$\nu^i + (1-\nu^i) = 1$$

everything that is consumed in country i is domestic or imported from country j, but also

$$\nu^i + (1 - \nu^j) = 1$$

the endowment of the domestic product is either sold locally—in a proportion ν^i—or exported to j in proportion $1 - \nu^j$—the weight of the imported input in country j—. The latter relations imply that $\nu^i = \nu^j$. The final result of trading is the current real balances

$$n_t^i \left(e_{i/€}^{t,PPP}\right)^{-1} := \left(\frac{P_i(s_{t-1}) \left(e_{i/€}^{t-1,PPP}\right)^{-1}}{P_i(s_t) \left(e_{i/€}^{t,PPP}\right)^{-1}} \right) y_{t-1}^i \tag{1}$$

After selling has taken place in the goods markets, households decide whether they will transfer cash from the asset market. Active households will add an amount x_t^i to their current real balance after paying γ^i. An indicator variable $z^i(s_t, \gamma^i)$ equals zero if transfers are zero and one if they are positive. At the end of the stage active households consume

$$c_t^i = n_t^i + x_t^i,$$

and inactive households consume $c_t^i = n_t^i$. The end of the stage household endowment of the local good is y_t^i.

Consumption of country i's and j's goods are determined by a CES Armington aggregator that combines the domestic input and the imported input in country i at time t, $c_t^{i,i}$ and $c_t^{j,i}$, respectively,

$$c^i(s_t, \gamma^i) = \left[\nu^i (c_t^{i,i})^\xi + (1 - \nu^i)(c_t^{j,i})^\xi I_t^i \right]^{\frac{1}{\xi}}$$

The Armington elasticity of substitution between $c_t^{i,i}$ and $c_t^{j,i}$ is defined as $\upsilon_{c^i,c^j} = \left| \frac{1}{\xi - 1} \right|$, and ν^i is the Armington weight of the domestic input in country i—the home good can be either consumed or used to pay the transfer cost γ^i from the asset market—and $(1 - \nu^i)$ is the weight of the imported input in country i. The imported input comes from the other country in the CA, country j, and it is entirely consumed in the same period it is bought. Note that consumption of the imported good is affected by default. If country i defaults at time t, its consumption of imported good is reduced to zero.

The following parameter constraints are assumed to hold $\xi < 1$, and $0 \leq \nu^i < 1$. Notice that $\nu^i < 1$ is necessary because without using imported inputs default would be costless. In addition, foreign and domestic inputs are assumed to be gross substitutes (i.e., $\xi < 1$). Consumption is subject to the CIA constraint (2) and the transition law (3)

$$c^i(s_t, \gamma^i) = n^i(s_t, \gamma^i) + x^i(s_t, \gamma^i) z^i(s_t, \gamma^i), \tag{2}$$

$$\begin{aligned} n^i(s_{t+1}, \gamma^i) &= \left(\frac{P_i(s_t)}{P_i(s_{t+1})} \right) y_t^i, \\ &= \left(\frac{P_i(s_t)}{P_i(s_{t+1})} \right) \left(\nu^i \pi_t^{i,i} + (1 - \nu^j) \pi_t^{i,j} \right) \end{aligned} \tag{3}$$

where in (2) at $t = 1$, the term $n^i(s_1, \gamma)$ is given by $M_0^i / P_i(s_1)$

Real balances (3) can be expressed in real euros as in equation (1). Analogously, transfers can be expressed in real euros as

$$x^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} = x^i(s_t, \gamma^i) \left(\frac{P_€(s_t)}{P_i(s_t)} \right) \tag{4}$$

Finally, combining (2) with (1) and (4) we have that the CIA constraint in real euros is

$$\begin{aligned} c^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} &= n^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} \\ &\quad + x^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} z^i(s_t, \gamma^i), \end{aligned} \tag{5}$$

If the sovereign government from country i defaults at time t—at the beginning of time t, before the goods market has opened—, consumption at t is reduced to

$$c^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} = \left(\frac{P_i(s_{t-1}) \left(e_{i/€}^{t-1,PPP} \right)^{-1}}{P_i(s_t) \left(e_{i/€}^{t,PPP} \right)^{-1}} \right) \pi_{t-1}^{i,i} \tag{6}$$

because households in i are excluded from the asset market, and because it is assumed that imports are reduced to zero.

3. THE ASSET MARKETS

In period 0 there is an initial round of trade in bonds in the asset market with no trade in goods markets. In the asset market in period 0, country i's households of type γ^i have:

1. M_0^i units of country i's money,
2. $\overline{B}^{i,i}(\gamma^i)$ units of country i's government debt (bonds) owed to country i households, i.e. *claims*[1] to $\overline{B}_i(\gamma^i)$ units of country i's currency, e.g. drachmas.
3. $\overline{B}_i^{j,i}$ units of country j's government debt owed to country i households—i.e. claims to $\overline{B}_i^{*,j}$ units of country j's currency, e.g. Deutsche mark—,

bonds are claims on $\overline{B}_i(\gamma^i)$ units of i's currency—the overbar distinguishes claims from payoffs as in (9)—and $\overline{B}_i^{*,j}$ units of j's currency both converted to euros in the asset market in that period. As in Alvarez, Atkeson and Kehoe (2008), we will call households of type γ^i the *active* households; and those that do not participate of the asset market as *inactive* households.

Likewise, in the asset market in period 0 country j's households start with

1. M_0^j units of country j money,
2. $\overline{B}^{i,j}$ units of country i's government debt owed to country j households—i.e. claims of country i's debt in hands of country j's households—, and
3. $\overline{B}^{j,j}(\gamma^j)$ units of country j's government debt owed to country j households.

We require that

$$\int \overline{B}^{i,i}(\gamma^i)f(\gamma^i)d\gamma^i + \overline{B}^{i,j} = \overline{B}^i$$

and

$$\overline{B}^{j,i} + \int \overline{B}^{j,j}(\gamma^j)f(\gamma^j)d\gamma^j = \overline{B}^j.$$

[1]A *claim* entitles a creditor to receive a payment, or payments, from a debtor in circumstances specified in a contract between them: interest payments on a bond contract times the number of contracts.

Where $\overline{B}^i$ denotes the stock of outstanding i currency bonds at $t = 0$ and $\overline{B}^j$ denotes the stock of outstanding j currency bonds at $t = 0$.

Agents must convert local currencies to euros using the time $t = 0$ exchange rate since all trades in the asset market are made in time $t = 0$ euros

$$\int \overline{B}^{i,i}(\gamma^i) f(\gamma^i) d\gamma^i \left(e^0_{i/€}\right)^{-1} + \overline{B}^{i,j} \left(e^0_{i/€}\right)^{-1} = \overline{B}^i \left(e^0_{i/€}\right)^{-1}$$

and

$$\overline{B}^{j,i} \left(e^0_{j/€}\right)^{-1} + \int \overline{B}^{j,j}(\gamma^j) f(\gamma^j) d\gamma^j \left(e^0_{j/€}\right)^{-1} = \overline{B}^j \left(e^0_{j/€}\right)^{-1}.$$

A typical CA household will hold a portfolio of country i's government bond and country j's government bond. The weight of government i's bond in country i household portfolios is ν^i_A and the weight of government j's bond in country i household portfolio is the complement $1 - \nu^i_A$. Equivalently, for country j: ν^j_A is the weight of government j's bond in j household portfolio and $1 - \nu^j_A$ is the weight of government i's bond in j's portfolio.

Each household in i must pay a real fixed cost γ^i in euros

$$\gamma^i \left(e^{t,PPP}_{i/€}\right)^{-1}$$

for each transfer of cash between the asset market and the goods market. This fixed cost is constant over time for any specific household, but it varies across households in both countries according to a probability distribution $F(\gamma^i)$ with density $f(\gamma^i)$; and according to the PPP exchange rate at time t. Households are indexed by their fixed cost γ^i. The fixed costs for households in each country are in units of the local good. We assume $F(0) > 0$, so that a positive mass of households has a zero fixed cost and will participate in the asset market.

Each government conducts an open market operation in the asset market, which determines the realizations of money growth rates μ^1_t and μ^2_t in the two countries and the current price levels in the two countries, $P_i(s_t)$ and $P_j(s_t)$. In the presence of real exchange rate equilibrium the equation of exchange (7) would hold

$$M^i \cdot V^i = P_i \cdot Q_i, \tag{7}$$

where M^i is money supply, V^i is velocity of money, P_i is price level and Q_i is quantity of assets, goods and services sold during the year. In equilibrium real money demand is simply Q_i/V^i. However, under real exchange rate disequilibrium, money growth affects prices according to (8) [see HVMG]

$$P_i(s_t) = \frac{M^i(s_t) - 2\alpha^i\alpha^j P_j(s_t)y_t^i \pm \sqrt{-M^i(s_t)}\sqrt{-M^i(s_t) + 4\alpha^i\alpha^j P_j(s_t)y_t^i}}{2y_t^i(\alpha^i)^2}. \quad (8)$$

As noted, households enter each stage t with claims on government debt. Once monetary policy has been decided and executed, and after country i decides whether to stay in the asset market or default, households receive their payoffs. This cash can be either reinvested in the asset market or, if the fixed cost is paid, transferred to the goods market. With $B_t^{i,i}$ and $B_t^{j,i}$ denoting the current payoffs of governments i and j to households in i of the state-contingent bonds purchased in the past, q_t^i and q_t^j denoting the current prices of the bonds; $\int q_t^i \theta_t^{i,i} ds_{t+1}$ country i household's purchases of new government i's bonds, and $\int q_t^j \theta_t^{i,j} ds_{t+1}$ country i household's purchases of new government j's bonds; the asset market constraint is

$$\nu_A^i B_t^{i,i}\left(e^0_{i/€}\right)^{-1} + (1-\nu_A^i)B_t^{j,i}\left(e^0_{j/€}\right)^{-1} = \nu_A^i \int q_t^i \left(e^0_{i/€}\right)^{-1}\theta_t^{i,i} ds_{t+1} + (1-\nu_A^i)\int q_t^j \left(e^0_{j/€}\right)^{-1}\theta_t^{i,j} ds_{t+1} + P_i(s_t)\left(x_t^i + \gamma^i\right)\left(e^0_{i/€}\right)^{-1},$$

where $\theta_t^{i,i}$ and $\theta_t^{i,j}$ are the quantities of government i and j's bonds bought at prices q_t^i and q_t^j by country i's households and payoff at some future time if the fixed cost is paid; and

$$\nu_A^i B_t^{i,i}\left(e^0_{i/€}\right)^{-1} + (1-\nu_A^i)B_t^{j,i}\left(e^0_{j/€}\right)^{-1} = \nu_A^i \int q_t^i \left(e^0_{i/€}\right)^{-1}\theta_t^{i,i} ds_{t+1} + (1-\nu_A^i)\int q_t^j \left(e^0_{j/€}\right)^{-1}\theta_t^{i,j} ds_{t+1} \quad (9)$$

otherwise, i.e. current payoff is entirely reinvested.

Throughout this research, we assume that the consumers are not allowed to store cash from one period to the next. This assumption implies that the CIA constraint holds with equality and greatly simplifies the analysis. For some models in which agents are allowed to store cash and end up doing so in equilibrium; see, for instance, the works of Alvarez, Atkeson, and Edmond (2003) and Khan and Thomas (2007).

4. THE MODEL

4.1. THE HOUSEHOLD'S PROBLEM

Let us describe the problem of the household of type γ^i in country i—the behavior of households in country j is analogous.
In each period $t \geq 1$, in the goods market, households start the period with currency i's real balances $n^i(s_t, \gamma^i)$—recall, by the time the goods market opens, monetary policy has been executed, and government i has defaulted or not—. They then choose transfers of real balances between the goods market and the asset market $x^i(s_t, \gamma^i)$, an indicator variable $z^i(s_t, \gamma^i)$ equal to zero if these transfers are 0, and 1 if they are positive.

If the sovereign government from country i defaults at the beginning of time t, consumption in the goods market at t is reduced to (6) because households in i are excluded from the asset market, and because it is assumed that imports are reduced to zero.
In the asset market at time $t \geq 1$, after monetary policy has been executed in the CA and sovereign i has decided whether to commit to repaying or not, country i's households begin by receiving cash payments $\nu_A^i B_t^{i,i}(s_t, \gamma^i) + (1 - \nu_A^i) B_t^{j,i}(s_t, \gamma^i)$ on their bonds. Next, they make decisions on cash transfers to the goods market and purchases of new bonds subject to the sequence of real euro budget constraints

$$\begin{aligned}
&\left(\nu_A^i B_t^{i,i}(s_t, \gamma^i) + (1 - \nu_A^i) B_t^{j,i}(s_t, \gamma^i)\right) \left(e_{i/€}^{t,PPP}\right)^{-1} \\
&= \nu_A^i \int q^i(s_t, s_{t+1}) \left(e_{i/€}^{t,PPP}\right)^{-1} \theta^{i,i}(s_t, s_{t+1}, \gamma^i) ds_{t+1} \\
&+ (1 - \nu_A^i) \int q^j(s_t, s_{t+1}) \left(e_{j/€}^{t,PPP}\right)^{-1} \theta^{i,j}(s_t, s_{t+1}, \gamma^i) ds_{t+1} \qquad (10) \\
&+ P_i(s_t) \left(e_{i/€}^{t,PPP}\right)^{-1} \left[x^i(s_t, \gamma^i) + \gamma^i\right] z^i(s_t, \gamma^i).
\end{aligned}$$

Here, $\theta^{i,i}(s_t, s_{t+1})$ is the quantity of bonds contracts issued by sovereign i and bought by i's households at time t at prices $q^i(s_t, s_{t+1})$, given s_t; $\theta^{i,j}(s_t, s_{t+1})$ is the quantity of bonds contracts issued by sovereign j and bought by country i households at time t at prices $q^j(s_t, s_{t+1})$, given s_t. Equation (10) states that cash payments received at the beginning of each stage—the left side of the equality—can either be reinvested—the first two terms on the right side of the equality—or transferred to the goods market if γ^i is paid—the last term on the right—.

Assume that both consumption $c^i(s_t, \gamma^i)$ and real bond holdings are uniformly bounded by some (possibly large) constants. The problem of country i's households of type γ^i is to maximize utility of real euro consumption

$$\sum_{t=1}^{\infty} \beta^t \int U\left(c^i(s_t, \gamma^i)\left(e_{i/€}^{t,PPP}\right)^{-1}\right) g(s_t) ds_t \tag{11}$$

subject to real euro balances (1), to the CIA constraint (5), and the asset market constraint (10). We assume that the period utility function takes the standard constant-relative risk-aversion, CRRA, form

$$U\left(c\left(e_{i/€}^{PPP}\right)^{-1}\right) = \frac{\left(c\left(e_{i/€}^{PPP}\right)^{-1}\right)^{1-\eta} - 1}{1-\eta}, \quad \text{with } \eta > 0, \eta \neq 1. \tag{12}$$

The parameter η measures the degree of relative risk aversion that is implicit in the utility function, $\eta = 1$ is risk neutrality and a greater η implies a greater risk aversion. We assume that η is a CA parameter, i.e. it does not vary across countries. Households in country j, solve an analogous problem. The main difference is that i's overvaluation implies j's undervaluation: country j's households benefit from country i's higher inflation.

Since each transfer of cash between the asset market and country i's goods market consumes γ^i units of i's good, the total goods cost in real euros of carrying out all transfers between i's households and the asset market in t is

$$\gamma^i \left(e_{i/€}^{t,PPP}\right)^{-1} \int z(s_t, \gamma^i) f(\gamma^i) d\gamma^i,$$

and likewise for the foreign households. The resource constraint in country i is given by

$$\int \left[c^i(s_t, \gamma^i)\left(e_{i/€}^{t,PPP}\right)^{-1} + \gamma^i \left(e_{i/€}^{t,PPP}\right)^{-1} z(s_t, \gamma^i)\right] f(\gamma^i) d\gamma^i$$

$$= y_t^i \left(e_{i/€}^{t,PPP} \right)^{-1}, \quad (13)$$

for all t, s_t, with the analogous constraint in country j. The fixed costs are paid for with cash obtained in the asset market. Thus, country i's money market-clearing condition in $t \geq 1$ is given by

$$\int \left(n^i(s_t, \gamma^i) \left(e_{i/€}^{t,PPP} \right)^{-1} + \left[x^i(s_t, \gamma^i) + \gamma^i \right] \left(e_{i/€}^{t,PPP} \right)^{-1} z^i(s_t, \gamma^i) \right) f(\gamma^i) d\gamma^i$$

$$= \frac{M^i(s_t)}{P_i(s_t) \left(e_{i/€}^{t,PPP} \right)^{-1}}, \quad (14)$$

for all s_t. The money market-clearing condition for the foreign country is analogous.

4.2. THE SOVEREIGN GOVERNMENT FROM COUNTRY i

Country i's sovereign cannot commit to repay its debt. As in the Eaton-Gersovitz (1981) model, if the country defaults at time t it does not repay at date t and the punishment is exclusion from *world* (*CA* in our case) asset markets in the same period, and all subsequent periods until its situation is resolved.

Stage t starts with country j's government setting the monetary policy, μ_t^j. Country i's government complies by setting its to $\mu_t^i = \kappa \mu_t^j$. Sovereign i then evaluates whether $v^{nd} \geq v^d$, if it does then country i's government pays off outstanding bonds issued in the past $\left(B^{i,i}(s_{t-1}) + B^{i,j}(s_{t-1}) \right) \left(e_{i/€}^{0} \right)^{-1}$ in euros to households in countries i and j. This determines the state s_t.

Given state s_t, sovereign i issues claims to euros in the next asset market of the form $\overline{B}^{i,i}(s_t) + \overline{B}^{i,j}(s_t)$. Otherwise, $v^{nd} < v^d$, then sovereign i reneges on its debt

$$\left(B^{i,i}(s_{t-1}) + B^{i,j}(s_{t-1}) \right) \left(e_{i/€}^{0} \right)^{-1},$$

and country i is entirely—both households and the government—excluded from the asset market. That is, sovereign i will be prevented from issuing further debt. Country i's households will be unable to purchase further sovereign i's or j's debt; and most importantly, country i's households will be precluded from making transfers from the asset market to the goods market $z^i(s_t, \gamma^i) = 0$. Further consequences of i's default are related to j's sanctions mentioned in the Introduction.

If country j decides to bailout country i, then country i reenters credit markets in the next period $t+1$ with a fresh record and zero debt. The probability of bailout is ϕ^i. We add to the Eaton and Gersovitz (1981) setup an endogenous link between the sovereign default and private economic activity. This link follows from the assumption that both households and the government are excluded from world asset markets when default occurs. Recall that with higher inflation comes diminishing spending capacity, thus more and more households need to be active in the asset market as overvaluation deepens. If they are excluded from asset markets then consumption is greatly reduced, $c = n$. This is consistent with the empirical evidence of severe adverse effects from sovereign default on private credit (asset) markets and foreign trade documented in Kaletsky (1985), Rose (2005), Kohlscheen and O'Connell (2008), Reinhart and Rogoff (2010), and Reinhart (2010), and with evidence on inefficient reallocation across foreign and domestic inputs in the aftermath of sovereign default found by Gopinath and Neiman (2010).

We now formalize sovereign i's problem. After monetary policy has been determined in the CA, the sovereign government chooses a debt policy (amounts and default or repayment) along with private consumption so as to solve a recursive social planner's problem. The state variables are the real euro bond position

$$\overline{B}^i(s_{t-1}) = \int_{s_t} q^i(s_{t-1}, s_t) \left(e_{i/€}^{t-1,PPP}\right)^{-1} \theta^i(s_{t-1}, s_t),$$

i.e., the amount to be repaid in real euros—opposed to nominal euros, $e^0_{i/€}$—, and s_t. Recall that $\overline{B}^i = \overline{B}^{i,i} + \overline{B}^{i,j}$. If a government purchases its own bonds, it increases the money supply $\mu^i > 0$, in effect creating money. When it sells bonds, money supply is decreasing, $\mu^i < 0$.

The planner's payoff is given by:

$$V^i(\overline{B}^i(s_{t-1}), s_t) = \max\left\{v^{nd}(\overline{B}^i(s_{t-1}), s_t), v^d(s_t)\right\}, \tag{15}$$

where $v^{nd}(\overline{B}^i(s_{t-1}), s_t)$ is the value of continuing in the credit relationship with foreign lenders (i.e., no default), and $v^d(s_t)$ is the value of default. If $\overline{B}^i_{t-1}(s_t) \geq 0$—i.e., the government is a creditor—, the payoff is simply

$v^{nd}(\overline{B}^i_{t-1}(s_t), s_t)$ since in this case the economy uses the credit market to save, receiving a return equal to the world's risk-free rate r^*_t. The payoff is given by the choice of $\overline{B}^i(s_t)$ and $c^{i,i}_t$ that solves this constrained maximization problem:

$$
\begin{aligned}
&V^i(\overline{B}^i(s_{t-1}), s_t)\\
&\quad= \max_{c^{i,i}_t, \overline{B}^i(s_t)} \left\{ U\left(\left(\nu^i (c^{i,i}_t)^{\xi} + (1-\nu^i)(c^{j,i}_t)^{\xi} I^i_t \right)^{1/\xi} \left(e^{t,PPP}_{i/€} \right)^{-1} \right) \right.\\
&\qquad\qquad \left. + \beta E \left[V^i(\overline{B}^i(s_t), s_{t+1}) \right] \right\}, \qquad (16)
\end{aligned}
$$

subject to:

$$
\begin{aligned}
&\frac{\overline{B}^i(s_t)}{M^i(s_{t-1})} \left(e^{t,PPP}_{i/€} \right)^{-1}\\
&\quad = \mu^i(s_t) \left(e^{t,PPP}_{i/€} \right)^{-1} - \left(e^{t-1,PPP}_{i/€} \right)^{-1}\\
&\quad + \int (M^i(s_{t-1}))^{-1} q^i(s_t, s_{t+1}) \theta^i(s_t, s_{t+1}) \left(e^{t,PPP}_{i/€} \right)^{-1} I^i_t ds_{t+1}, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&\int \left[\left(\nu^i (c^{i,i}_t)^{\xi} + (1-\nu^i)(c^{j,i}_t)^{\xi} I^i_t \right)^{1/\xi} + \gamma^i z^i(s_t, \gamma^i) \right] \left(e^{t,PPP}_{i/€} \right)^{-1} f(\gamma^i) d\gamma^i\\
&\qquad = (\pi^{i,i}_t + \pi^{i,j}_t I^i_t) \left(e^{t,PPP}_{i/€} \right)^{-1}, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&\left\{ \int \left[\left(\frac{P_i(s_{t-1}) \left(e^{t-1,PPP}_{i/€} \right)^{-1}}{P_i(s_t) \left(e^{t,PPP}_{i/€} \right)^{-1}} \right) (\pi^{i,i}_t + \pi^{i,j}_t I^i_t) \right. \right.\\
&\quad \left. + \left(x^i(s_t, \gamma^i) + \gamma^i \right) z^i(s_t, \gamma^i) I^i_t \right] \left(e^{t,PPP}_{i/€} \right)^{-1} f(\gamma^i) d\gamma^i \Bigg\} (M^i(s_{t-1}))^{-1}\\
&\qquad = \frac{\mu^i(s_t)}{P^i(s_t) \left(e^{t,PPP}_{i/€} \right)^{-1}}, \qquad (19)
\end{aligned}
$$

where (17) is the government budget constraint at s_t with $t \geq 1$; (18) is the resource constraint in country i's economy; (19) is country i's money market clearing condition; and $P^i(s_t)$ are disequilibrium prices given by (8)

We assume that i has not defaulted by the beginning of time t, i.e., $I^i_{t-1} = 1$. First order conditions are given by the derivative of the Lagrangian with respect to domestic consumption

$$c_t^{i,i} = \left(- \frac{(c_t^{j,i})^{\xi} I_t^i - \nu^i (c_t^{j,i})^{\xi} I_t^i - \left(e_{i/€}^{t,PPP} (f(\gamma^i))^{-1/\eta} \lambda_2^{-1/\eta} \right)^{\xi}}{\nu^i} \right)^{1/\xi}, \tag{20}$$

where λ_2 is the lagrange multiplier of the resource constraint (18), the marginal utility of income.

The derivative with respect to the bond position $\overline{B}^i(s_t)$ is given by

$$\frac{\partial \mathcal{L}_{v^{nd}}}{\partial \overline{B}_t^i(s_t)} = - \frac{\lambda_1 \left(e_{i/€}^{t,PPP} \right)^{-1}}{M^i(s_{t-1})} = 0, \tag{21}$$

where λ_1 is the lagrange multiplier of the government budget constraint (17). This suggests that the λ_1 is equal to zero.

In order to evaluate whether to default or not, government i substitutes the value of optimal consumption $c_t^{i,i}$ from equation (20) in $V^i(\cdot,\cdot)$ and compares the value in two cases:

- If i does not default at t, v^{nd}, against
- If i defaults at t, v^d.

$$V^i(\overline{B}_{t-1}^i(s_{t-1}), s_t)$$
$$= \max_{c_t^{i,i}, \overline{B}_t^i(s_t)} \left\{ \frac{\left(\left(\nu^i (c_t^{i,i})^{\xi} + (1-\nu^i)(c_t^{j,i})^{\xi} \right)^{1/\xi} \left(e_{i/€}^{t,PPP} \right)^{-1} \right)^{1-\eta} - 1}{1-\eta} \right. \tag{22}$$
$$+\beta \left\{ \max \left\{ \max \left\{ \frac{\left(\left(\nu^i (c_{t+1}^{i,i})^{\xi} + (1-\nu^i)(c_{t+1}^{j,i})^{\xi} \right)^{1/\xi} \left(e_{i/€}^{t+1,PPP} \right)^{-1} \right)^{1-\eta} - 1}{1-\eta} \right\} \right\},\right.$$
$$\left. \left. \left. \left\{ \max \frac{\left(\nu^i (c_{t+1}^{i,i}) \left(e_{i/€}^{t+1,PPP} \right)^{-1} \right)^{1-\eta} - 1}{1-\eta} \right\} \right\} \right\} \right\},$$

The second max after the discount factor is the value v^{nd} without default at t, after the monetary policies have been executed—$I_t^i = 1$; and the third corresponds to v^{nd} with default, $I_t^i = 0$. We substitute (20) in (22) to obtain equations (41) and (42) in the Appendix.

The difference between (41) and (42) reduces to: in the absence of default

$$\left(\left(-\frac{(c_{t+1}^{j,i})^{\xi} - \nu^i (c_{t+1}^{j,i})^{\xi} - \left(\lambda_2^{-1/\eta}(f(\gamma^i))^{-1/\eta} e_{i/€}^{t+1,PPP}\right)^{\xi}}{\nu^i}\right)^{1/\xi}\right)^{\xi} + (1-\nu^i)\left(c_{t+1}^{j,i}\right)^{\xi}$$

and, if i defaults at t

$$\frac{\left(\lambda_2^{-1/\eta}(f(\gamma^i))^{-1/\eta} e_{i/€}^{t+1,PPP}\right)^{\xi}}{\nu^i}.$$

4.3. THE SOVEREIGN GOVERNMENT FROM COUNTRY j

The sovereign government from country j aims to maximize his value function, and he will never default. Government j's problem is analogous to the v^{nd} problem that country i's government faces. However, government j's problem is complex because given his leading role in the CA, he sets the monetary policy for the CA, and in doing so, he must consider the followers optimization problem. From a formal point of view the leader faces a Stackelberg problem. In particular, the leader faces a *bilevel optimization problem* [see Sinha et al. (2013)] coupled with disequilibrium prices.

It is important to note that as long as disequilibrium prices are present, i.e. while one country is overvalued and the other undervalued, the horizon instead of being infinity is basically equal to 1. That is, the leader is unable set a long-term horizon policy because the follower can default at any stage. We assume that our leader will never default. This does not imply that $\overline{B}_t^j \geq 0$, at all time t, it means that he is fully committed to repaying its debts. In case $\overline{B}_t^j < 0$, we simply assume that j obtained his financing elsewhere. In order to state his problem, our leader must first consider if

a) The follower is not in default yet; or

b) The follower is already in default.

We now present the problem for each case.

A) THE FOLLOWER IS NOT IN DEFAULT YET

The leader's problem is to maximize the utility of real euro consumption subject to government i's objective function and its constraints, government j's budget constraint, its resource constraint, and country j's money market clearing condition. We have two cases: 1) Government j acknowledges the presence of disequilibrium prices; 2) It does not. We will obtain a monetary policy rule for the CA for each case. Formally, we have the following

$$V^j(\overline{B}^j_{t-1}(s_{t-1}), s_t) = \max_{c^j(s_t), \overline{B}^j_t(s_{t+1})} \left\{ U\left(c^j(s_t, \gamma^j) \left(e^{t,PPP}_{j/€} \right)^{-1} \right) + \beta E\left[V^j(\overline{B}^j_{t+1}(s_{t+2}), s_{t+1}) \right] \right\}, \tag{23}$$

subject to i's objective function

$$V^i(\overline{B}^i_{t-1}(s_{t-1}), s_t) = \max \left\{ \left\{ \frac{\left(c^i(s_t, \gamma^i) \left(e^{t,PPP}_{i/€} \right)^{-1} \right)^{1-\eta} - 1}{1-\eta} \right\} + \beta \left\{ \frac{\left(c^i(s_{t+1}, \gamma^i) \left(e^{t+1,PPP}_{i/€} \right)^{-1} \right)^{1-\eta} - 1}{1-\eta} \right\} \right\}, \tag{24}$$

plus government i's constraints (17), (18) and (19).

Government j must also include its own economy's constraints: its budget constraint at s_t with $t \geq 1$

$$\overline{B}^j_t(s_t) \left(e^{t,PPP}_{j/€} \right)^{-1} = M^j(s_t) \left(e^{t,PPP}_{j/€} \right)^{-1} - M^j(s_{t-1}) \left(e^{t-1,PPP}_{j/€} \right)^{-1} + \int q^j(s_t, s_{t+1}) \theta^{j,j}(s_t, s_{t+1}) \left(e^{t,PPP}_{j/€} \right)^{-1} ds_{t+1} \tag{25}$$

$$+\int q^j(s_t,s_{t+1})\theta^{j,i}(s_t,s_{t+1})\left(e^{t,PPP}_{j/€}\right)^{-1}ds_{t+1}$$

its resource constraint,

$$\int\left\{c^j(s_t,\gamma^j)+\gamma^j z^j(s_t,\gamma^j)\right\}\left(e^{t,PPP}_{j/€}\right)^{-1}f(\gamma^j)d\gamma^j$$
$$=\left(\pi_t^{j,j}+\pi_t^{j,i}\right)\left(e^{t,PPP}_{j/€}\right)^{-1}, \quad (26)$$

and country j's money market clearing condition.

$$\int\left[\left(\frac{P_j(s_{t-1})\left(e^{t-1,PPP}_{j/€}\right)^{-1}}{P_j(s_t)\left(e^{t,PPP}_{j/€}\right)^{-1}}\right)\left(\pi_t^{j,j}+\pi_t^{j,i}\right)+\left(x^j(s_t,\gamma^j)+\gamma^j\right)z^j(s_t,\gamma^j)\right]$$
$$\times\left(e^{t,PPP}_{j/€}\right)^{-1}f(\gamma^j)d\gamma^j=\frac{M^j(s_t)}{P_j(s_t)\left(e^{t,PPP}_{j/€}\right)^{-1}}. \quad (27)$$

Recall that $x^j(s_t,\gamma^j)$, stands for the transfers from the asset market to the goods market, it can come from two sources $x^j(s_t,\gamma^j)=B^{j,j}(s_{t-1})+B^{i,j}(s_{t-1})I_t^i$, payoffs made by government j or payoffs made by government i—$B^{j,j}$ and $B^{i,j}$, respectively—. Payoffs from government i are multiplied by the default indicator variable. The first order condition with respect to consumption is

$$c^j(s_t,\gamma^j)=e^{t,PPP}_{j/€}\left(\lambda_6 f(\gamma^j)e^{t,PPP}_{j/€}\right)^{-1/\eta}.$$

where λ_6 is the resource constraint's Lagrange multiplier.

Country j's payoff without default,

$$V^j(\overline{B}^j_{t-1}(s_{t-1}),s_t)=$$
$$\frac{1+\beta-\left(\left(\lambda_6 f(\gamma^j)e^{t,PPP}_{j/€}\right)^{-1/\eta}\right)^{1-\eta}-\beta\left(\left(\lambda_6 f(\gamma^j)e^{t+1,PPP}_{j/€}\right)^{-1/\eta}\right)^{1-\eta}}{1-\eta}. \quad (28)$$

If country i decides to default during this stage, j's consumption is reduced to $c_t^{j,j}$. Optimal $c_t^{j,j}$ satisfies

$$c_t^{j,j}$$

$$= \left(-\frac{(c_{t+1}^{i,j})^{\xi} I_{t+1}^{i} - \left(e_{j/€}^{t+1,PPP} \left(\lambda_6 f(\gamma^j) e_{j/€}^{t+1,PPP} \right)^{-1/\eta} \right)^{\xi} - (c_{t+1}^{i,j})^{\xi} I_{t+1}^{i} \nu^j}{\nu^j} \right)^{1/\xi}.$$

Government j's payoff in case i defaults is

$$V^j(\overline{B}_{t-1}^j(s_{t-1}), s_t)$$
$$= \frac{\left[\left(\frac{\left(e_{j/€}^{t,PPP} \left(\lambda_6 f(\gamma^j) e_{j/€}^{t,PPP} \right)^{-1/\eta} \right)^{\xi} + (c_t^{i,j})^{\xi} (\nu^j - 1)}{\nu^j} \right)^{1/\xi} \nu^j \left(e_{j/€}^{t,PPP} \right)^{-1} \right]^{1-\eta} - 1}{1-\eta}$$
$$+ \beta \frac{\left[\left(\frac{\left(e_{j/€}^{t,PPP} \left(\lambda_6 f(\gamma^j) e_{j/€}^{t,PPP} \right)^{-1/\eta} \right)^{\xi}}{\nu^j} \right)^{1/\xi} \nu^j \left(e_{j/€}^{t,PPP} \right)^{-1} \right]^{1-\eta} - 1}{1-\eta}. \quad (29)$$

B) THE FOLLOWER IS ALREADY IN DEFAULT

As stated in the Introduction, I_t^j is an indicator function that is equal to 1 if j decides to bailout i. If country i's government is already in default, then government j's problem is as follows

$$V^d(\overline{B}_t^j(s_{t+1}), s_t) =$$
$$\max_{c^j(s_t), \overline{B}_t^j(s_{t+1})} \left\{ U \left(\left[\nu^j (c_t^{j,j})^{\xi^j} + (1-\nu^j)(c_t^{i,j})^{\xi^j} I_t^j \right]^{\frac{1}{\xi^j}} \left(e_{j/€}^{t,PPP} \right)^{-1} \right) \right.$$
$$\left. + \beta E \left[V^d(\overline{B}_{t+1}^j(s_{t+2}), s_{t+1}) \right] \right\}, \quad (30)$$

subject to

$$\begin{aligned} \overline{B}_t^j(s_{t+1}) \left(e_{j/€}^{t,PPP} \right)^{-1} &= M^j(s_t) \left(e_{j/€}^{t,PPP} \right)^{-1} - M^j(s_{t-1}) \left(e_{j/€}^{t-1,PPP} \right)^{-1} \\ &+ \int q^j(s_t, s_{t+1}) \theta^{j,j}(s_t, s_{t+1}) \left(e_{j/€}^{t,PPP} \right)^{-1} ds_{t+1} \quad (31) \\ &+ I_t^j \int q^j(s_t, s_{t+1}) \theta^{j,i}(s_t, s_{t+1}) \left(e_{j/€}^{t,PPP} \right)^{-1} ds_{t+1} \end{aligned}$$

$$\int \left[\left[\nu^j (c_t^{j,j})^{\xi^j} + (1-\nu^j)(c_t^{i,j})^{\xi^j} I_t^j \right]^{\frac{1}{\xi^j}} + \gamma^j z^j(s_t, \gamma^j) \right] \left(e_{j/€}^{t,PPP} \right)^{-1} f(\gamma^j) d\gamma^j$$

$$= \left(\pi_t^{j,j} + \pi_t^{j,i} I_t^j\right) \left(e_{j/€}^{t,PPP}\right)^{-1}, \tag{32}$$

$$\begin{aligned}
&\int \left[\left(\frac{P_j(s_{t-1})}{P_j(s_t)} \right) (\pi_t^{j,j} + \pi_t^{j,i} I_t^j) \right. \\
&\quad \left. + \left(B_t^{j,j}(s_t, \gamma^j) + \gamma^j \right) z^j(s_t, \gamma^j) \right] \left(e_{j/€}^{t,PPP}\right)^{-1} f(\gamma^j) d\gamma^j \\
&\quad = \frac{M^j(s_t)}{P^j(s_t) \left(e_{j/€}^{t,PPP}\right)^{-1}}.
\end{aligned} \tag{33}$$

The follower's restriction is related only to the case of default, v^d, namely equations (34), (35) and (36).

$$\begin{aligned}
&v^d(0, s_t) \\
&\quad = \max_{c_t, \theta^i(s_t, s_{t+1})} \left\{ U \left(\nu^i (c_t^{i,i}) \left(e_{i/€}^{t,PPP}\right)^{-1} \right) + \beta(1 - \phi^i) E \left[v^d(0, s_{t+1}) \right] \right. \\
&\qquad \left. + \beta \phi^i E \left[V(0, s_{t+1}) \right] \right\},
\end{aligned} \tag{34}$$

subject to

$$\nu^i (c_t^{i,i}) \left(e_{i/€}^{t,PPP}\right)^{-1} = \left(\frac{P_i(s_{t-1})}{P_i(s_t)} \right) \pi_t^{i,i} \left(e_{i/€}^{t,PPP}\right)^{-1}, \tag{35}$$

$$\frac{M^{i,*}(s_t) \left(e_{i/€}^{t,PPP}\right)^{-1} - M^{i,*}(s_{t-1}) \left(e_{i/€}^{t-1,PPP}\right)^{-1}}{M^{i,*}(s_{t-1}) \left(e_{i/€}^{t-1,PPP}\right)^{-1}} = 2\%. \tag{36}$$

NO-BAILOUT, $I_t^j = 0$

Optimal consumption in this scenario is given by

$$c_t^{j,j} = \frac{e_{j/€}^{t,PPP} \left(\lambda_5 f(\gamma^j)\right)^{-1/\eta}}{\nu^j}, \tag{37}$$

while country j's payoff is

$$V_j^{d,nb}() = \frac{1 + \beta - \left(\left(\lambda_5 f(\gamma^j)\right)^{-1/\eta} \right)^{1-\eta} - \beta \left(\left(\lambda_5 f(\gamma^j)\right)^{-1/\eta} \right)^{1-\eta}}{\eta - 1}. \tag{38}$$

Observe that $\eta = 1$ is excluded, i.e., the case of risk-neutrality is not possible in the no-bailout scenario.

Country i's payoff in the case i is in default and j does not bailout is

$$v_i^{d,nb}(0, s_t) = \frac{\left(\nu^i c_t^{i,i}\left(e_{i/€}^{t,PPP}\right)^{-1}\right)^{1-\eta} - 1}{1-\eta} + \beta \frac{\left(\nu^i c_{t+1}^{i,i}\left(e_{i/€}^{t+1,PPP}\right)^{-1}\right)^{1-\eta} - 1}{1-\eta}. \quad (39)$$

BAILOUT, $I_t^j = 1$

Optimal consumption if country j decides to bailout is

$$c^j(s_t, \gamma^j) = e_{j/€}^{t,PPP}\left(\lambda_6 f(\gamma^j)\right)^{-1/\eta}. \quad (40)$$

Recall that by definition $c^j(s_t)$ is equal to

$$c^j(s_t, \gamma^i) = \left[\nu^j (c_t^{j,j})^\xi + (1-\nu^j)(c_t^{i,j})^\xi\right]^{1/\xi},$$

substituting into (40)

$$\left[\nu^j (c_t^{j,j})^\xi + (1-\nu^j)(c_t^{i,j})^\xi\right]^{1/\xi} = e_{j/€}^{t,PPP}\left(\lambda_6 f(\gamma^j)\right)^{-1/\eta}.$$

Solving for $c_t^{j,j}$ we have:

$$c_t^{j,j} = \left(-\frac{(c_t^{i,j})^\xi - \nu^j (c_t^{i,j})^\xi - \left(e_{j/€}^{t,PPP}(\lambda_6 f(\gamma^j))^{-1/\eta}\right)^\xi}{\nu^j}\right)^{1/\xi}.$$

Government j's payoff is equation (43) and country i's payoff in the case i is in default and j decides to bailout is (44) both in the Appendix. We may conclude that bailout is optimal for both countries i and j since $V_j^{d,b} > V_j^{d,nb}$ and $v_i^{d,b} > v_i^{d,nb}$. It would be interesting to analyze a repeated game. Because there is no incentive for country i to repay its obligations if bailout will always be the answer.

Table 1: Sovereign default game results.

	Bailout	No-Bailout
Default	$v_i^d > v_i^{nd}$ and $V_j^b > V_j^{nb}$	$v_i^d > v_i^{nd}$ and $V_j^b \leq V_j^{nb}$
No-Default	$v_i^d \leq v_i^{nd}$ and $V_j^b > V_j^{nb}$	$v_i^d \leq v_i^{nd}$ and $V_j^b \leq V_j^{nb}$

5. EQUILIBRIUM

Summarizing if country i is not in default at the beginning of t

1. Case v_i^{nd}: If country i is not in default at the beginning of t and has no incentive to default by the end of t its payoff is given by equation (41); while country j's payoff in this scenario is equation (28).

2. Case v_i^d: If, on the contrary, country i decides to default by the end of t, it receives payoff (42); and j receives (29).

If country i is in default at the beginning of t

1. Case V_j^{nb}: If country j decides against bailout of i it receives payoff (38); and i receives (39).

2. Case V_j^b: If country j decides in favor of bailout it receives (43); and i receives (44).

Table 1 summarizes our findings. The case Default-Bailout is read as follows: country i decides to default; country j decides to bailout and in turn country i receives a certain payoff for the default-bailout interaction. This is an equilibrium: default was optimal for i and bailout is optimal for both j and i.

The case Default-No-Bailout is not an equilibrium since bailout dominates no-bailout.

The case No-Default-Bailout is not an equilibrium. There can be no bailout without default.

The case No-Default-No-Bailout is the case where i does not need to default and hence j does not need to bailout. This is also an equilibrium because both are maximizing their objective functions.

Proposition 1. *The equilibria generated in the sovereign default game are both Nash and Pareto optimal.*

Proof. Both equilibria in Table 1 are Nash optimal. Both equilibria are competitive by construction, hence they are Pareto optimal. □

There is a problem with equilibrium Default-Bailout: it induces moral hazard. Hence the model predicts more defaults in the future: what caused default was over-valuation, but bailout does not move sovereign i to control inflation (it motivates the opposite), therefore, the problem is not solved and Default-Bailout can be expected to repeat. If government j wants to avoid moral hazard it can expel i, but this would be the end of the CA, hence this result is not an equilibrium.

6. CONCLUSIONS

In order to explore the existence and the nature of equilibria in the event of default in a CA, we develop a two player dynamic game. We find two equilibria: no-default/no-bailout and default/bailout. Both are maxima of maxima: Nash and Pareto optimal. However, the default/bailout equilibrium is "perverse" in the sense that it generates moral hazard. There is an alternative to bailout, expulsion of the defaulter, but this is not optimal for the CA.

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APPENDIX

Government i's payoff without default at t, v^{nd}
Substituting (20) in (22) and taking $I_t^i = 1$ we have

$$
\begin{aligned}
&v^{nd}(\overline{B}^i_{t-1}(s_{t-1}), s_t)\\
&= \frac{\left(\left(\nu^i\left(\left(-\frac{(c_t^{j,i})^\xi-\nu^i(c_t^{j,i})^\xi-\left(e_{i/€}^{t,PPP}(f(\gamma^i))^{-1/\eta}\lambda_2^{-1/\eta}\right)^\xi}{\nu}\right)^{1/\xi}\right)^{\xi}+(1-\nu^i)(c_t^{j,i})^\xi\right)^{1/\xi}\left(e_{i/€}^{t,PPP}\right)^{-1}\right)^{1-\eta}-1}{1-\eta}\\
&+\beta\left\{\frac{\left(\left(\nu^i\left(\left(-\frac{(c_{t+1}^{j,i})^\xi-\nu^i(c_{t+1}^{j,i})^\xi-\left(e_{i/€}^{t+1,PPP}(f(\gamma^i))^{-1/\eta}\lambda_2^{-1/\eta}\right)^\xi}{\nu}\right)^{1/\xi}\right)^{\xi}+(1-\nu^i)(c_{t+1}^{j,i})^\xi\right)^{1/\xi}\left(e_{i/€}^{t+1,PPP}\right)^{-1}\right)^{1-\eta}-1}{1-\eta}\right\},
\end{aligned}
\tag{41}
$$

If country i's government does not default at t, then country j receives its optimal value.
Government i's payoff with default at t, v^d

Substituting (20) in (22) and taking $I_t^i = 0$ we have

$$
\begin{aligned}
&v^d(\overline{B}^i_{t-1}(s_{t-1}), s_t) \\
&= \frac{\left(\left(\nu^i \left(\left(-\frac{(c_t^{j,i})^\xi - \nu^i (c_t^{j,i})^\xi - \left(e_{i/€}^{t,PPP} (f(\gamma^i))^{-1/\eta} \lambda_2^{-1/\eta}\right)^\xi}{\nu}\right)^{1/\xi}\right)^\xi + (1-\nu^i)(c_t^{j,i})^\xi\right)^{1/\xi} \left(e_{i/€}^{t,PPP}\right)^{-1}\right)^{1-\eta} - 1}{1-\eta} \\
&+ \beta \left\{ \frac{\left(\nu^i \left(\frac{\left(e_{i/€}^{t+1,PPP} (f(\gamma^i))^{-1/\eta} \lambda_2^{-1/\eta}\right)^\xi}{\nu}\right)^{1/\xi} \left(e_{i/€}^{t+1,PPP}\right)^{-1}\right)^{1-\eta} - 1}{1-\eta} \right\},
\end{aligned}
\tag{42}
$$

Government j**'s payoff** is

$$V_j^{d,b}() = \frac{\left(-1+\left(e_{j/€}^{t,PPP}\right)^{-1}\left(\left((c_t^{i,j})^{\xi}-(c_t^{i,j})^{\xi}\nu^j+\left(\left(\frac{1}{\nu^j}\left(\left(e_{j/€}^{t,PPP}(\lambda_6 f(\gamma^j))^{-1/\eta}\right)^{\xi}+c_t^{i,j}(\nu^j-1)\right)\right)^{1/\xi}\right)^{\xi}\nu^j\right)^{1/\xi}\right)\right)^{1-\eta}}{1-\eta}$$

$$+\frac{\beta\left(-1+\left(\left(e_{j/€}^{t+1,PPP}\right)^{-1}\left(\left(-(c_{t+1}^{i,j})^{\xi}I_{t+1}^{i}(\nu^j-1)+\left(\left(\frac{1}{\nu^j}\left(\left(e_{j/€}^{t+1,PPP}(\lambda_6 f(\gamma^j))^{-1/\eta}\right)^{\xi}+(c_{t+1}^{i,j})^{\xi}(\nu^j-1)\right)\right)^{1/\xi}\right)^{\xi}\nu^j\right)^{1/\xi}\right)\right)^{1-\eta}\right)}{1-\eta} \tag{43}$$

Country i**'s payoff** in the case i is in default and j decides to bailout is

$$v_i^{d,b}(0,s_t) = \frac{\left(\nu^i c_t^{i,i}\left(e_{i/€}^{t,PPP}\right)^{-1}\right)^{1-\eta}-1}{1-\eta}$$

$$+\beta\frac{\left(\left(\nu^i(c_{t+1}^{i,i})^{\xi}+(1-\nu^i)(c_{t+1}^{j,i})^{\xi}\right)^{1/\xi}\left(e_{i/€}^{t+1,PPP}\right)^{-1}\right)^{1-\eta}-1}{1-\eta} \tag{44}$$